Chapter 1

Concept Application Exercises

Exercise 1.1

1. We have set, $A = \{1, 2, \{3, 4\}, 5\}$

(a) The statement $\{3, 4\} \subset A$ is incorrect because $3 \in \{3, 4\}$, however, $3 \notin A$.

(b) We know that the element of the set can be a set.
So, the statement $\{3, 4\} \in A$ is correct because $\{3, 4\}$ is an element of A.

(c) The statement $\{\{3, 4\}\} \subset A$ is correct because $\{3, 4\} \in \{\{3, 4\}\}$ and $\{3, 4\} \in A$.

(d) The statement $1 \in A$ is correct because 1 is an element of A.

(e) The statement $1 \subset A$ is incorrect because an element of a set can never be a subset of the set.

(f) The statement $\{1, 2, 5\} \subset A$ is correct because each element of $\{1, 2, 5\}$ is element of the set A.

(g) The statement $\{1, 2, 5\} \in A$ is incorrect because set $\{1, 2, 5\}$ is not an element of the set A.

(h) The statement $\{1, 2, 3\} \subset A$ is incorrect because $3 \in \{1, 2, 3\}$, however $3 \notin A$.

(i) The statement $\phi \in A$ is incorrect because ϕ is a not an element of the set A.

(j) The statement $\phi \subset A$ is correct because ϕ is a subset of every set.

(k) The statement $\{\phi\} \subset A$ is incorrect because $\phi \in \{\phi\}$, however $\phi \notin A$.

2. Given, $X = \{1, 2, 3, \ldots, 10\}$

(a) Since $1^2, 2^2, 3^2 \in X$

$\therefore \quad \{a \mid a \in X \text{ and } a^2 \notin X\} = \{4, 5, 6, 7, 8, 9, 10\}$

(b) $a \in X$ but $a/2 \notin X$

$\Rightarrow \quad a = 1, 3, 5, 7, 9$

$\therefore \quad \{a \mid a \in X \text{ but } a/2 \notin X\} = \{1, 3, 5, 7, 9\}$

(c) a is factor of 24

$\Rightarrow \quad a = 1, 2, 3, 4, 6, 8$

$\therefore \quad \{a \mid a \text{ is factor of } 24\} = \{1, 2, 3, 4, 6, 8\}$

3. (a) The subsets of $\{p\}$ are ϕ and $\{p\}$.

(b) The subsets of $\{p, q\}$ are ϕ, $\{p\}$, $\{q\}$, and $\{p, q\}$.

(c) The subsets of $\{1, 2, 3, 4\}$ are ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{2, 3, 4\}$, $\{3, 4, 1\}$, $\{4, 1, 2\}$, $\{1, 2, 3, 4\}$

(d) The only subset of ϕ is ϕ.

4. Let set A have n elements.
Then number of elements in its power set $P(A)$ is 2^n.
According to the question,

$2^n = 128 = 2^7$

$\therefore \quad n = 7$

5. Let $A = \{2\}$, $B = \{0, 2\}$, and $C = \{1, \{0, 2\}, 3\}$
Here, $A \subset B$, $B \in C$
However, $A \notin C$
So, the given statement is false.

6. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$

(a) $A' = \{5, 6, 7, 8, 9\}$

(b) $B' = \{1, 3, 5, 7, 9\}$

(c) $A \cup C = \{1, 2, 3, 4, 5, 6\}$
$\therefore \quad (A \cup C)' = \{7, 8, 9\}$

(d) $A \cup B = \{1, 2, 3, 4, 6, 8\}$
$\therefore \quad (A \cup B)' = \{5, 7, 9\}$

(e) $(A')' = A = \{1, 2, 3, 4\}$

(f) $B - C = \{2, 8\}$
$\therefore \quad (B - C)' = \{1, 3, 4, 5, 6, 7, 9\}$

7. We have $A - B = \phi$, which means that all the elements of set A are common with elements of set B.
Also, $B - A = \phi$, which means that all the elements of set B are common with elements of set A.
So, $A = B$.

8. Let $A = \{0, 1\}$ and $B = \{1, 2\}$

$\therefore \quad A \cup B = \{0, 1, 2\}$
$P(A) = \{\phi, \{0\}, \{1\}, \{0, 1\}\}$
$P(B) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$
$P(A \cup B) = \{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}$
$P(A) \cup P(B) = \{\phi, \{0\}, \{1\}, \{0, 1\}, \{2\}, \{1, 2\}\}$

$\therefore \quad P(A) \cup P(B) \neq P(A \cup B)$

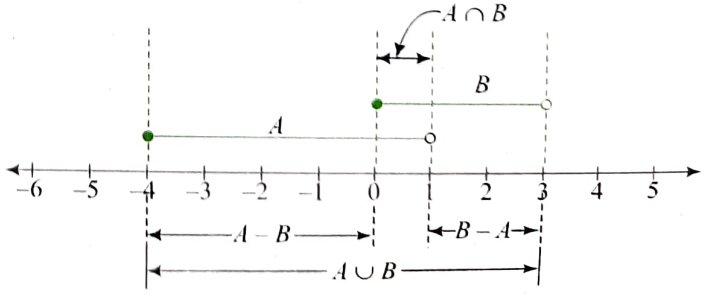
9. We have $A \cap B = B \cap X = \phi$ and $A \cup X = B \cup X$
It can be seen that

$A = A \cap (A \cup X) = A \cap (B \cup X)$ $\quad [A \cup X = B \cup X]$
$= (A \cap B) \cup (A \cap X)$ $\quad$ [Distributive law]
$= (A \cap B) \cup \phi$ $\quad [A \cap X = \phi]$
$= A \cap B$ $\quad \ldots (1)$

Now, $B = B \cap (B \cup X)$
$= B \cap (A \cup X)$ $\quad [A \cup X = B \cup X]$
$= (B \cap A) \cup (B \cap X)$ $\quad$ [Distributive law]
$= (B \cap A) \cup \phi$ $\quad [B \cap X = \phi]$
$= B \cap A$
$= A \cap B$ $\quad \ldots (2)$

Hence, from (1) and (2), we obtain $A = B$.

10.



(a) $A \cap B = [0, 1)$

(b) $A \cup B = [-4, 3)$

(c) $A - B = [-4, 0)$

(d) $B - A = [1, 3)$

(e) $(A \cup B)' = (-\infty, -4) \cup [3, \infty)$

(f) $(A \cap B)' = (-\infty, 0) \cup [1, \infty)$

11. Let

U be the set of all students who took part in the survey,

T be the set of students who like tea, and

C be the set of students who like coffee.

According to the question,

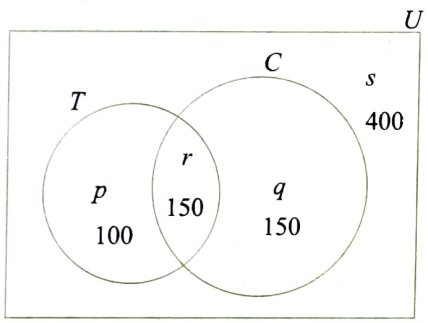
$n(U) = 800,\ n(T) = 250,\ n(C) = 300,\ n(T \cap C) = 150$

To find the number of students taking neither tea nor coffee i.e., we have to find $n(T' \cap C')$

$$\begin{aligned} n(T' \cap C') &= n((T \cup C)') \\ &= n(U) - n(T \cup C) \\ &= n(U) - [n(T) + n(C) - n(T \cap C)] \\ &= 800 - [250 + 300 - 150] = 400 \end{aligned}$$

Hence, 400 students like neither tea nor coffee.

Alternative method:



We can use Venn diagram as shown in the figure.

In the above figure regions are

$r \rightarrow$ students liking tea and coffee both $= n(T \cap C)$

$p \rightarrow$ students liking tea only $= n(T) - n(T \cap C)$

$q \rightarrow$ students liking coffee only $= n(C) - n(T \cap C)$

$s \rightarrow$ students liking neither tea nor coffee $= n(T' \cap C')$

Here, $r = 150$ (given)

Now, $p + r =$ students liking tea $= 250$ (given)

$\Rightarrow \quad p = 250 - 150 = 100$

$q + r =$ students liking coffee $= 300$ (given)

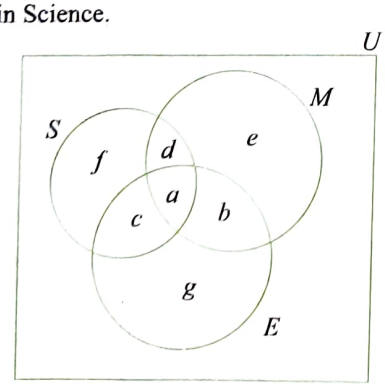
$\Rightarrow \quad q = 300 - 150 = 150$

$p + q + r =$ students liking at least one of tea and coffee

$= 100 + 150 + 150 = 400$

$s =$ students liking neither tea nor coffee $= 800 - 400 = 400$

12. Let M be the set of students who passed in Mathematics, E be the set of students who passed in English and S be the set of students who passed in Science.



Given

$n(U) = 100,$

$n(E) = 15,\ n(M) = 12,\ n(S) = 8,$

$n(E \cap M) = 6,\ n(M \cap S) = 7,$

$n(E \cap S) = 4,$ and $n(E \cap M \cap S) = 4,$

From the Venn diagram, we have

$a = n(E \cap M \cap S) = 4$ and $a + d = n(M \cap S) = 7 \Rightarrow d = 3$

$a + b = n(M \cap E) = 6 \quad \Rightarrow b = 2$

$a + c = n(S \cap E) = 4 \quad \Rightarrow c = 0$

$a + b + d + e = n(M) = 12$

$\Rightarrow \quad 4 + 2 + 3 + e = 12 \quad \Rightarrow e = 3$

$a + b + c + g = 15$

$\Rightarrow \quad 4 + 2 + 0 + g = n(E) = 15 \quad \Rightarrow g = 9$

$a + c + d + f = n(S) = 8$

$\Rightarrow \quad 4 + 0 + 3 + f = 8 \quad \Rightarrow f = 1$

Number of students passed in

(a) English and Mathematics but not in Science $= b = 2$

(b) Mathematics and Science but not in English $= d = 3$

(c) Mathematics only $= e = 3$

(d) more than one subject $= a + b + c + d = 4 + 2 + 0 + 3 = 9$

Exercise 1.2

1. $-3 < 2x - 1 < 19$

or $\quad -2 < 2x < 20$

or $\quad -1 < x < 10 \qquad (1)$

$$-1 \le \frac{2x+3}{5} \le 3$$

or $\quad -5 \le 2x + 3 \le 15$

or $\quad -8 \le 2x \le 12$

or $\quad -4 \le x \le 6 \qquad (2)$

Now from (1) and (2), common values are $(-1, 6]$

2. Let the length of the shortest side of the triangle be x cm.

Then, length of the longest side $= 3x$ cm

Length of the third side $= (3x - 2)$ cm

Since the perimeter of the triangle is at least 61 cm.

$x + 3x + (3x - 2) \ge 61$

$\Rightarrow \quad 7x - 2 \ge 61$

$\Rightarrow \quad 7x \ge 61 + 2$

$\Rightarrow \quad 7x \ge 63$

$\Rightarrow \quad x \ge 9$

Thus, the minimum length of the shortest side is 9 cm.

3. (a) $-5 \le x \le -1$

$\Rightarrow \quad x^2 \in [1, 5]$

(b) $3 < x < 6$

$\Rightarrow \quad x^2 \in (9, 36)$

(c) $-2 < x \le 3$

$\Rightarrow \quad -2 < x < 0$ or $0 \le x \le 3$

For $-2 < x < 0,\ x^2 \in (0, 4) \qquad (1)$

and for $0 \le x \le 3,\ x^2 \in [0, 9] \qquad (2)$

From (1) and (2), $x^2 \in [0, 9]$

Alternatively, $x \in (-2, 3]$, now least value of x^2 is 0 which occurs when $x = 0$

Greatest value of x^2 is 9 for $x = 3$

$\Rightarrow \quad x^2 \in [0, 9]$

(d) $(-3, \infty)$

Here least value of x^2 is 0 for $x = 0$, and when x goes up to infinity, x^2 also goes up to infinity

$\Rightarrow \quad x^2 \in [0, \infty)$

(e) $(-\infty, 4)$

Here least value of x^2 is 0 for $x = 0$ and $x^2 \rightarrow \infty$, when $x \rightarrow -\infty$

Hence, $x^2 \in [0, \infty)$

4. (a) $2 < x < 5$

$\Rightarrow \frac{1}{2} > \frac{1}{x} > \frac{1}{5}$

$\Rightarrow \frac{1}{x} \in \left(\frac{1}{5}, \frac{1}{2}\right)$

(b) $-5 \le x < -1$

$\Rightarrow -\frac{1}{5} \ge \frac{1}{x} > -1$

$\Rightarrow \frac{1}{x} \in \left(-1, -\frac{1}{5}\right]$

(c) $x > 3$ or $3 < x < \infty$

$\Rightarrow \frac{1}{3} > \frac{1}{x} > 0$

$\Rightarrow \frac{1}{x} \in \left(0, \frac{1}{3}\right)$

(d) $x \le -2$ or $-\infty < x \le -2$

$\Rightarrow 0 > \frac{1}{x} \ge -\frac{1}{2}$

$\Rightarrow \frac{1}{x} \in \left[-\frac{1}{2}, 0\right)$

(e) $x \in [-3\ 4]$ or $-3 \le x \le 4$

Now for $\frac{1}{x}$ to get defined, we must have $-3 \le x < 0$ or $0 < x \le 4$

$\Rightarrow -\frac{1}{3} \ge \frac{1}{x} > -\infty$ or $\frac{1}{4} \le \frac{1}{x} < \infty$

$\Rightarrow \frac{1}{x} \in \left(-\infty, -\frac{1}{3}\right] \cup \left[\frac{1}{4}, \infty\right)$

5. (a) $4x^2 + 28x + 41 = (2x+7)^2 - 8$

Now, $(2x+7)^2 \ge 0\ \forall\ x \in R$.

$\Rightarrow (2x+7)^2 - 8 \ge -8\ \ \forall\ x \in R$

So, the values of the expression are $[-8, \infty)$.

For $x \in [-3, 2]$

$-3 \le x \le 2$

$\Rightarrow -6 \le 2x \le 4$

$\Rightarrow 1 \le 2x + 7 \le 11$

$\Rightarrow 1 \le (2x+7)^2 \le 121$

$\Rightarrow -7 \le (2x+7)^2 - 8 \le 113$

Thus, for $x \in [-3, 2]$, the values of the expression are $[-7, 113]$.

(b) $1 + 6x - x^2 = 10 - (x-3)^2$

Now, $(x-3)^2 \ge 0\ \ \forall\ x \in R$.

$\Rightarrow -(x-3)^2 \le 0\ \forall\ x \in R$

$\Rightarrow 10 - (x-3)^2 \le 10\ \forall\ x \in R$

So, the values of the expression are $(-\infty, 10]$.

For $x \in [-3, 2]$

$-3 \le x \le 2$

$\Rightarrow -6 \le x - 3 \le -1$

$\Rightarrow 1 \le (x-3)^2 \le 36$

$\Rightarrow -36 \le -(x-3)^2 \le -1$

$\Rightarrow -26 \le 10 - (x-3)^2 \le 9$

Thus, for $x \in [-3, 2]$ values of the expression are $[-26, 9]$.

6. $\frac{2+x^2}{4-x^2} = \frac{6-(4-x^2)}{4-x^2} = \frac{6}{4-x^2} - 1$

Now, $x^2 \ge 0$

$\therefore \quad -x^2 \le 0$

$\Rightarrow \quad 4 - x^2 \le 4$

$\Rightarrow \quad \frac{4-x^2}{6} \le \frac{2}{3}$

$\Rightarrow \quad -\infty < \frac{4-x^2}{6} < 0$ or $0 < \frac{4-x^2}{6} \le \frac{2}{3}$

$\Rightarrow \quad 0 > \frac{6}{4-x^2} > -\infty$ or $\infty > \frac{6}{4-x^2} \ge \frac{3}{2}$

$\Rightarrow \quad \frac{6}{4-x^2} \in (-\infty, 0) \cup \left[\frac{3}{2}, \infty\right)$

$\Rightarrow \quad \left(\frac{6}{4-x^2} - 1\right) \in (-\infty, -1) \cup \left[\frac{1}{2}, \infty\right)$

7. $\frac{\sqrt{x-1}}{x-2} < 0$ is meaningful for $x - 1 \ge 0$ or $x \ge 1$ (1)

Also $\frac{\sqrt{x-1}}{x-2} < 0$

$\Rightarrow \quad x - 2 < 0$ or $x < 2$ (2)

$\therefore \quad 1 \le x < 2$

8. $\sqrt{x-2} \le 3$ is meaningful for $x - 2 \ge 0$ or $x \ge 2$ (1)

Also $\sqrt{x-2} \le 3$

$\Rightarrow \quad x - 2 \le 9$

$\Rightarrow \quad x \le 11$ (2)

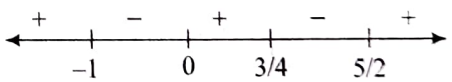
From (1) and (2), we have $2 \le x \le 11$

Exercise 1.3

1. $\frac{x(3-4x)(x+1)}{(2x-5)} < 0$

or $-\frac{x(4x-3)(x+1)}{(2x-5)} < 0$

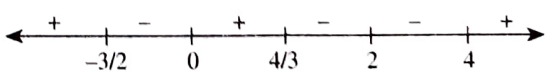
or $\frac{x(4x-3)(x+1)}{(2x-5)} > 0$



$\Rightarrow \quad x \in (-\infty, -1) \cup (0, 3/4) \cup (5/2, \infty)$

2. $\frac{(2x+3)(4-3x)^3(x-4)}{(x-2)^2 x^5} \le 0$

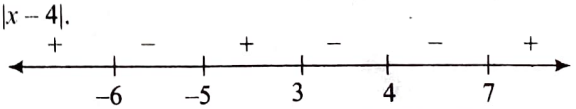
or $\frac{(2x+3)(3x-4)^3(x-4)}{(x-2)^2 x^5} \ge 0$



From the sign scheme, we have

$x \in (-\infty, -3/2) \cup (0, 4/3] \cup [4, \infty)$

3. Sign of expression does not change while crossing $x = 4$ as we have $|x-4|$.



From the sign scheme, we have $x \in (-6, -5] \cup [3, 4) \cup (4, 7]$

4. $\frac{5x+1}{(x+1)^2} - 1 < 0$

or $\frac{5x+1-(x+1)^2}{(x+1)^2} < 0$

or $\frac{-x^2+3x}{(x+1)^2} < 0$

or $\frac{x(x-3)}{(x+1)^2} > 0$

$\Rightarrow$ $x < 0$ or $x > 3, x \neq -1$

5. $\frac{x+2}{x^2+1} > \frac{1}{2}$

or $\frac{x+2}{x^2+1} - \frac{1}{2} > 0$

or $\frac{-x^2-1+2x+4}{2(x^2+1)} > 0$

or $\frac{3+2x-x^2}{2(x^2+1)} > 0$

or $x^2 - 2x - 3 < 0$

or $(x-3)(x+1) < 0$

$\Rightarrow$ $-1 < x < 3$

6. $4^{-x+0.5} - 7.2^{-x} < 4$ (1)

Let $2^{-x} = t$

$\therefore$ $t > 0$ (2)

Now (1) reduces to $2t^2 - 7t < 4$

or $2t^2 - 7t - 4 < 0$

or $(2t+1)(t-4) < 0$

or $\frac{-1}{2} < t < 4$ (3)

From (2) and (3), we have

$0 < t < 4$

$\Rightarrow$ $0 < 2^{-x} < 4$

$\Rightarrow$ $-\infty < -x < 2$

$\Rightarrow$ $-2 < x < \infty$

$\Rightarrow$ $x \in (-2, \infty)$

7. $\frac{x^4}{(x-2)^2} > 0$ is always true except for $x = 0, 2$.

Hence, $x \in R - \{0, 2\}$

8. $\frac{6x^2-5x-3}{x^2-2x+6} \leq 4$

or $\frac{6x^2-5x-3}{x^2-2x+6} - 4 \leq 0$

or $\frac{2x^2+3x-27}{(x-1)^2+5} \leq 0$

or $2x^2 + 3x - 27 \leq 0$

or $(2x+9)(x-3) \leq 0$

or $-\frac{9}{2} \leq x \leq 3$

9. $\frac{(x+2)(x^2-2x+1)}{-x^2+3x-4} \geq 0$

or $\frac{(x+2)(x-1)^2}{x^2-3x+4} \leq 0$

Now $x^2 - 3x + 4 > 0 \; \forall \; x \in R$.

$\therefore$ $(x+2)(x-1)^2 \leq 0$

$\Rightarrow$ $x \in (-\infty, -2] \cup \{1\}$

10. $\sqrt{x+2} \geq x$

$\Rightarrow$ $x + 2 \geq x^2$ for $x \geq 0$

$\Rightarrow$ $x^2 - x - 2 \leq 0$

$\Rightarrow$ $(x-2)(x+1) \leq 0$

$\Rightarrow$ $x \in [0, 2]$

$x < 0$ always valid but $x + 2 \geq 0$

$\Rightarrow$ $x \geq -2$

$x \in [-2, 2]$

11. Since $\sqrt{2+x-x^2}$ is defined only if

$2 + x - x^2 \geq 0$

or $(x-2)(x+1) \leq 0$

or $-1 \leq x \leq 2$.

For these values of x, R.H.S. < 0; hence, $x \in [-1, 2]$

12. $\sqrt{-x^2+10x-16} < x - 2$

We must have $-x^2 + 10x - 16 \geq 0$

or $x^2 - 10x + 16 \leq 0$

$\Rightarrow$ $2 \leq x \leq 8$ (1)

Also $-x^2 + 10x - 16 < x^2 - 4x + 4$

or $2x^2 - 14x + 20 > 0$

or $x^2 - 7x + 10 > 0$ (2)

$\Rightarrow$ $x > 5$ or $x < 2$

From (1) and (2), $5 < x \leq 8 \Rightarrow x = 6, 7, 8$

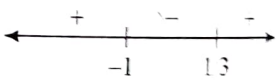
13. $f(x) = y = \frac{1-x^2}{x^2+3}$

or $yx^2 + 3y = 1 - x^2$

or $x^2 = \frac{1-3y}{y+1}$

Now $x^2 \geq 0$

$\Rightarrow$ $\frac{1-3y}{y+1} \geq 0$



$\Rightarrow$ $\frac{3y-1}{y+1} \leq 0$

Hence, $y \in (-1, 1/3]$

Exercise 1.4

1. **(a)** is not always true: for $a = -4$ and $b = 3$, given statement is not true

(b) is not always true: for $a = -3$ and $b = 2$, given statement is not true

(c) is not always true: for $a = -5$ and $b = 3$, given statement is not true

2. $x^2 - |x| - 2 = 0$

$\Rightarrow$ $|x|^2 - |x| - 2 = 0$

$\Rightarrow$ $(|x| - 2)(|x| + 1) = 0$

$\Rightarrow$ $|x| = 2 \Rightarrow x = \pm 2$.

$x^2 - 2|x| + 3 = 0$

Thus, $|x|^2 - 2|x| + 3 = 0$ has no roots as discriminant is negative.

$x^2 - 3|x| + 2 = 0$

$\Rightarrow$ $|x|^2 - 3|x| + 2$

$\Rightarrow$ $(|x| - 2)(|x| - 1) = 0$

$\Rightarrow$ $|x| = 1, 2 \Rightarrow x = \pm 1, \pm 2$.

$\Rightarrow$ $x^2 + 3|x| + 2 = 0$

$\Rightarrow$ $|x|^2 + 3|x| + 2 = 0$

$\Rightarrow$ $(|x| + 2)(|x| + 1) = 0$

$\Rightarrow$ $|x| = -1, -2$ (not possible)

Thus, the equation (c) has maximum real roots.

3. **Case I:** Let $x \geq 3$

$$\left.\begin{aligned} x + 2y &= 6 \\ \therefore\quad x - 3 &= y \end{aligned}\right\}$$

$\Rightarrow$ $3x = 12$

$\Rightarrow$ $x = 4, y = 1$

Case II : Let $x < 3$

$$\left.\begin{aligned} x + 2y &= 6 \\ 3 - x &= y \end{aligned}\right\}$$

$\Rightarrow$ $-x = 0$ or $x = 0, y = 3$

Hence, the solutions are (0, 3) and (4, 1)

4. $|x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x < 2 \end{cases}$

$\Rightarrow$ $|x-2| - (x-2) = \begin{cases} 0, & x \geq 2 \\ 4-2x, & x < 2 \end{cases}$

Therefore, given expression is defined for $(-\infty, 2)$

5. $f(x) = x + \sqrt{x^2} = x + |x|$

$$= \begin{cases} x+x, & x \geq 0 \\ x-x, & x < 0 \end{cases}$$

$$= \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Hence, $f(x)$ takes values $[0, \infty)$

6. $\left|\frac{x+2}{x-1}\right| = 2$

or $\frac{x+2}{x-1} = \pm 2$

$\Rightarrow$ $x + 2 = 2x - 2$ or $x + 2 = 2 - 2x$

$\Rightarrow$ $x = 4$ or $x = 0$

Here, both $x = 0$ and 4 satisfy the original equation.

7. $2x - 1 = x$ if $x \geq 0$

For which $x = 1$

Also $2x - 1 = -x$ if $x < 0$

For which $x = 1/3$ which is not the solution as $x < 0$

Hence, the only solution is $x = 1/3$

8. $2^x + 1 > 0, x \in R$

Therefore, given equation is $|2^x - 1| + 2^x + 1 = 2$

or $|2^x - 1| = 1 - 2^x$

or $2^x - 1 \leq 0$

or $2^x \leq 1$

or $x \leq 0$

9. $|x^2 + 4x + 3| = x + 1$

or $|(x + 1)(x + 3)| = x + 1$

$|(x + 1)(x + 3)| = (x + 1)(x + 3)$, when $(x + 1)(x + 3) \geq 0$

or $x \leq -3$ or $x \geq -1$

Hence, given equation reduces to $(x + 1)(x + 3) = x + 1$

$\Rightarrow$ $x = -1$ ($x = -2$ is rejected as $x \leq -3$ or $x \geq -1$)

$|(x + 1)(x + 3)| = -(x + 1)(x + 3)$, when $(x + 1)(x + 3) < 0$

or $-3 < x < -1$

Hence, given equation reduces to $-(x + 1)(x + 3) = x + 1$

$\Rightarrow$ $x = -4$ which is rejected as $-3 < x < -1$

10. $|4 - |x - 1|| = 3$

$\Rightarrow$ $|x - 1| - 4 = \pm 3$

$\Rightarrow$ $|x - 1| = 7, |x - 1| = 1$

$\Rightarrow$ $x = -6, 0, 2, 8$

11. Let $f(x) = |x - 1| - |2x - 5|$

A. x	B. $f(x)$	C. x for which $f(x) = 2x$	D. $A \cap C$
$x < 1$	$1 - x - (5 - 2x)$	$x - 4 = 2x$ or $x = -4$	$x = -4$
$1 \leq x \leq 5/2$	$x - 1 - (5 - 2x)$	$3x - 6 = 2x$ or $x = 6$	No such x exists
$x > 5/2$	$x - 1 - (2x - 5)$	$4 - x = 2x$ or $x = 4/3$	No such x exists

Hence, solutions set is $\{-4\}$

Exercise 1.5

1. $|x^2 - 7| \leq 9$

or $-9 \leq x^2 - 7 \leq 9$

or $-2 \leq x^2 \leq 16$

or $x^2 \leq 16$

$\Rightarrow$ $x \in [-4, 4]$

2. $||x - 2| - 3| < 5$

or $-5 < |x - 2| - 3 < 5$

or $-2 < |x - 2| < 8$

or $|x - 2| < 8$

or $-8 < x - 2 < 8$

or $-6 < x < 10$

3. $|x + y| = |x| + |y|$

$\Rightarrow$ x and y have same sign or at least one of x and y is zero.

$\Rightarrow$ (x, y) lies in 1st or 3rd quadrant or any of the x-axis or y-axis

$|x + y| < |x| + |y|$

$\Rightarrow$ x and y have opposite sign

$\Rightarrow$ (x, y) lies in 2nd or 4th quadrant

$|x - y| = |x| + |y|$

$\Rightarrow$ x and $-y$ have same sign

$\Rightarrow$ x and y have opposite sign

$\Rightarrow$ (x, y) lies in 2nd or 4th quadrant at least one of x and y is zero.

4. We must have $5 - |2x - 3| \geq 0$

or $|2x - 3| \leq 5$

or $-5 \leq 2x - 3 \leq 5$

$\Rightarrow$ $x \in [-1, 4]$

5. $|x^2 - x - 2| + |x + 6| = |x^2 - 2x - 8|$

or $|x^2 - x - 2| + |x + 6| = |(x^2 - x - 2) - (x + 6)|$

or $|x^2 - x - 2| + |-x - 6| = |(x^2 - x - 2) + (-x - 6)|$

or $(x^2 - x - 2)(-x - 6) \geq 0$

or $(x^2 - x - 2)(x + 6) \leq 0$

or $(x - 2)(x + 1)(x + 6) \leq 0$

or $x \in (-\infty, -6] \cup [-1, 2]$

6. $|x^2 + x - 6| < 6$

$\Rightarrow \quad -6 < x^2 + x - 6 < 6$

Now $-6 < x^2 + x - 6$

or $\quad x^2 + x > 0$

or $\quad x(x+1) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (0, \infty) \quad (1)$

$x^2 + x - 6 < 6$

or $\quad x^2 + x - 12 < 0$

or $\quad (x+4)(x-3) < 0$

$\Rightarrow \quad x \in (-4, 3) \quad (2)$

From (1) and (2), common values are $x \in (-4, -1) \cup (0, 3)$

7. $|x| + \left|\frac{4 - x^2}{x}\right| = \left|\frac{4}{x}\right|$

$\Rightarrow \quad |x| + \left|\frac{4}{x} - x\right| = \left|x + \left(\frac{4}{x} - x\right)\right|$

$\therefore \quad x\left(\frac{4}{x} - x\right) \geq 0, x \neq 0$

$\Rightarrow \quad x^2 - 4 \leq 0$

$\Rightarrow \quad x \in [-2, 2] - \{0\}$

8. $\frac{1}{|x| - 3} - \frac{1}{2} < 0$

or $\quad \frac{2 - |x| + 3}{|x| - 3} < 0$

or $\quad \frac{|x| - 5}{|x| - 3} > 0$

$\Rightarrow \quad |x| > 5$ or $|x| < 3$

$\Rightarrow \quad x > 5$ or $x < -5$ or $-3 < x < -3$

$\Rightarrow \quad x \in (-\infty, -5) \cup (-3, 3) \cup (5, \infty)$

9. $|x - \pi| + |x^2 - \pi^2| \leq 0$

or $\quad |x - \pi|(1 + |x + \pi|) \leq 0$

or $\quad |x - \pi| = 0$

or $\quad x = \pi$

10. $\left|1 - \frac{|x|}{1 + |x|}\right| \geq \frac{1}{2}$

or $\quad \left|\frac{1}{1 + |x|}\right| \geq \frac{1}{2}$

or $\quad 1 + |x| \leq 2$

or $\quad |x| \leq 2$

$-1 \leq x \leq 1$

11. $\frac{|x-1|}{x+2} > 1$

$\Rightarrow \quad \frac{|x-1| - (x+2)}{x+2} > 0$

Case I: when $x \geq 1$

$$\frac{(x-1) - (x+2)}{(x+2)} > 0$$

Hence, $x < -2$, which is not possible as $x > 1$.

Case II: when $x < 1$

$\Rightarrow \quad \frac{(1-x) - (x+2)}{(x+2)} > 0$

or $\quad \frac{-1 - 2x}{(x+2)} > 0$

or $\quad \frac{2x + 1}{(x+2)} < 0$

$\Rightarrow \quad x \in \left(-2, -\frac{1}{2}\right) \quad (\text{as } x < 1)$

12. We have $|x^2 - 1| + |x^2 - 4| > 3$

$\Rightarrow \quad |x^2 - 1| + |4 - x^2| > |x^2 - 1 + 4 - x^2|$

$\Rightarrow \quad (x^2 - 1)(4 - x^2) < 0$

$\Rightarrow \quad (x^2 - 1)(x^2 - 4) > 0$

$\Rightarrow \quad x^2 < 1$ or $x^2 > 4$

$\Rightarrow \quad -1 < x < 1$ or $x < -2$ or $x > 2$

Exercises

Single Correct Answer Type

1. (4) Every rectangle, rhombus, square in a plane is a parallelogram but every trapezium is not a parallelogram.

$\therefore \quad F_1 = F_2 \cup F_3 \cup F_4 \cup F_1$

2. (3) Since $A \subseteq B$, $A \cup B = B$

So, $n(A \cup B) = n(B) = 6$

3. (3) For $A \cap B$, we have

$$-x = \frac{1}{x}$$

$\Rightarrow \quad -x^2 = 1$

$\Rightarrow \quad x^2 = -1$, which is not possible

$\therefore \quad A \cap B = \phi$

4. (2) According to the question,

$$2^m - 2^n = 112$$

$\Rightarrow \quad 2^n(2^{m-n} - 1) = 2^4 \times 7$

Comparing exponents on both sides, we get

$$2^n = 2^4 \text{ and } 2^{m-n} - 1 = 7$$

Now, $2^n = 2^4$

$\Rightarrow \quad n = 4$

And $\quad 2^{m-n} = 8$

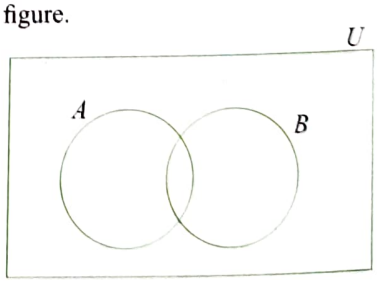
$\Rightarrow \quad 2^{m-n} = 2^3$

$\Rightarrow \quad m - n = 3$

$\Rightarrow \quad m - 4 = 3$

$\therefore \quad m = 7$

5. (4) Given that $A \not\subset B$. Then it is not necessary that B is subset of A as shown in figure.



Also, it is not necessary that A and B are disjoint.

Obviously, A and complement of B are not disjoint.

6. (3) $4N = \{4, 8, 12, 16, 20, 24, \ldots\}$ and $6N = \{6, 12, 18, 24, \ldots\}$

Hence, $4N \cap 6N = \{12, 24, 36, \ldots\} = 12N$

7. (2) $(A \cap B')' \cup (B \cap C) = ((A' \cup (B')') \cup (B \cap C)$

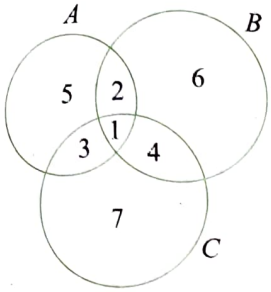
$= (A' \cup B) \cup (C \cap B)$

$= A' \cup (B \cup (C \cap B))$

$= A' \cup B$

8. (1) $(A \cup B)' \cup (A' \cap B)$
$= (A' \cap B') \cup (A' \cap B)$
$= (A' \cap A') \cap (A' \cup B) \cap (B' \cup A') \cap (B' \cup B)$
$= A' \cap \{A' \cup (B \cap B')\} \cap U$
$= A' \cap \{A' \cup \phi\} \cap U$
$= A' \cap A' \cap U$
$= A' \cap U = A'$

9. (2)

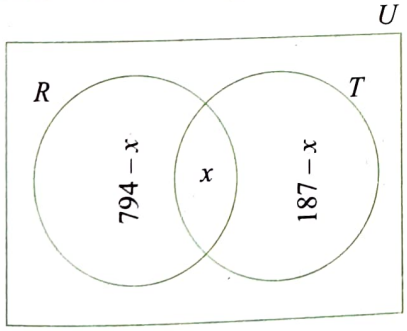


$A - B \rightarrow$ Regions 3 and 5.
$B - C \rightarrow$ Regions 2 and 6.
$C - A \rightarrow$ Regions 4 and 7.
$\therefore$ $(A - B) \cup (B - C) \cup (C - A) \rightarrow$ Regions 2, 3, 4, 5, 6, 7
$\therefore$ $[(A - B) \cup (B - C) \cup (C - A)]' \rightarrow$ Region 1, which is $A \cap B \cap C$

10. (4) Shaded region in the figure is clearly $A - (B \cup C)$.

11. (3) $[(A' \cap B' \cap C)] \cup [(B \cap C) \cup (A \cap C)]$
$= [(A \cup B)' \cap C)] \cup [(B \cup A) \cap C]$
$= [(A \cup B)' \cup (B \cup A)] \cap C$
$= X \cap C$
$= C$

12. (2) Let R be the set of families having a radio and T be the set of families having a TV.
Then
$n(R \cup T)$
= Number of families having at least one of the radio and TV
$= 1003 - 63 = 940$
$n(R) = 794$ and $n(T) = 187$
Let x be the number of families having both a radio and a TV.
Then number of families who have only Radio $= 794 - x$
Number of families who have only TV $= 187 - x$



From Venn diagram,
$794 - x + x + 187 - x = 940$
$\Rightarrow$ $981 - x = 940$
$\therefore$ $x = 981 - 940 = 41$
Hence, the required number of families having both a radio and a TV is 41.

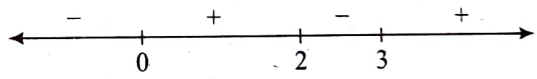
13. (3) Let A be the set of those people who watch news channel and B be the set of those people who watch entertainment channel.
Then
$n(A) = 63$, $n(B) = 76$, and $n(A \cap B) = x$
$\because$ $n(A \cup B) \le 100$
$\Rightarrow$ $n(A) + n(B) - n(A \cap B) \le 100$
$\Rightarrow$ $63 + 76 - x \le 100$
$\Rightarrow$ $x \ge 39$
Also, $n(A \cap B) \le \min.\{n(A), n(B)\}$
$\Rightarrow$ $x \le 63$
$\therefore$ $39 \le x \le 63$

14. (2) $n(A) = 40\%$ of $10,000 = 4,000$
$n(B) = 20\%$ of $10,000 = 2,000$
$n(C) = 10\%$ of $10,000 = 1,000$
$n(A \cap B) = 5\%$ of $10,000 = 500$
$n(B \cap C) = 3\%$ of $10,000 = 300$
$n(C \cap A) = 4\%$ of $10,000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10,000 = 200$
Number of families which buy newspaper A only
$= n(A \cap B^c \cap C^c)$
$= n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap B \cap C)]$
$= 4000 - [500 + 400 - 200]$
$= 4000 - 700$
$= 3300$

15. (2) $\dfrac{(x+2)(x+3)}{(x-2)(x-3)} \le 1$

$\Rightarrow$ $\dfrac{x^2+5x+6}{x^2-5x+6} - 1 \le 0$

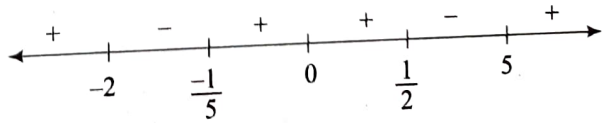
$\Rightarrow$ $\dfrac{10x}{(x-2)(x-3)} \le 0$



$x \in (-\infty, 0] \cup (2, 3)$

16. (2) We have $(x + 1)^2 > 5x - 1$
$\Rightarrow$ $x^2 - 3x + 2 > 0$
$\Rightarrow$ $(x - 1)(x - 2) > 0$
$\Rightarrow$ $x < 1$ or $x > 2$ ... (i)
And $(x + 1)^2 < 7x - 3$
$\Rightarrow$ $x^2 - 5x + 4 < 0$
$\Rightarrow$ $(x - 1)(x - 4) < 0$
$\Rightarrow$ $1 < x < 4$ ... (ii)
From (i) and (ii), common values are, $x \in (2, 4)$
Therefore, there is only one integral value of x which is 3.

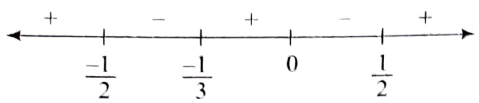
17. (2) We have, $\dfrac{x^2(x-5)(2x-1)}{(5x+1)(x+2)} < 0$



From the sign scheme, solution is
$\left(-2, -\dfrac{1}{5}\right) \cup \left(\dfrac{1}{2}, 5\right)$ ... (i)

Now $\dfrac{3x+1}{x(6x^2+x-1)} > 0$

$$\Rightarrow \quad \frac{3x+1}{x(3x-1)(2x+1)} > 0$$



From the sign scheme solution is

$$\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{3}, 0\right) \cup \left(\frac{1}{3}, \infty\right) \qquad \ldots(ii)$$

Clearly, from (i) and (ii), option(s) (1) and (3) are correct.

18. (4) We have $x^4 - 29x^2 + 100 \le 0$

$\Rightarrow \quad (x^2 - 4)(x^2 - 25) \le 0$

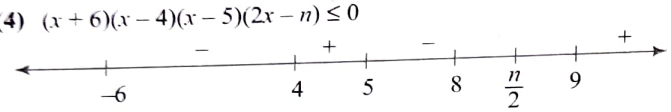
$\Rightarrow \quad 4 \le x^2 \le 25$

$\Rightarrow \quad x \in [-5, -2] \cup [2, 5]$

Integers satisfying above are $\pm 5, \pm 4, \pm 3, \pm 2$.

So, there are eight integers.

19. (4) $(x + 6)(x - 4)(x - 5)(2x - n) \le 0$



Hence, $x \in [-6, 4] \cup \left[5, \frac{n}{2}\right]$

Now, there are 11 integers in $[-6, 4]$.

So, there must be 4 integers in $\left[5, \frac{n}{2}\right]$

So, $\frac{n}{2} \in [8, 9)$

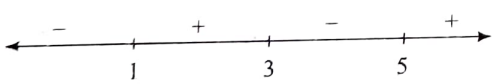
$\Rightarrow \quad n \in [16, 18)$

20. (1) We have $\frac{x+7}{x-5} + \frac{3x+1}{2} \ge 0$

$$\Rightarrow \quad \frac{2x+14+3x^2-14x-5}{2(x-5)} \ge 0$$

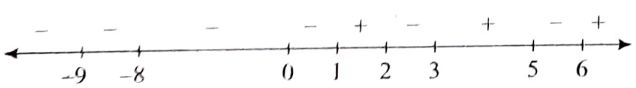
$$\Rightarrow \quad \frac{3x^2-12x+9}{2(x-5)} \ge 0$$

$$\Rightarrow \quad \frac{(x-1)(x-3)}{(x-5)} \ge 0$$



From the sign scheme, $x \in [1, 3] \cup (5, \infty)$.

21. (3) We have $\frac{(x-5)^{1005}(x+8)^{1008}(x-1)}{x^{1006}(x-2)^3(x-3)^5(x-6)(x+9)^{1010}} \le 0$



From the sign scheme,

$x \in (-\infty, 9) \cup (-9, 0) \cup (0, 1] \cup (2, 3) \cup [5, 6)$

22. (3) We have $|x|^3 - 4|x|^2 + 3|x| = 0$

$\Rightarrow \quad |x|(|x|^2 - 4|x| + 3) = 0$

$\Rightarrow \quad |x|(|x| - 1)(|x| - 3) = 0$

$\Rightarrow \quad |x| = 0, 1, 3$

$\Rightarrow \quad x = 0, 1, 3, -1, -3$

23. (4) We have $|x - 1||x^2 - 2| = 2$

Since x is an integer, we have

$|x - 1| = 1$ and $|x^2 - 2| = 2$

or $|x - 1| = 2$ and $|x^2 - 2| = 1$

$\therefore \quad x = 0, 2$ and $x = 0, \pm 2$

or $\quad x = -1, 3$ and $x = \pm 1$

Therefore, solutions are $-1, 0, 2$.

24. (3) We have $\frac{|x+2|-x}{x} < 2$

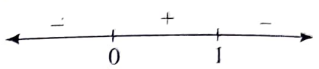
$\Rightarrow \quad \frac{|x+2|-3x}{x} < 0$

Let $\quad |x + 2| - 3x = 0$

$\Rightarrow \quad x + 2 = \pm 3x$

$\Rightarrow \quad x = 1$

Sign scheme of the expression $\frac{|x+2|-3x}{x}$ is as shown in the following figure:



From the sign scheme, $x \in (-\infty, 0) \cup (1, \infty)$.

25. (2) $|2 - |x|| = x + 4$ ... (i)

$\Rightarrow \quad 2 - |x| = x + 4$ ... (ii)

or $\quad |x| - 2 = x + 4$ ... (iii)

From (i), $2 - x = x + 4$ or $x = -1$.

But this value does not satisfy (i).

From (iii), $-x - 2 = x + 4$ or $x = -3$, which satisfies (i).

So, equation has only one solution.

26. (1) We have $\frac{x^2+6x-7}{|x+2||x+3|} < 0.$

Clearly, $x \ne -2, -3$.

Now, $x^2 + 6x - 7 < 0$

$\Rightarrow \quad (x + 7)(x - 1) < 0$

$\Rightarrow \quad x \in (-7, 1) - \{-3, -2\}$

So, integers are $\{-6, -5, -4, -1, 0\}$

27. (3) We have $-4 \le x < 2$

$\Rightarrow \quad -2 \le x + 2 < 4$

$\Rightarrow \quad 0 \le |x + 2| < 4$

$\Rightarrow \quad -3 \le |x + 2| - 3 < 1$

$\Rightarrow \quad 0 \le ||x + 2| - 3| \le 3$

28. (2) We have, $||x - 1| - 5| < 2x - 5$

$\therefore \quad 2x - 5 > 0$

$\Rightarrow \quad x > \frac{5}{2}$ (1)

Now from (1),

$-2x + 5 < |x - 1| - 5 < 2x - 5$

$\Rightarrow \quad -2x + 10 < |x - 1| < 2x$

Case I: $x \ge 1$

$\therefore \quad -2x + 10 < x - 1 < 2x$

$\Rightarrow \quad 3x > 11$ and $x > -1$ (2)

$\Rightarrow \quad x > 11/3$

Case II: $x < 1$

$\therefore \quad -2x + 10 < 1 - x < 2x$

$\Rightarrow \quad x > 9$ and $x > 1/3$

$\Rightarrow \quad x > 9$, not possible as $x < 1$

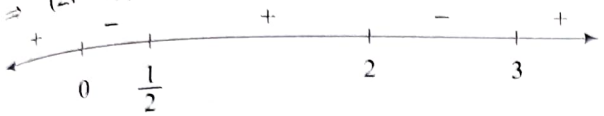
Therefore, $x \in (11/3, \infty)$

(2) We have,

$$|2x^2 - 5x + 2| + |x^2 - 3x| = |x^2 - 2x + 2|$$
$$= |(2x^2 - 5x + 2) - (x^2 - 3x)|$$

$\therefore$ $(2x^2 - 5x + 2)(x^2 - 3x) \le 0$

$\Rightarrow$ $(2x - 1)(x - 2)x(x - 3) \le 0$



From the sign scheme, we have $x \in \left[0, \frac{1}{2}\right] \cup [2, 3]$

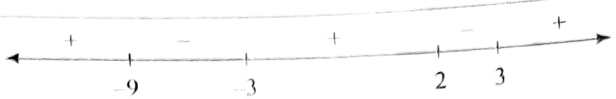
(3) We have $|x^2 - 5x + 6| + |x^2 + 12x + 27| = |17x + 21|$

or $|x^2 - 5x + 6| + |x^2 + 12x + 27|$

$= |(x^2 + 12x + 27) - (x^2 - 5x + 6)|$

$\Rightarrow$ $(x^2 - 5x + 6)(x^2 + 12x + 27) \le 0$

$\Rightarrow$ $(x - 2)(x - 3)(x + 3)(x + 9) \le 0$



From the sign scheme, $x \in [-9, -3] \cup [2, 3]$

Archives

JEE Main

Single Correct Answer Type

1. (3) $B = B \cap (A \cup B)$

$= B \cap (A \cup C)$ $(\because A \cup B = A \cup C)$

$= (B \cap A) \cup (B \cap C)$

$= (A \cap B) \cup (B \cap C)$

$= (A \cap C) \cup (B \cap C)$ $(\because A \cap B = A \cap C)$

$= (A \cup B) \cap C$

$= (A \cup C) \cap C$

$= C$

Chapter 2

Concept Application Exercises

Exercise 2.1

1. Since $x = 1$ is a root of the given equation, it satisfies the equation.
Hence, putting $x = 1$ in the given equation, we get

$$a + b + c = -1 \quad (1)$$

but given that

$$a + b = 1 \quad (2)$$

$\Rightarrow \quad c = -2$

Now put $x = 2$ in the given equation, we have

$$8 + 4a + 2b - 2 = 0$$

or $\quad 6 + 2a + 2(a + b) = 0$

or $\quad 6 + 2a + 2 = 0$

or $\quad a = -4$

$\Rightarrow \quad b = 5$

2. Since $(1 - p)$ is the root of quadratic equation,

$$x^2 + px + (1 - p) = 0 \quad (1)$$

So $(1 - p)$ satisfies the above equation. Therefore,

$$(1 - p)^2 + p(1 - p) + (1 - p) = 0$$

or $\quad (1-p)[1 - p + p + 1] = 0$

or $\quad (1 - p)(2) = 0$

or $\quad p = 1$

On putting this value of p in Eq. (1), we get

$$x^2 + x = 0$$

or $\quad x(x + 1) = 0$

or $\quad x = 0, -1$

3. $p(x)$ is positive or zero for all real numbers.
Also $\quad p(1) = 0$
Then we have $p(x) = k(x - 1)^2$, where $k > 0$
Now $\quad p(2) = 2$

$\Rightarrow \quad k = 2$

$\therefore \quad p(x) = 2(x - 1)^2$

Exercise 2.2

1. Let $f(x) = 2x^3 + 3px^2 - 4x + p$. Given

$$f(-2) = 2(-2)^3 + 3(-2)^2 p - 4(-2) + p = 5$$

or $\quad 13p - 8 = 5$

or $\quad p = 1$

2. Let $f(x) = (x + 1)^7 + (2x + k)^3$. Now

$$f(-2) = (-2 + 1)^7 + (-4 + k)^3 = 0$$

or $\quad (k - 4)^3 - 1 = 0$

or $\quad k = 5$

3. $f(x)$ is divisible by $x - 1$

$\Rightarrow \quad f(1) = 0$

$\Rightarrow \quad 1 - 3 + 2 + a = 0$

$\Rightarrow \quad a = 0$

$\Rightarrow \quad f(x) = x^3 - 3x^2 + 2x$

Now remainder when $f(x)$ is divided by $x - 2$ is $f(2)$

$\Rightarrow \quad f(2) = (2)^3 - 3(2)^2 + 2(2) = 0$

4. $f(x)$ is divisible by $x^2 - x$ or $x(x - 1)$

$\Rightarrow \quad f(0) = 0$ or $b = 0$

Also $\quad f(1) = 0$ or $1 - 1 + a + b = 0$ or $a = 0$

$\Rightarrow \quad f(x) = x^3 - x^2$

$\Rightarrow \quad f(2) = (2)^3 - (2)^2 = 4$

5. $x^5 + x^3 + x^2 + 2 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)$

Putting $x = 1$, we get,

$$5 = (1 - x_1)(1 - x_2)(1 - x_3)(1 - x_4)(1 - x_5)$$

Putting $x = -1$, we get,

$$1 = (-1 - x_1)(-1 - x_2)(-1 - x_3)(-1 - x_4)(-1 - x_5)$$

Multiplying, we get

$$5 = (x_1^2 - 1)(x_2^2 - 1)(x_3^2 - 1)(x_4^2 - 1)(x_5^2 - 1)$$

Exercise 2.3

1. We must have $a^2 - 3a + 2 = 0$

$\Rightarrow \quad a = 1, 2$

and $\quad a^2 - 5a + 6 = 0 \quad \Rightarrow a = 2, 3,$

and $\quad a^2 - 4 = 0 \quad \Rightarrow a = \pm 2$

Therefore, when $a = 2$, the equation will be true for all real x.

2. $x^2 + ax + 1$ must divide $ax^3 + bx + c$. Now,

$$\frac{ax^3 + bx + c}{x^2 + ax + 1} = a(x - a) + \frac{(b - a + a^3)x + c + a^2}{x^2 + ax + 1}$$

The remainder must be zero. Hence, $(b - a + a^3)x + c + a^2 = 0 \ \forall x \in R$

So, $\quad b - a + a^3 = 0, a^2 + c = 0$

3. $(a + b + c)^2 = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$

$\Rightarrow \quad ab + bc + ca = -2$

On squaring, we get

$$a^2b^2 + b^2c^2 + c^2a^2 + 2ab^2c + 2a^2bc + 2abc^2 = 4$$

$\Rightarrow \quad a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a + b + c) = 4$

$\Rightarrow \quad a^2b^2 + b^2c^2 + c^2a^2 = 4$

Now, $\quad a^2 + b^2 + c^2 = 4$

On squaring, we get

$$a^4 + b^4 + c^4 + 2(a^2b^2 + a^2c^2 + b^2c^2) = 16$$

$\Rightarrow \quad a^4 + b^4 + c^4 = 8$

Exercise 2.4

1. Graphs of $y = x^2 + 2$ and $y = 3x + 4$ intersect when $x^2 + 2 = 3x - 4$ or $x^2 - 3x + 6 = 0$. But this equation has no real roots; hence, graphs never intersect.

2. When graphs of $y = -14$ and $y = -x(x^2 + x + 1) = 0$ intersect, we have

$$-(x^3 + x^2 + x) = -14 \text{ or } x^3 + x^2 + x - 14 = 0$$

Now $x = 2$ satisfies the equation, then one root is $x = 2$.
Dividing $x^3 + x^2 + x - 14$ by $x - 2$,
we have $(x - 2)(x^2 + 3x + 7) = 0$

$\Rightarrow \quad x = 2$ or $x^2 + 3x + 7 = 0$

Now $x^2 + 3x + 7 = 0$ has non-real roots.
Hence, graphs cut in only one real point.

3. (a) Graph of $y = f(x)$ intersect x-axis at $x = -2$ and $x = 1$, which are roots of the equation $f(x) = 0$.

(b) Graph of $y = f(x)$ intersect the line $y = 4$ intersect at points whose x-values are -3 and 2, which are the roots of the equation $f(x) = 4$.

(c) For the roots of the equation $f(x) = x + 2$, we draw the graph of $y = x + 2$, which passes through the points $(2, 4)$ and $(-2, 0)$ which lies on the graph of $y = f(x)$ also.

So, roots of the equation $f(x) = x + 2$ are $x = 2$ and $x = -2$.

Exercise 2.5

1. $\frac{x^2+3x+2}{x^2-6x-7} = 0$ or $\frac{(x+1)(x+2)}{(x-7)(x+1)} = 0$ is solvable over $R - \{7, -1\}$

Hence, from given equation, $x = -2$, which is the only root of the equation.

2. $\sqrt{x-2} + \sqrt{4-x} = 2$

Squaring we get $x - 2 + 4 - x + 2\sqrt{(x-2)(4-x)} = 4$

or $\sqrt{(x-4)(2-x)} = 1$

or $-(x^2 - 6x + 8) = 1$

or $x^2 - 6x + 9 = 0$

$\Rightarrow$ $(x-3)^2 = 0$ or $x = 3$

3. Given equation is solvable for $x \in [2, \infty)$. Now,

$\sqrt{x-2}\,(x^2 - 4x - 5) = 0$

or $\sqrt{x-2}\,(x-5)(x+1) = 0$

$\Rightarrow$ $x = 2$ or 5 ($x \neq -1$, as $x \in [2, \infty)$)

4. $\sqrt{x+5} + \sqrt{x+21} = \sqrt{6x+40}$

or $(\sqrt{x+5} + \sqrt{x+21})^2 = 6x + 40$

or $(x+5) + (x+21) + 2\sqrt{(x+5)(x+21)} = 6x + 40$

or $\sqrt{(x+5)(x+21)} = 2x + 7$

or $(x+5)(x+21) = (2x+7)^2$

or $3x^2 + 2x - 56 = 0$

or $(3x + 14)(x - 4) = 0$

$\Rightarrow$ $x = 4, -14/3$

Clearly, $x = -14/3$ does not satisfy the original equation. Hence, $x = 4$ is the only root of the given equation.

Exercise 2.6

1. Let $f(x) = 3x^4 + 6x^3 + x^2 + 6x + 3$.

$\therefore$ $f'(x) = 12x^3 + 18x^2 + 2x + 6$

and $f''(x) = 36x^2 + 36x + 2x$

$= 2(18x^2 + 18x + 2) \neq 0$ for any real x.

So, graph of $f'(x)$ intersects x-axis only once.

Hence, $f(x)$ has only one turning point.

So, $f(x) = 0$ has maximum two real roots.

Now, $f(0) = 3 > 0$ and $f(-1) = 3 - 6 + 1 - 6 + 3 < 0$

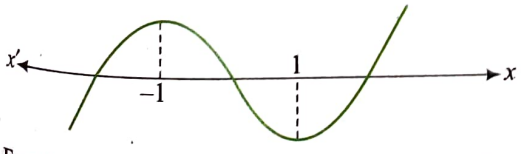
So, graph cuts x-axis between -1 and 1.

Hence, $f(x) = 0$ has two real roots.

2. Let $f(x) = x^3 - 3x + a$

Let $f'(x) = 0$

$\Rightarrow$ $3x^2 - 3 = 0$ or $x = \pm 1$



For three distinct roots, $f(1) < 0$ and $f(-1) > 0$.

$\Rightarrow$ $1 - 3 + a < 0$ and $-1 + 3 + a > 0$

or $a < 2$ and $a > -2$

or $-2 < a < 2$

3. Let $f(x) = (x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 + (x-5)^3$

$\Rightarrow$ $f'(x) = 3(x-1)^2 + 3(x-2)^2 + 3(x-3)^2 + 3(x-4)^2 + 3(x-5)^2$

Now for $f'(x) = 0$ we have $(x-1)^2 + (x-2)^2 + (x-3)^2 + (x-4)^2 + (x-5)^2 = 0$, which has no real roots.

Also coefficient of x^3 is 5; hence, when $x \to \infty$, $f(x) \to \infty$ and when $x \to -\infty$, $f(x) \to -\infty$

Hence, graph of $y = f(x)$ meets the x-axis only once; hence, equation $f(x) = 0$ has one real root.

4. $f(x) = x^3 + 2x^2 + 3x + 4$

$\Rightarrow$ $f'(x) = 3x^2 + 4x + 3$

Now $f'(x) = 3x^2 + 4x + 3 = 0$ has nonreal roots.

Hence, graph has no turning point.

Also when $x \to \infty$, $f(x) \to \infty$ and when $x \to -\infty$, $f(x) \to -\infty$

Hence, graph of $y = f(x)$ meets the x-axis only once.

Exercise 2.7

1. We have

$x(x+2)(x^2 - 1) = -1$

or $x(x+2)(x+1)(x-1) = -1$

or $(x(x+1))(x+2)(x-1) = -1$

or $(x^2 + x)(x^2 + x - 2) = -1$

or $(x^2 + x)^2 - 2(x^2 + x) + 1 = 0$

Putting $x^2 + x = y$, we get $y^2 - 2y + 1 = 0$

or $(y-1)^2 = 0$

or $y = 1$

Now $x^2 + x = 1$

or $x^2 + x - 1 = 0$

or $x = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

Hence, the roots of the given equation are

$\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$, and $\frac{-1-\sqrt{5}}{2}$

2. Given equation is $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$

Dividing throughout by x^2, we get

$$\left(\frac{x^2+2}{x}\right)^2 + 8 = 6\left(\frac{x^2+2}{x}\right)$$

$\Rightarrow$ $t^2 - 6t + 8 = 0$ where $t = \frac{x^2+2}{x}$

$\therefore$ $t = 4, t = 2$

$\therefore$ $\frac{x^2+2}{x} = 4$ or $x = 2 \pm \sqrt{2}$

and $\frac{x^2+2}{x} = 2$ or $x = 1 \pm i$

3. Let the given expression be equal to x. Then,

$x = 2 + \frac{1}{x}$

or $x^2 - 2x - 1 = 0$

or $x = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

But, the given expression is positive. Therefore, $x = 1 + \sqrt{2}$ is the value of the given expression.

4. $4^x + 6^x = 9^x$

or $\left(\frac{4}{9}\right)^x + \left(\frac{2}{3}\right)^x = 1$

Putting $\left(\frac{2}{3}\right)^x = y$, we have

$y^2 + y - 1 = 0$

or $y = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \left(\frac{2}{3}\right)^x = \frac{\sqrt{5}-1}{2}$ $\left(\because \left(\frac{2}{3}\right)^x > 0\right)$

$\Rightarrow x = \log_{2/3}\left(\frac{\sqrt{5}-1}{2}\right)$

5. $3^{2x^2-7x+7} = 9$

or $3^{2x^2-7x+7} = 3^2$

or $2x^2 - 7x + 7 = 2$

or $2x^2 - 7x + 5 = 0$ or $x = 1, 5/2$

6. $\frac{8^x + 27^x}{12^x + 18^x} = \frac{7}{6}$

or $\frac{(8)^x}{(12)^x}\frac{(1+(27/8)^x)}{(1+(18/12)^x)} = \frac{7}{6}$

or $\left(\frac{2}{3}\right)^x\left(\frac{1+(3/2)^{3x}}{1+(3/2)^x}\right) = \frac{7}{6}$

Let $\left(\frac{3}{2}\right)^x = t$. Then

$\frac{1+t^3}{t(1+t)} = \frac{7}{6}$

or $\frac{(1+t)(t^2+1-t)}{t(1+t)} = \frac{7}{6}$

or $\frac{t^2+1-t}{t} = \frac{7}{6}$ $(\because t + 1 \neq 0)$

or $6t^2 - 13t + 6 = 0$

or $(2t - 3)(3t - 2) = 0$

$\Rightarrow t = \frac{2}{3}$ or $\frac{3}{2}$

$\Rightarrow x = -1$ or 1

7. $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$

or $\sqrt{3x^2 - 7x - 30} = (x + 5) - \sqrt{2x^2 - 7x - 5}$

On squaring, we get

$3x^2 - 7x - 30 = x^2 + 10x + 25 - 2(x + 5) \times \sqrt{2x^2 - 7x - 5} + 2x^2 - 7x - 5$

or $10x + 50 = 2(x + 5)\sqrt{2x^2 - 7x - 5}$

$\Rightarrow x = -5$ or $\sqrt{2x^2 - 7x - 5} = 5$

$\Rightarrow x = -5$ or $2x^2 - 7x - 30 = 0$

$\Rightarrow x = -5$ or $x = 6$ or $x = -5/2$

But $x = -5$ does not satisfy the original equation.

8. Let $L = \sqrt{5x^2 - 6x + 8}$ and $M = \sqrt{5x^2 - 6x - 7}$. Hence,

$L - M = 1$ and $L^2 - M^2 = 15$

$\Rightarrow L + M = 15$

Adding, we get

$2L = 16$

or $L^2 = 64$

or $5x^2 - 6x + 8 = 64$

or $5x^2 - 6x - 56 = 0$

or $(x - 4)(5x + 14) = 0$

or $x = 4, -14/5$

9. We have

$\sqrt{x^2 + 4x - 21} + \sqrt{x^2 - x - 6} = \sqrt{6x^2 - 5x - 39}$

or $\sqrt{(x+7)(x-3)} + \sqrt{(x-3)(x+2)} = \sqrt{(x-3)(6x+13)}$ (1)

or $\sqrt{x-3}\,(\sqrt{x-7} + \sqrt{x+2} - \sqrt{6x+13}) = 0$

or $\sqrt{x-3} = 0$ or $\sqrt{x+7} + \sqrt{x+2} - \sqrt{6x+13} = 0$

$\Rightarrow x = 3$ or $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

Now, $\sqrt{x+7} + \sqrt{x+2} = \sqrt{6x+13}$

or $(\sqrt{x+7} + \sqrt{x+2})^2 = 6x + 13$

or $x + 7 + x + 2 + 2\sqrt{(x+7)(x+2)} = 6x + 13$

or $2x + 9 + 2\sqrt{(x+7)(x+2)} = 6x + 13$

or $2\sqrt{(x+7)(x+2)} = 4x + 4$

or $\sqrt{(x+7)(x+2)} = 2(x+1)$

or $(x + 7)(x + 2) = 4(x + 1)^2$ (squaring both sides)

or $x^2 + 9x + 14 = 4(x^2 + 2x + 1)$

or $3x^2 - x - 10 = 0$

or $(x - 2)(3x + 5) = 0$

$\Rightarrow x = 2$ or $x = \frac{-5}{3}$, which does not satisfy (1).

Hence, $x = 3$ only.

Exercise 2.8

1. Given equation is $ax^2 + 2bx + c = 0$. Hence,

$D = 4b^2 - 4ac$

$= (a + c)^2 - 4ac$

$= (a - c)^2 > 0$

Thus, the roots are real and distinct.

2. The equations $ax^2 + 2bx + c = 0$ and $bx^2 - 2\sqrt{ac}\,x + b = 0$ have real roots. Therefore,

$4b^2 - 4ac \geq 0$ and $4ac - 4b^2 \geq 0$

$\Rightarrow b^2 = ac$

3. The discriminant of the given equation is

$D = 4(a + b - 2c)^2 - 4(a - b)^2$

$= 4(a + b - 2c - a + b)(a + b - 2c + a - b)$

$= 16(a - c)(b - c) < 0$ $[\because a < c < b]$

Hence, the roots of the given equation are complex.

4. For the given equation $4ax^2 + 3bx + 2c = 0$, we have

$D = (3b)^2 - 4(4a)(2c)$

$= 9b^2 - 32ac$

$= 9(-a - c)^2 - 32ac$

$= 9a^2 - 14ac + 9c^2$

$$=9c^2\left(\left(\frac{a}{c}\right)^2-\frac{14}{9}\frac{a}{c}+1\right)$$

$$=9\left(\left(\frac{a}{c}-\frac{7}{9}\right)^2-\frac{49}{81}+1\right)$$

which is always positive. Hence, the roots are real and distinct.

5. For real root,
$(\lambda-1)^2-64\geq 0$ and $64-4(\lambda+4)\geq 0$
$\Rightarrow$ $(\lambda-1)^2\geq 64$ and $48-4\lambda\geq 0$
$\Rightarrow$ $\lambda-1\geq 8$ or $\lambda-1\leq -8$ and $12\geq\lambda$
$\Rightarrow$ $\lambda\geq 9$ or $\lambda\leq -7$ and $\lambda\leq 12$
Hence, the greatest value of λ is 12.

6. Given equation is
$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$
or $(-2a)x^2+(-2b)x+(-2c)=0$
or $ax^2+bx+c=0$
$\Rightarrow$ $D=b^2-4ac$
$=(-c-a)^2-4ac$
$=(c-a)^2>0$
Hence, roots are real and distinct.

7. $\because$ $D=q^4-4p^2\geq 0$
$=(q^2+2p)(q^2-2p)\geq 0$
$\Rightarrow$ $q^2\geq 2p$
If $p=1$, then $q=2, 3, 4, 5$
$p=2$, then $q=2, 3, 4, 5$
$p=3$, then $q=3, 4, 5$
$p=4$, then $q=3, 4, 5$
$p=5$, then $q=4, 5$
So, number of possible equations is 16.

8. $f(x)=x^2-x-3=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-3=\left(x-\frac{1}{2}\right)^2-\frac{13}{4}$

Now, $\left(x-\frac{1}{2}\right)^2\geq 0, \forall\, x\in R$

$\Rightarrow$ $\left(x-\frac{1}{2}\right)^2-\frac{13}{4}\geq\frac{-13}{4}, \forall\, x\in R$

Hence, the range is $[-13/4, \infty)$.

9. (a) Let $\frac{x^2+34x-71}{x^2+2x-7}=y$
or $x^2(1-y)+2(17-y)x+(7y-71)=0$
For real value of x,
$b^2-4ac\geq 0$
$\Rightarrow$ $y^2-14y+45\geq 0$
$\Rightarrow$ $(y-5)(y-9)\geq 0$
$\Rightarrow$ $y\leq 5$ or $y\geq 9$
Hence, the range is $(-\infty, 5]\cup[9, \infty)$.

(b) Let $y=\frac{x^2-x+1}{x^2+x+1}$
or $(1-y)x^2-(1+y)x+1-y=0$
Now, if x is real, then
$D\geq 0$
$\Rightarrow$ $(1+y)^2-4(1-y)^2\geq 0$
or $(1+y-2+2y)(1+y+2-2y)\geq 0$
or $(3y-1)(3-y)\geq 0$
or $3\left(y-\frac{1}{3}\right)(y-3)\leq 0$
or $\frac{1}{3}\leq y\leq 3$
Hence, the range is $[1/3, 3]$.

10. Let $y=\sqrt{x-1}+\sqrt{5-x}$
$\Rightarrow$ $y^2=x-1+5-x+2\sqrt{(x-1)(5-x)}$
$=4+2\sqrt{-x^2-5+6x}$
$=4+2\sqrt{4-(x-3)^2}$
Then, y^2 has minimum value 4 [when $4-(x-3)^2=0$] and maximum value 8 when $x=3$. Thus, $y\in[2, 2\sqrt{2}]$

11. $f(x, y)=(x-2)^2+(y-1)^2=0$
$\Rightarrow$ $x=2$ and $y=1$
$\therefore$ $\frac{\left(\sqrt{x}-\sqrt{y}\right)^2+4\sqrt{xy}}{x+\sqrt{xy}}$
$=\frac{\left(\sqrt{2}-1\right)^2+4\sqrt{2}}{2+\sqrt{2}}=\frac{\left(\sqrt{2}+1\right)^2}{\sqrt{2}\left(\sqrt{2}+1\right)}=\frac{\sqrt{2}+1}{\sqrt{2}}$

Exercise 2.9

1. It is given that
$\alpha\beta=2$
$\Rightarrow$ $\frac{3a+4}{a+1}=2$
or $3a+4=2a+2$
or $a=-2$
Also, $\alpha+\beta=-\frac{2a+3}{a+1}$
Putting the value of a, we get the sum of roots as
$$\alpha+\beta=-\frac{2a+3}{a+1}=-\frac{-4+3}{-2+1}=-1$$

2. Let α, β be the roots of the given equation. Then,
$\alpha+\beta=a-2$ and $\alpha\beta=-(a+1)$
Now, $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$
$=(a-2)^2+2(a+1)$
$=a^2-2a+6$
$=(a-1)^2+5$
Clearly, $\alpha^2+\beta^2\geq 5$. So, the minimum value of $\alpha^2+\beta^2$ is 5 which it attains at $a=1$.

3. $x_1+x_2=(1-\sin\theta), x_1x_2=-\frac{1}{2}\cos^2\theta$
Now, $x_1^2+x_2^2=(x_1+x_2)^2-2x_1x_2$
$=(1-\sin\theta)^2+\cos^2\theta$
$=2-2\sin\theta$
$\Rightarrow$ $x_1^2+x_2^2|_{\max}=2+2=4$

4. $\tan\theta+\sec\theta=-\frac{b}{a}$ (1)

$\tan\theta\sec\theta=\frac{c}{a}$ (2)

Now, $\sec^2\theta-\tan^2\theta=1$
$\Rightarrow$ $\sec\theta-\tan\theta=-\frac{a}{b}$ (3)

From (1) and (3),

$\Rightarrow \quad \sec\theta = -\dfrac{(a^2+b^2)}{2ab}$ and $\tan\theta = \dfrac{(a^2-b^2)}{2ab}$

Substituting these values in Eq. (2), we have

$$\frac{(a^2+b^2)(b^2-a^2)}{4a^2b^2} = \frac{c}{a}$$

or $\quad b^4 - a^4 = 4acb^2$

or $\quad a^4 = b^2(b^2 - 4ac)$

5. Let α, $\alpha+1$ be the roots of $x^2 - bx + c = 0$, where $\alpha \in Z$.

$\therefore \quad \alpha + (\alpha+1) = b \qquad (1)$

$\alpha(\alpha+1) = c \qquad (2)$

Form (1), we have

$$\alpha = \frac{b-1}{2}$$

Putting in Eq. (2), we have

$$\left(\frac{b-1}{2}\right)\left(\frac{b-1}{2}+1\right) = c$$

or $\quad \dfrac{b^2-1}{4} = c$

or $\quad b^2 - 1 = 4c$

or $\quad b^2 - 4c = 1$

6. Let the roots be α, β. Then,

$$\frac{\alpha}{\beta} = \frac{2}{3} \text{ or } \alpha = \frac{2\beta}{3}$$

Now, $\alpha + \beta = \dfrac{m}{12}$

$\Rightarrow \quad \dfrac{2\beta}{3} + \beta = \dfrac{m}{12}$

$\Rightarrow \quad \dfrac{5\beta}{3} = \dfrac{m}{12} \qquad (1)$

and $\quad \alpha\beta = \dfrac{5}{12}$

$\Rightarrow \quad \dfrac{2\beta}{3}\beta = \dfrac{5}{12}$

$\Rightarrow \quad \beta^2 = \dfrac{5}{8}$

$\Rightarrow \quad \beta = \sqrt{5/8}$

Putting the value of β in (1), we get

$$\frac{5}{3}\sqrt{\frac{5}{8}} = \frac{m}{12} \text{ or } m = 5\sqrt{10}$$

7. The given equation is $x^2 - px - (p+c) = 0$. Therefore,

$\alpha + \beta = p$, $\alpha\beta = -(p+c)$

So, $\quad (\alpha+1)(\beta+1) = \alpha\beta + (\alpha+\beta) + 1$

$= -(p+c) + p + 1$

$= 1 - c \qquad (1)$

Now, $\dfrac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \dfrac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)}$$

[Using (1)]

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1$$

8. Let α, β be the roots of the given equation $ax^2 + bx + c = 0$.

Then, $\alpha + \beta = -\dfrac{b}{a}$; $\alpha\beta = \dfrac{c}{a}$

Now roots of equation $2x^2 + 8x + 2 = 0$ are $\alpha - 1$, $\beta - 1$. Their sum is

$$\alpha + \beta - 2 = -\frac{b}{a} - 2 = -\frac{8}{2} = -4$$

$\Rightarrow \quad \dfrac{b}{a} = 2$

Their product is

$$(\alpha-1)(\beta-1) = \alpha\beta - (\alpha+\beta) + 1$$

$$= \frac{c}{a} + \frac{b}{a} + 1 = 1$$

($\because$ New equation is $2x^2 + 8x + 2 = 0$)

or $\quad c + b = 0$

or $\quad b = -c$

$\Rightarrow \quad \dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{-2}$

9. Let α, β be the roots of the equation $x^2 - a(x-1) + b = 0$. Then

$\alpha^2 - a\alpha + a + b = 0$ and $\beta^2 - a\beta + a + b = 0$

$\therefore \quad \alpha^2 - a\alpha = \beta^2 - a\beta = -a - b$

$\Rightarrow \quad \dfrac{1}{\alpha^2-a\alpha} + \dfrac{1}{\beta^2-a\beta} + \dfrac{2}{a+b}$

$$= \frac{1}{-(a+b)} + \frac{1}{-(a+b)} + \frac{2}{a+b} = 0$$

10. $(\alpha+\beta)^2 = (a+b)^2$

and $\quad (\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta$

$$= (a+b)^2 - 4\left(\frac{a^2+b^2}{2}\right)$$

$$= 2ab - (a^2+b^2)$$

$$= -(a-b)^2$$

Now, the required equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$ is

$$x^2 - \{(\alpha+\beta)^2 + (\alpha-\beta)^2\}x + (\alpha+\beta)^2(\alpha-\beta)^2 = 0$$

or $\quad x^2 - \{(a+b)^2 - (a-b)^2\}x - (a+b)^2(a-b)^2 = 0$

or $\quad x^2 - 4abx - (a^2-b^2)^2 = 0$

11. Since α, β are roots of $x^2 + bx + 1 = 0$, we have $\alpha + \beta = -b$, $\alpha\beta = 1$.

Now, $\left(-\alpha - \dfrac{1}{\beta}\right) + \left(-\beta - \dfrac{1}{\alpha}\right) = -(\alpha+\beta) - \left(\dfrac{1}{\beta} + \dfrac{1}{\alpha}\right)$

$$= b + b = 2b$$

and $\left(-\alpha - \dfrac{1}{\beta}\right)\left(-\beta - \dfrac{1}{\alpha}\right) = \alpha\beta + 2 + \dfrac{1}{\alpha\beta} = 1 + 2 + 1 = 4$

Thus, the equation whose roots are

$-\alpha - 1/\beta$ and $-\beta - 1/\alpha$ is $x^2 - x(2b) + 4 = 0$.

12. Let the roots be α and β. Then,

$\alpha + \beta = 2$ and $\alpha^3 + \beta^3 = 98$

Now, $\alpha^3 + \beta^3 = (\alpha+\beta)(\alpha^2 - \alpha\beta + \beta^2)$

$\Rightarrow \quad 98 = 2[(\alpha+\beta)^2 - 3\alpha\beta]$

or $\quad 49 = (4 - 3\alpha\beta)$

or $\quad \alpha\beta = -15$

Thus, the equation is $x^2 - 2x - 15 = 0$.

Exercise 2.10

1. Subtracting the given equations, we get
$(a-b)x + c(b-a) = 0 \Rightarrow x = c$ is the common root
Thus, roots of $x^2+ax+bc=0$ are b and c and those of $x^2+bx+ca=0$ are c and a. Also, $a + b = -c$. Thus, the required equation is
$x^2 - (a+b)x + ab = 0$
$\Rightarrow \quad x^2 + cx + ab = 0$

2. $y = mx$ is a factor of $ax^2 + bxy + c^2$
Hence, $ax^2 + bxy + cy^2$ will be zero when $y - mx = 0$ or $y = mx$
$\Rightarrow \quad ax^2 + bx \cdot mx + cm^2x^2 = 0$
or $\quad cm^2 + bm + a = 0 \qquad (1)$
Since $my - x$ is a factor of $a_1x^2 + b_1xy + c_1y^2$, so
$a_1x^2 + b_1xy + c_1y^2 = 0$ when $my - x = 0$
$\Rightarrow \quad a_1m^2y^2 + b_1 \cdot my \cdot y + c_1y^2 = 0 \qquad$ [putting $x = my$]
$\Rightarrow \quad a_1m^2 + b_1m + c_1 = 0 \qquad (2)$
Eliminating m from (1) and (2), we get
$(bc_1 - ab_1)(cb_1 - ba_1) = (aa_1 - cc_1)^2$

3. Given equations are
$x^2 + 2x + 9 = 0 \qquad (1)$
and $\quad ax^2 + bx + c = 0 \qquad (2)$
Clearly, roots of Eq. (1) are imaginary. Eqs. (1) and (2) have a common root which must be imaginary; hence, both the roots will be common. Therefore, Eqs. (1) and (2) are identical.
$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{9}$ or $a:b:c = 1:2:9$

4. We have
$x^3 - mx^2 - 4 = 0 \qquad (1)$
and $\quad x^3 + mx + 2 = 0 \qquad (2)$
Multiplying (2) by x and adding with (1), we get
$x^4 + x^3 + 2x - 4 = 0$
$\Rightarrow \quad (x-1)(x+2)(x^2+2) = 0$
So, common root can be $x = 1$ or $x = -2$.
If $x = 1$, then $m = -3$.
If $x = -2$, then $m = -3$.

5. Roots of equation $5x^2 + 12x + 13 = 0$ are imaginary
Hence, both equations have same roots
$\therefore \quad a : b : c = 5 : 12 : 13$
$\therefore \quad a = 5k,\ b = 12k,\ c = 13k$
Here $c^2 = a^2 + b^2$
$\Rightarrow \quad \angle C = 90°$

Exercise 2.11

1. Let $a + \frac{1}{a} = t$.
$\therefore \quad t^3 - 3t - 18 = 0$
$t = 3$ satisfies this equation
$\therefore \quad (t-3)(t^2 + 3t + 6) = 0$
Clearly, $t = 3$ is the only solution.
$\therefore \quad a + \frac{1}{a} = 3$
$\Rightarrow \quad a^2 + \frac{1}{a^2} = 7$
$\Rightarrow \quad a^4 + \frac{1}{a^4} = 47$

2. Let the roots be $x_1, -x_1, x_2$. Then,
$x_1 - x_1 + x_2 = a$
$\Rightarrow \quad x_2 = a$
Hence, $x = a$ is a root of the given equation.
$\therefore \quad a^3 - a^3 + ab - c = 0$
$\Rightarrow \quad ab = c$

3. Let $y = x^2$. Then $x = \sqrt{y}$
$\therefore \quad x^3 + 8 = 0 \Rightarrow y^{3/2} + 8 = 0$
or $\quad y^3 = 64$ or $y^3 - 64 = 0$
Thus, the equation having roots α^2, β^2 and γ^2 is $x^3 - 64 = 0$.

4. Let $y = \frac{\alpha}{1+\alpha} \Rightarrow \alpha = \frac{y}{1-y}$
Since α is a root of the equation $x^3 - px + q = 0$, so
$\alpha^3 - p\alpha + q = 0$
$\Rightarrow \quad \frac{y^3}{(1-y)^3} - p\frac{y}{1-y} + q = 0$
or $\quad y^3 - py(1-y)^2 + q(1-y)^3 = 0$
or $\quad (1-p-q)y^3 + (2p+3q)y^2 - (p+3q)y + q = 0$
Therefore, the required cubic equation is
$(p+q-1)x^3 - (2p+3q)x^2 + (p+3q)x - q = 0$

5. $\frac{\alpha_1\alpha_2 + \alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3} = \frac{-\alpha_1\alpha_3}{\alpha_1\alpha_2\alpha_3} = -\frac{1}{\alpha_2}$
Therefore, the required equation has roots $-1/\alpha_1, -1/\alpha_2, -1/\alpha_3$.
$\therefore \quad y = -\frac{1}{x}$ or $x = -\frac{1}{y}$
Hence, the required equation is
$$\left(-\frac{1}{y}\right)^3 + a\left(-\frac{1}{y}\right)^2 + b = 0$$
$\Rightarrow \quad by^3 + ay - 1 = 0$
$\Rightarrow \quad bx^3 + ax - 1 = 0$

6. Equation $2x^3 + x^2 - 7 = 0$ has roots α, β, γ.
$\therefore \quad \alpha + \beta + \gamma = -1/2,\ \alpha\beta + \alpha\gamma + \beta\gamma = 0,\ \alpha\beta\gamma = 7/2$
$$\sum\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$$
$$= \frac{1}{\beta}(\alpha+\gamma) + \frac{1}{\alpha}(\beta+\gamma) + \frac{1}{\gamma}(\alpha+\beta)$$
$$= \frac{1}{\beta}\left(-\frac{1}{2} - \beta\right) + \frac{1}{\alpha}\left(-\frac{1}{2} - \alpha\right) + \frac{1}{\gamma}\left(-\frac{1}{2} - \gamma\right)$$
$$= -\frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 1 - 1 - 1$$
$$= -\frac{1}{2}\left(\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}\right) - 3$$
$$= -3$$

7. Equation $8x^3 + 1001x + 2008 = 0$ has roots r, s, and t. Thus,
$$r + s + r = 0,\ rst = -\frac{2008}{8} = -251$$
Now let $r + s = A;\ s + t = B;\ t + r = C$
$\therefore \quad A + B + C = 2(r + s + t) = 0$
Hence $A^3 + B^3 + C^3 = 3ABC$

$\therefore \quad (r+s)^3 + (s+t)^3 + (t+r)^3$

$= 3(r+s)(s+t)(t+r)$

$= 3(r+s+t-t)(s+t+r-r)(t+r+s-s)$

$= -3rst \qquad (\text{as } r+s+t=0)$

$= 3(251) = 753$

8. If a polynomial has real coefficients, then roots occur in complex conjugate and roots are $\pm 2i$, $2 \pm i$. Hence,

$f(x) = (x+2i)(x-2i)(x-2-i)(x-2+i)$

$\therefore \quad f(1) = (1+2i)(1-2i)(1-2-i)(1-2+i)$

$= 5 \times 2 = 10$

Also, $f(1) = 1 + a + b + c + d$

$\therefore \quad 1 + a + b + c + d = 10$

$\Rightarrow \quad a + b + c + d = 9$

Exercise 2.12

1. $f(x) = \sqrt{x^2 + ax + 4}$ is defined for all x. Hence,

$x^2 + ax + 4 \geq 0$ for all x

$\Rightarrow \quad D = a^2 - 16 \leq 0$

$\Rightarrow \quad a \in [-4, 4]$

2. Let $f(x) = ax^2 + bx + c$. Since $f(x)$ has no real roots either $f(x) > 0$ or $f(x) < 0$ for all $x \in R$. Since $f(0) = c < 0$, we get $f(x) < 0$, $\forall\, x \in R$. Therefore, $a < 0$ as the parabola $y = f(x)$ opens downwards. Also,

$f(1) = a + b + c < 0$

3. $ax^2 + bx + c = 0$ has imaginary roots. Hence,

$ax^2 + bx + c < 0$, $\forall\, x \in R$, if $a < 0$

or $\quad ax^2 + bx + c > 0$, $\forall\, x \in R$, if $a > 0$

But, given

$a + c < b$

or $\quad a - b + c < 0$ or $f(-1) < 0$

$\Rightarrow \quad f(x) = ax^2 + bx + c < 0 \;\forall\, x \in R$

$\Rightarrow \quad f(-2) = 4a - 2b + c < 0$

$\Rightarrow \quad 4a + c < 2b$

4. $x, y, z \in R$

$x + y + z = 6$ and $xy + yz + zx = 7$

$\Rightarrow \quad y(6-y-z) + yz + z(6-y-z) = 7$

$\Rightarrow \quad -y^2 + (6-z+z-z)y + z(6-z) - 7 = 0$

$\Rightarrow \quad y^2 + (z-6)y + 7 + z(z-6) = 0$

Now, y is real. Therefore,

$(z-6)^2 - 4[7 + z(z-6)] \geq 0$

or $\quad 3z^2 - 12z - 8 \leq 0$

or $\quad \dfrac{12 - \sqrt{144+96}}{6} \leq z \leq \dfrac{12 + \sqrt{144+96}}{6}$

or $\quad \dfrac{6 - 2\sqrt{15}}{3} \leq z \leq \dfrac{6 + 2\sqrt{15}}{3}$

From symmetry, x and y have same range.

5. Let,

$$y = \frac{x^2 + 2x + c}{x^2 + 4x + 3c}$$

or $\quad (y-1)x^2 + (4y-2)x + 3cy - c = 0$

Now, x is real. Hence,

$D = (4y-2)^2 - 4(y-1)(3cy-c) \geq 0 \;\forall\, y \in R$

or $\quad (2y-1)^2 - (y-1)(3cy-c) \geq 0$, $\forall\, y \in R$

or $\quad (4-3c)y^2 + (-4+c+3c)y + 1 - c \geq 0$, $\forall\, y \in R$

or $\quad 4 - 3c > 0$ and $(4c-4)^2 - 4(4-3c)(1-c) \leq 0$

or $\quad c < \dfrac{4}{3}$ and $4(c-1)^2 - (4-3c)(1-c) \leq 0$

or $\quad c < \dfrac{4}{3}$ and $(c-1) \times (4c - 4 + 4 - 3c) \leq 0$

or $\quad c < \dfrac{4}{3}$ and $(c-1)(c) \leq 0$

or $\quad c < \dfrac{4}{3}$ and $0 \leq c \leq 1$

or $\quad 0 \leq c \leq 1$

6. Let $x^2 + 2xy + 3y^2 - 6x - 2y + 11 \geq 0$, $\forall\, x, y \in R$

or $\quad x^2 + (2y-6)x + 3y^2 - 2y + 11 \geq 0$, $\forall\, x \in R$

$\Rightarrow \quad (2y-6)^2 - 4(3y^2 - 2y + 11) \leq 0$, $\forall\, y \in R$

or $\quad (y-3)^2 - (3y^2 - 2y + 11) \leq 0$, $\forall\, y \in R$

or $\quad 2y^2 + 4y + 2 \geq 0$, $\forall\, y \in R$

$\Rightarrow \quad (y+1)^2 \geq 0$, $\forall\, y \in R$, which is always true

7. Let $y = \dfrac{x^2 - x}{1 - ax}$

or $\quad x^2 - x = y - axy$

or $\quad x^2 + x(ay - 1) - y = 0$

Since x is real, so

$(ay-1)^2 + 4y \geq 0$

$\Rightarrow \quad a^2y^2 + 2y(2-a) + 1 \geq 0$, $\forall\, y \in R$

$\Rightarrow \quad a^2 > 0$, $4(2-a)^2 - 4a^2 \leq 0 \Rightarrow 4 - 4a \leq 0 \Rightarrow a \in [1, \infty)$

8. For the equation $ax^2 + bx + 6 = 0$, roots are not real. Hence,

$D < 0 \Rightarrow b^2 - 24a < 0$ or $a > \dfrac{b^2}{24}$

Also, $f(0) = 6 > 0$. So,

$f(-1) > 0 \Rightarrow a - b + 6 > 0$

$\Rightarrow \quad a > \max\left\{\dfrac{b^2}{24}, b - 6\right\}$

9. Given, roots of $ax^2 + bx + c = 0$ are imaginary. Hence,

$b^2 - 4ac < 0 \qquad (1)$

Let us consider $f(x) = a^2x^2 + abx + ac$. Here, coefficient of $f(x)$ is a^2 which is +ve, which makes graph concave upward. Also,

$D = (ab)^2 - 4a^2(ac) = a^2(b^2 - 4ac) < 0$

Hence, $f(x) > 0$, $\forall\, x \in R$

10. Let $f(x) = x^2 + \dfrac{b}{a}x + \dfrac{c}{a}$.

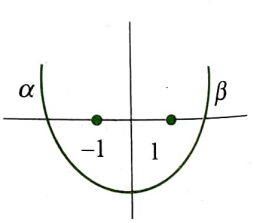
According to the question, we have the following graph.

From the graph,

$f(-1) < 0$ and $f(1) < 0$

$\Rightarrow \quad 1 + \dfrac{c}{a} - \dfrac{b}{a} < 0$ and $1 + \dfrac{c}{a} + \dfrac{b}{a} < 0$

$\Rightarrow \quad 1 + \dfrac{c}{a} + \left|\dfrac{b}{a}\right| < 0$



11. The given equation is, $x^2 + (a-b)x + (1-a-b) = 0$

$a, b \in R$

For this equation to have unequal real roots for all b, $D > 0$

$\Rightarrow \quad (a-b)^2 - 4(1-a-b) > 0$

$\Rightarrow \quad a^2 + b^2 - 2ab - 4 + 4a + 4b > 0$

$\Rightarrow \quad b^2 + b(4-2a) + a^2 + 4a - 4 > 0 \qquad (1)$

This is a quadratic expression in b, and it must be true $\forall\ b \in R$.

So, Discriminant < 0

$\Rightarrow\ (4-2a)^2 - 4(a^2+4a-4) < 0$

$\Rightarrow\ (2-a)^2 - (a^2+4a-4) < 0$

$\Rightarrow\ 4 - 4a + a^2 - a^2 - 4a + 4 < 0$

$\Rightarrow\ -8a + 8 < 0$

$\Rightarrow\ a > 1$

Exercise 2.13

1. Let $f(x) = x^2 - 2(a-1)x + (2a+1)$. Then, $f(x) = 0$ will have both roots positive, if

(i) $D > 0$

$\Rightarrow\ 4(a-1)^2 - 4(2a+1) \geq 0$

$\Rightarrow\ a^2 - 4a \geq 0$

$\Rightarrow\ a \leq 0$ or $a \geq 4$ (1)

(ii) Sum of the roots > 0

$\Rightarrow\ 2(a-1) > 0 \Rightarrow a > 1$ (2)

(iii) Product of roots > 0

$\Rightarrow\ (2a+1) > 0 \Rightarrow a > -\frac{1}{2}$ (3)

From (1), (2), and (3), we get $a \geq 4$.

2. Roots of $(a-5)x^2 + 2(a-10)x + a + 10 = 0$ are opposite in sign.

Then, product of roots is negative. That is, $\frac{a+10}{a-5} < 0$

or $-10 < a < 5$ (using sign scheme method)

3. Both the roots of $x^2 - ax + a = 0$ are greater than 2. Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 4a \geq 0 \Rightarrow a \in (-\infty, 0) \cup [4, \infty)$

(ii) $Af(2) > 0 \Rightarrow 1 \cdot (4 - 2a + a) > 0 \quad \Rightarrow a \in (-\infty, 4)$

(iii) $-\frac{B}{2A} > 2 \Rightarrow \frac{a}{2} > 2 \Rightarrow a > 4$

Hence, no such a can be obtained.

4. Both roots of $ax^2 + ax + 1 = 0$ are less than 1. Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 4a \geq 0 \Rightarrow a \in (-\infty, 0] \cup [4, \infty)$ (1)

(ii) $-\frac{B}{2A} > 1 \Rightarrow -\frac{a}{2a} < 1$ (which is always true as $a \neq 0$)

(iii) $Af(1) > 0$

$\Rightarrow\ a(2a+1) > 0$

$\Rightarrow\ a \in \left(-\infty, -\frac{1}{2}\right) \cup (0, \infty)$ (2)

From (1) and (2), $a \in (-\infty, -1/2) \cup [4, \infty)$.

5. Both roots of $x^2 + ax + 2 = 0$ lies in the interval $(0, 3)$. Compare this equation with $Ax^2 + Bx + C = 0$. Then, the required conditions are

(i) $D = a^2 - 8 \geq 0 \Rightarrow a \in (-\infty, -2\sqrt{2}\,] \cup [2\sqrt{2}, \infty)$ (1)

(ii) $Af(0) > 0$ and $Af(3) > 0$

$\Rightarrow\ 2 > 0,\ 9 + 3a + 2 > 0$

$\Rightarrow\ a \in \left(-\frac{11}{3}, \infty\right)$ (2)

(iii) $0 < -\frac{B}{2A} < 3$

$\Rightarrow\ 0 < -\frac{a}{2} < 3$

$\Rightarrow\ -6 < a < 0$ (3)

From (1), (2) and (3), $a \in (-11/3, -2\sqrt{2}]$.

6. According to question, 1 lies between the roots; therefore, $f(1) < 0$

$\Rightarrow\ 1 - 3 + a < 0$

$\Rightarrow\ a < 2$

7. Let $f(x) = x^2 - bx - 1\ (b \in R^+)$

$f(-1) = b = +\text{ve}$

$f(0) = -1 = -\text{ve}$

$f(1) = -b = -\text{ve}$

Clearly, one root lies in $(-1, 0)$ and the other in $(1, \infty)$. Now, sum of roots is b, which is positive. So, α (having the least absolute value) $\in (-1, 0)$.

8. $a < b < c < d$

$\mu(x-a)(x-c) + \lambda(x-b)(x-d) = 0$

Let the corresponding expression be

$f(x) = \mu(x-a)(x-c) + \lambda(x-b)(x-d)$

$f(a) = \lambda(a-b)(a-d)$

$f(c) = \lambda(c-b)(c-d)$

$\Rightarrow\ f(a)f(c) = \lambda^2(a-b)(a-d)(c-b)(c-d) < 0$

$\Rightarrow\ f(a)$ and $f(c)$ are of opposite signs

Hence, root of the equation lies between a and c. Therefore, the roots are real for all real μ and λ.

9. Let $t = \sin^2 x \Rightarrow t \in [0, 1]$

Hence, $t^2 + at + 1 = 0$ should have at least one solution in $[0, 1]$. Since product of roots is positive and equal to one, equation must have exactly one root in $[0, 1]$. Hence,

$f(1) < 0$

$\Rightarrow\ 2 + a < 0$

$\Rightarrow\ a \in (-\infty, -2)$

Alternative Solution:

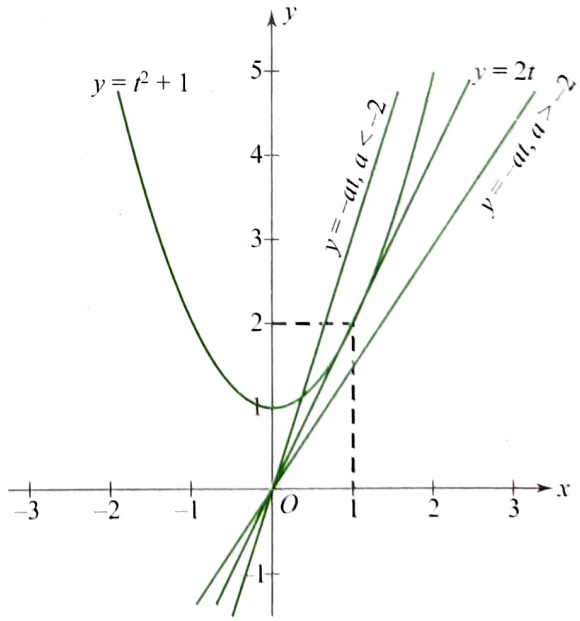
We have $t^2 + at + 1 = 0$ or $t^2 + 1 = -at$, where $t \in [0, 1]$

So, the graph of $y = -at$ must intersect the graph of $y = t^2 + 1$ at least once for $t \in [0, 1]$.

Graph of $y = -at$ is a line through origin and has variable slope.

Graph of $y = t^2 + 1$ is a parabola with vertex at $(0, 1)$ and concave upward.

Graph of both the relations are as shown in the following figure.



For parabola, $\frac{dy}{dt} = 2t$. So, $\left(\frac{dy}{dt}\right)_{(1,2)} = 2$.

Therefore, equation of tangent to parabola at $(1, 2)$ is $y = 2t$, which is line through origin.

Now, if slope of line $y = -at$ is less than 2, then it does not intersect parabola.

If slope of line is more than 2, then it intersects parabola for $t \in (0, 1)$.

Thus, for solution of the equation $t^2 + 1 = -at$, we must have $-a > 2$.

So, $a < -2$ or $a \in (-\infty, -2)$.

Exercises

Single Correct Answer Type

1. (2) $x = 2 + \sqrt{3}$

or $(x-2)^2 = 3$

or $x^2 - 4x + 1 = 0$ (1)

or $(x-2)^4 = 9$

or $x^4 - 8x^3 + 24x^2 - 32x + 16 = 9$

or $x^4 - 8x^3 + 18x^2 - 8x + 2 + 6(x^2 - 4x + 1) - 1 = 0$

Using (1), we get

$x^4 - 8x^3 + 18x^2 - 8x + 2 = 1$

2. (4) $x = 1 + \cfrac{1}{3 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{2...\infty}}}}$

$= 1 + \cfrac{1}{3 + \cfrac{1}{1+x}}$

$= 1 + \cfrac{1+x}{4+3x}$

or $3x^2 = 5$

or $x = \sqrt{\cfrac{5}{3}}$ (as x is positive)

3. (1) Put $x^2 + x = y$, so that Eq. (1) becomes

$(y-2)(y-3) = 12$

or $y^2 - 5y - 6 = 0$

or $(y-6)(y+1) = 0$ or $y = 6, -1$

When $y = 6$, we get

$x^2 + x - 6 = 0$

$\Rightarrow$ $(x+3)(x-2) = 0$ or $x = -3, 2$

When $y = -1$, we get

$x^2 + x + 1 = 0$

which has nonreal roots and sum of roots is -1.

4. (4) Here, $x = 0$ is not a root. Divide both the numerator and denominator by x and put $x + 3/x = y$ to obtain

$$\frac{4}{y+1} + \frac{5}{y-5} = -\frac{3}{2} \Rightarrow y = -5, 3$$

$x + 3/x = -5$ has two irrational roots and $x + 3/x = 3$ has imaginary roots.

5. (3) As $(\lambda+1)x^2 + 2 = \lambda x + 3$ has only one solution, so

$D = 0$

$\Rightarrow$ $\lambda^2 - 4(\lambda+1)(-1) = 0$

or $\lambda^2 + 4\lambda + 4 = 0$

or $(\lambda+2)^2 = 0$

$\therefore$ $\lambda = -2$

6. (1) Given quadratic expression is $x^2 + 2(a+b+c)x + 3(bc+ca+ab)$. This quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero. Hence,

$4(a+b+c)^2 - 4 \times 3(bc+ca+ab) = 0$

or $(a+b+c)^2 - 3(bc+ca+ab) = 0$

or $a^2 + b + c^2 + 2ab + 2bc + 2ca - 3(bc+ca+ab) = 0$

or $a^2 + b^2 + c^2 - ab\ \ bc - ca = 0$

or $\frac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$

or $\frac{1}{2}[(a^2+b^2-2ab) + (b^2+c^2-2bc) + (c^2+a^2-2ca)] = 0$

or $\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

which is possible only when $(a-b)^2 = 0, (b-c)^2 = 0$ and $(c-a)^2 = 0$, i.e., $a = b = c$.

7. (2) Given, $(ax^2 + c)y + (a'x^2 + c') = 0$

or $x^2(ay + a') + (cy + c') = 0$.

If x is rational, then the discriminant of the above equation must be a perfect square. Hence,

$0 - 4(ay + a')(cy + c')$ must be a perfect square

$\Rightarrow$ $-acy^2 - (ac' + a'c)y - a'c'$ must be a perfect square

$\Rightarrow$ $(ac' + a'c)^2 - 4ac\,a'c' = 0$ $[\because D = 0]$

$\Rightarrow$ $(ac' - a'c)^2 = 0$

$\Rightarrow$ $ac' = a'c$

$\Rightarrow$ $\frac{a}{a'} = \frac{c}{c'}$

8. (3) Here a, b, c are positive. So,

$|x| = -b + \sqrt{b^2 + ac}$

Hence, x has two real values, neglecting $|x| = -b - \sqrt{b^2 + ac}$, as $|x| \geq 0$.

9. (3) Clearly, $x = 2$ is a root of the equation and imaginary roots always occur in pairs. Therefore, the other root is also real.

10. (1) $(a-1)x^2 + ax + \sqrt{1-a^2} = 0$

$\therefore$ $D = a^2 - 4(a-1)\sqrt{1-a^2}$

$= a^2 - 4a\sqrt{1-a^2} + 4\sqrt{1-a^2}$

$= (a - 2\sqrt{1-a^2})^2 + 4\sqrt{1-a^2}(1 - \sqrt{1-a^2})$

> 0 for $a \in (-1, 1)$

Hence roots are real but not equal.

11. (3) Discriminant $D = (2m-1)^2 - 4(m-2)m = 4m + 1$ must be perfect square. Hence,

$4m + 1 = k^2$, for some $k \in I$

$\Rightarrow$ $m = \frac{(k-1)(k+1)}{4}$

Clearly, k must be odd. Let $k = 2n + 1$.

$\therefore$ $m = \frac{2n(2n+2)}{4} = n(n+1), n \in I$

12. (1) Given equation is

$x^2 - (y+4)x + y^2 - 4y + 16 = 0$

Since x is real, we have

$D \geq 0$

$\Rightarrow$ $(y+4)^2 - 4(y^2 - 4y + 16) \geq 0$

or $-3y^2 + 24y - 48 \geq 0$

or $y^2 - 8y + 16 \leq 0$

or $(y-4)^2 \leq 0$

or $y - 4 = 0$

or $y = 4$

Since the equation is symmetric in x and y, we have $x = 4$ only.

13. (3) Let the roots be α, β.

$\therefore \quad \alpha+\beta=-2a$ and $\alpha\beta=b$

Given, $|\alpha-\beta|\le 2m$

or $\quad |\alpha-\beta|^2\le (2m)^2$

or $\quad (\alpha+\beta)^2-4ab\le 4m^2$

or $\quad 4a^2-4b\le 4m^2$

$\Rightarrow \quad a^2-m^2\le b$ and discriminant $D>0$ or $4a^2-4b>0$

$\Rightarrow \quad a^2-m^2\le b$ and $b<a^2$

Hence, $b\in[a^2-m^2, a^2)$.

14. (2) Let $\dfrac{x}{x^2-5x+9}=y$

or $\quad yx^2-5yx+9y=x$

or $\quad yx^2-(5y+1)x+9y=0$

Now, x is real, so

$D\ge 0$

$\Rightarrow \quad (-(5y+1))^2-4\cdot y\cdot(9y)\ge 0$

or $\quad -11y^2+10y+1\ge 0$

or $\quad 11y^2-10y-1\le 0$

or $\quad (11y+1)(y-1)\le 0$

or $\quad -\dfrac{1}{11}\le y\le 1$

15. (3) $D>0$

$\Rightarrow \quad (a-3)^2+4(a+2)>0$

or $\quad a^2-6a+9+4a+8>0$

or $\quad a^2-2a+17>0$

$\Rightarrow \quad a\in R$

$\therefore \quad \dfrac{a^2+1}{a^2+2}=1-\dfrac{1}{a^2+2}\ge\dfrac{1}{2}$

16. (2) The discriminants of the given equations are

$D_1=a^2+12b$,

$D_2=c^2-4b$ and $D_3=d^2-8b$.

$\therefore \quad D_1+D_2+D_3=a^2+c^2+d^2\ge 0$

Hence, at least one of D_1, D_2, D_3 is non-negative. Therefore, the equation has at least two real roots.

17. (2) $x^4+x^2+1=(x^2+1)^2-x^2$

$=(x^2+x+1)(x^2-x+1)$

$$x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}\ne 0 \quad \forall x$$

Therefore, we can cancel this factor and we get

$(a-1)(x^2-x+1)=(a+1)(x^2-x+1)$

or $\quad x^2-ax+1=0$

It has real and distinct roots if $D=a^2-4>0$.

18. (2) Let D_1 and D_2 be discriminants of $x^2+b_1x+c_1=0$ and $x^2+b_2x+c_2=0$, respectively. Then,

$D_1+D_2=b_1^2-4c_1+b_2^2-4c_2$

$=(b_1^2+b_2^2)-4(c_1+c_2)$

$=b_1^2+b_2^2-2b_1b_2 \qquad [\because b_1b_2=2(c_1+c_2)]$

$=(b_1-b_2)^2\ge 0$

$\Rightarrow \quad D_1\ge 0$ or $D_2\ge 0$ or D_1 and D_2 both are positive

Hence, at least one of the equations has real roots.

19. (3) $A=a(b-c)(a+b+c)$

$B=b(c-a)(a+b+c)$

$C=c(a-b)(a+b+c)$

Now, $Ax^2+Bx+C=0$

$\Rightarrow \quad (a+b+c)\{a(b-c)x^2+b(c-a)x+c(a-b)\}=0$

Given that roots are equal. Hence,

$D=0$

$\Rightarrow \quad b^2(c-a)^2-4ac(b-c)(a-b)=0$

or $\quad b^2c^2-2ab^2c+b^2a^2-4a^2bc+4acb^2+4a^2c^2-4abc^2=0$

or $\quad b^2c^2+b^2a^2+4a^2c^2+2ab^2c-4a^2bc-4abc^2=0$

or $\quad (bc+ab-2ac)^2=0$

or $\quad bc+ab=2ac$

$\Rightarrow \quad \dfrac{1}{a}+\dfrac{1}{c}=\dfrac{2}{b}$

Hence, a, b, c are in H.P.

20. (1) Since α, β are the roots of $x^2-px+q=0$ and α', β' are the roots of $x^2-p'x+q'=0$, then

$\alpha+\beta=p, \alpha\beta=q, \alpha'+\beta'=p', \alpha'\beta'=q'$

Now, $(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$

$=2(\alpha^2+\beta^2)+2(\alpha'^2+\beta'^2)-2\alpha'(\alpha+\beta)-2\beta'(\alpha+\beta)$

$=2\{(\alpha+\beta)^2-2\alpha\beta+(\alpha'+\beta')^2-2\alpha'\beta'-(\alpha+\beta)(\alpha'+\beta')\}$

$=2(p^2-2q+p'^2-2q'-pp')$

21. (3) We have $a\alpha^2+c=-b\alpha$ or $a\alpha+b=-\dfrac{c}{\alpha}$

Similarly, $a\beta+b=-\dfrac{c}{\beta}$

Hence, the given expression is

$$\frac{b}{c}(\alpha^2+\beta^2)=\frac{b(b^2-2ac)}{a^2c}$$

22. (1) $x^2+ax+b+1=0$ has positive integral roots α and β.

Now, $(\alpha+\beta)=-a$ and $\alpha\beta=b+1$

$\Rightarrow \quad (\alpha+\beta)^2+(\alpha\beta-1)^2=a^2+b^2$

$\Rightarrow \quad a^2+b^2=(\alpha^2+1)(\beta^2+1)$.

$\Rightarrow \quad a^2+b^2$ can be equal to 50 (since other options have prime numbers)

23. (2) $ax^2-bx+c=0$

$$\alpha+\beta=\frac{b}{a}, \alpha\beta=\frac{c}{a}$$

Also, $(a+cy)^2=b^2y$

$\Rightarrow \quad c^2y^2-(b^2-2ac)y+a^2=0$

$$\Rightarrow \quad \left(\frac{c}{a}\right)^2y^2-\left(\left(\frac{b}{a}\right)^2-2\left(\frac{c}{a}\right)\right)y+1=0$$

$\Rightarrow \quad (\alpha\beta)^2y^2-(\alpha^2+\beta^2)y+1=0$

$\Rightarrow \quad y^2-(\alpha^{-2}+\beta^{-2})y+\alpha^{-2}\beta^{-2}=0$

$\Rightarrow \quad (y-\alpha^{-2})(y-\beta^{-2})=0$

Hence, the roots are α^{-2}, β^{-2}.

24. (2) $a\dfrac{(2x+1)^2}{(x-3)^2}+b\dfrac{(2x+1)}{(x-3)}+c=0$

$\Rightarrow \quad \dfrac{2x+1}{x-3}=\alpha$ or $\dfrac{2x+1}{x-3}=\beta$

or $\quad 2x+1=\alpha x-3\alpha$

or $\quad x(\alpha-2)=1+3\alpha$

or $\quad x=\dfrac{1+3\alpha}{\alpha-2}, \dfrac{1+3\beta}{\beta-2}$

25. (2) Multiplying the given equation by c/a^3, we get

$$\frac{b^2c^2}{a^3}x^2-\frac{b^2c}{a^2}x+c=0$$

or $a\left(\frac{bc}{a^2}x\right)^2 - b\left(\frac{bc}{a^2}\right)x + c = 0$

$\Rightarrow \quad \frac{bc}{a^2}x = \alpha, \beta$

$\Rightarrow \quad (\alpha+\beta)\alpha\beta\, x = \alpha, \beta$

$\Rightarrow \quad x = \frac{1}{(\alpha+\beta)\alpha}, \frac{1}{(\alpha+\beta)\beta}$

26. (2) Given, $a(p+q)^2 + 2bpq + c = 0$ and $a(p+r)^2 + 2bpr + c = 0$

$\Rightarrow$ q and r satisfy the equation $a(p+x)^2 + 2bpx + c = 0$

$\Rightarrow$ q and r are the roots of $ax^2 + 2(ap+bp)x + c + ap^2 = 0$

$\Rightarrow \quad qr$ = Product of roots $= \frac{c+ap^2}{a} = p^2 + \frac{c}{a}$

27. (1) α, β are roots of $ax^2 + bx + c = 0$. Hence,

$$\alpha+\beta = -\frac{b}{a} \tag{1}$$

$$\alpha\beta = \frac{c}{a} \tag{2}$$

α^2, β^2 are roots of $a^2x^2 + b^2x + c^2 = 0$. Hence,

$$\alpha^2+\beta^2 = -\frac{b^2}{a^2} \tag{3}$$

$$\alpha^2\beta^2 = \frac{c^2}{a^2} \tag{4}$$

Now, from (3), we have

$$(\alpha+\beta)^2 - 2\alpha\beta = -\frac{b^2}{a^2}$$

$\Rightarrow \quad \left(\frac{-b}{a}\right)^2 - 2\frac{c}{a} = \frac{-b^2}{a^2}$

or $\quad 2\frac{b^2}{a^2} = \frac{2c}{a}$

or $\quad b^2 = ac \Rightarrow a, b, c$ are in G.P.

28. (3) We have,

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \tag{1}$$

and $\quad \frac{k+1}{k}\ \frac{k+2}{k+1} = \frac{c}{a}$

$\Rightarrow \quad \frac{k+2}{k} = \frac{c}{a}$ or $\frac{2}{k} = \frac{c}{a} - 1 = \frac{c-a}{a}$ or $k = \frac{2a}{c-a}$ (2)

Now, eliminate k. Putting the value of k in Eq. (1), we get

$$\frac{c+a}{2a} + \frac{2c}{c+a} = -\frac{b}{a}$$

or $\quad (c+a)^2 + 4ac = -2b\,(a+c)$

or $\quad (a+c)^2 + 2b\,(a+c) = -4ac$

Adding b^2 to both sides, we have

$$(a+b+c)^2 = b^2 - 4ac$$

29. (3) We have

$$\alpha+\beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a},$$

$$\alpha+h+\beta+h = -\frac{q}{p}$$

$\Rightarrow \quad \alpha+\beta+2h = -\frac{q}{p}$

$\Rightarrow \quad -\frac{b}{a} + 2h = \frac{-q}{p}$

$\Rightarrow \quad h = \frac{1}{2}\left(\frac{b}{a} - \frac{q}{p}\right)$

30. (1) Let α and α^2 be the roots of $x^2 - x - k = 0$. Then,

$\alpha + \alpha^2 = 1$ and $\alpha^3 = -k$

or $\quad (-k)^{1/3} + (-k)^{2/3} = 1$

or $\quad -k^{1/3} + k^{2/3} = 1$

or $\quad (k^{2/3} - k^{1/3})^3 = 1$

or $\quad k^2 - k - 3k\,(k^{2/3} - k^{1/3}) = 1$

or $\quad k^2 - k - 3k\,(1) = 1$

or $\quad k^2 - 4k - 1 = 0$

or $\quad k = 2 \pm \sqrt{5}$

31. (4) Here,

$$\alpha^4 + \beta^4 = (\alpha^2+\beta^2)^2 - 2\alpha^2\beta^2$$
$$= \{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2$$
$$= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4}$$
$$= p^4 + \frac{1}{2p^4} + 2$$
$$= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \geq 2 + \sqrt{2}$$

Thus, the minimum value of $\alpha^4 + \beta^4$ is $2 + \sqrt{2}$.

32. (1) α, β be the roots of $x^2 + px + q = 0$

γ, δ be the roots of $x^2 + px + r = 0$

$\alpha+\beta = -p$

$\alpha\beta = q$

$\gamma+\delta = -p$

$\gamma\delta = r$

Now, $(\alpha-\gamma)(\alpha-\delta) = \alpha^2 - \alpha(\gamma+\delta) + \gamma\delta$

$= \alpha^2 - \alpha(\alpha+\beta) + r$

$= -\alpha\beta + r = -q + r$

By symmetry $(\beta-\gamma)(\beta-\delta) = -q + r$

Hence, $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} = 1$

33. (1) Let α be the root of $x^2 - x + m = 0$ and 2α be the root of $x^2 - 3x + 2m = 0$. Then,

$\alpha^2 - \alpha + m = 0$ and $4\alpha^2 - 6\alpha + 2m = 0$

Eliminating α, $m^2 = -2m \Rightarrow m = 0, m = -2$

34. (3) According to the question

α, β are roots of equation $x^2 - 3px + 2q = 0$

and $\quad \alpha, \frac{1}{\beta}$ are roots of equation $x^2 - 3ax + 2b = 0$

Therefore, we have $\alpha+\beta = 3p$, $\alpha\beta = 2q$

and $\quad \alpha + \frac{1}{\beta} = 3a, \frac{\alpha}{\beta} = 2b$

$\therefore \quad (2q-2b)^2 = \left(\alpha\beta - \frac{\alpha}{\beta}\right)^2 = \alpha^2\left(\beta - \frac{1}{\beta}\right)^2$

$= \frac{\alpha}{\beta}\cdot\beta\alpha\left[(\alpha+\beta) - \left(\alpha+\frac{1}{\beta}\right)\right]^2$

$= (2b)\,(2q)\,[3p - 3a]^2$

$= 36\,bq\,(p-a)^2$

35. (3) Given, α, β are roots of equation

$x^2 - 2x + 3 = 0$

$\Rightarrow \quad \alpha^2 - 2\alpha + 3 = 0 \quad (1)$

and $\beta^2 - 2\beta + 3 = 0 \quad (2)$

$\Rightarrow \quad \alpha^2 = 2\alpha - 3$ or $\alpha^3 = 2\alpha^2 - 3\alpha$

$\Rightarrow \quad P = (2\alpha^2 - 3\alpha) - 3\alpha^2 + 5\alpha - 2$

$= -\alpha^2 + 2\alpha - 2 = 3 - 2 = 1$ [Using (1)]

Similarly, we have $Q = 2$.

Now, sum of roots is 3 and product of roots is 2. Hence, the required equation is $x^2 - 3x + 2 = 0$.

36. (2) Let equation is $a(x - \alpha)(x - \beta) = 0$, where α, β are prime numbers

Sum of coefficients $= a - a(\alpha + \beta) + a\alpha\beta$

$= a(\alpha - 1)(\beta - 1)$, which is prime number

$\Rightarrow \quad a = 1$ and one of $(\alpha - 1)$ and $(\beta - 1)$ will be 1

Let $\alpha - 1 = 1$ or $\alpha = 2$.

then $\beta - 1$ will be 2.

So $\beta = 3$

Hence, sum of roots = 5

37. (1) Roots of $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ are reciprocals of each other.

Hence, $\alpha < \frac{1}{2}$ and $\beta > -1$.

38. (3) Let equation $x^2 - 6x + a = 0$ have roots α and β.

Also, let equation $x^2 - cx + 6 = 0$ have roots α and γ.

Given $\frac{\beta}{\gamma} = \frac{4}{3} \Rightarrow \frac{\alpha\beta}{\alpha\gamma} = \frac{4}{3} \Rightarrow \frac{a}{6} = \frac{4}{3} \Rightarrow a = 8$

$\therefore$ First equation is $x^2 - 6x + 8 = 0$

$\Rightarrow \quad (x - 2)(x - 4) = 0$

$\Rightarrow \quad x = 2, 4$

If $\beta = 4$, then $\gamma = \frac{3\beta}{4} = 3$.

If $\beta = 2$, then $\gamma = \frac{3\beta}{4} = \frac{3}{2}$, which is not an integer.

Now, $x^2 - 3x - 4 = 0$

$\Rightarrow \quad (x - 4)(x + 1) = 0$

$\Rightarrow \quad x = -1, 4$

Clearly, one root is less than 2 and other is more than 2.

39. (3) Since α is root of all equations

$a\alpha^2 + 2b\alpha + c = 0$

$2b\alpha^2 + c\alpha + a = 0$

$c\alpha^2 + a\alpha + 2b = 0$

Adding we get $(a + 2b + c)(\alpha^2 + \alpha + 1) = 0$

$a + 2b + c \neq 0$ as $a, b, c > 0$

$\Rightarrow \quad \alpha^2 + \alpha + 1 = 0$ or $\alpha^2 + \alpha = -1$

40. (1) By observation $x = -2$ satisfies equation $x^3 + 3x^2 + 3x + 2 = 0$

then we have $(x + 2)(x^2 + x + 1) = 0$

$x^2 + x + 1 = 0$ has nonreal roots.

Since nonreal roots occur in conjugate pair, $x^2 + x + 1 = 0$

and $ax^2 + bx + c = 0$ are identical

$\Rightarrow \quad a = b = c$

41. (2) Given equations are

$x^3 + ax + 1 = 0 \quad (1)$

or $x^4 + ax^2 + x = 0 \quad (2)$

and $x^4 + ax^2 + 1 = 0$

From (1) – (2), we get $x = 1$. Thus, $x = 1$ is the common root. Hence,

$1 + a + 1 = 0 \Rightarrow a = -2$

42. (2) For given situation, $x^2 - (k - 2)x + k^2 = 0$ and $x^2 + kx + 2k - 1 = 0$ should have both roots common or each should have equal roots. If both roots are common, then

$$\frac{1}{1} = \frac{-(k-2)}{k} = \frac{k^2}{2k-1}$$

$\Rightarrow \quad k = -k + 2$ and $2k - 1 = k^2 \Rightarrow k = 1$

If both the equations have equal roots, then

$(k - 2)^2 - 4k^2 = 0$ and $k^2 - 4(2k - 1) = 0$

$\Rightarrow \quad (3k - 2)(-k - 2) = 0$ and $k^2 - 8k + 4 = 0$

There is no common value of k.

Therefore, $k = 1$ is the only possible value.

43. (2) $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$

or $(31 + 8\sqrt{15})^{x^2-3} + 1^{x^2-3} = (32 + 8\sqrt{15})^{x^2-3}$

$\Rightarrow \quad x^2 - 3 = 1$ or $x = \pm 2$ $[\because a^n + b^n = (a + b)^n]$

44. (4) $(x^2 + x + 1)^2 + 1 = (x^2 + x + 1)(x^2 - x - 5)$

Since $x^2 + x + 1 > 0 \ \forall \ x \in R$,

Dividing both sides by $x^2 + x + 1$, we get

$$(x^2 + x + 1) + \frac{1}{(x^2 + x + 1)} = x^2 - x - 5$$

$\therefore \quad$ L.H.S. ≥ 2

But $x^2 - x - 5 < 1$ for $x \in (-2, 3)$

Therefore, the equation has no solution.

45. (3) We have,

$\alpha + \beta = -p$ and $\alpha\beta = q \quad (1)$

Also, since α, β are the roots of $x^{2n} + p^n x^n + q^n = 0$, we have

$\alpha^{2n} + p^n\alpha^n + q^n = 0$ and $\beta^{2n} + p^n\beta^n + q^n = 0$

Subtracting the above relations, we get

$(\alpha^{2n} - \beta^{2n}) + p^n(\alpha^n - \beta^n) = 0$

$\therefore \quad \alpha^n + \beta^n = -p^n \quad (2)$

Given, α/β or β/α is a root of $x^n + 1 + (x + 1)^n = 0$. So,

$(\alpha/\beta)^n + 1 + [(\alpha/\beta) + 1]^n = 0$

$\Rightarrow \quad (\alpha^n + \beta^n) + (\alpha + \beta)^n = 0$

$\Rightarrow \quad -p^n + (-p)^n = 0$ [Using (1) and (2)]

It is possible only when n is even.

46. (4) $P(a) = P(b) = P(c) = P(d) = 3$

$\Rightarrow \quad P(x) = 3$ has a, b, c, d as its roots

$\Rightarrow \quad P(x) - 3 = (x - a)(x - b)(x - c)(x - d)Q(x)$

$[\because Q(x)$ has integral coefficient$]$

Given $P(e) = 5$, then

$(e - a)(e - b)(e - c)(e - d)Q(e) = 5$

This is possible only when at least three of the five integers $(e - a), (e - b), (e - c), (e - d), Q(e)$ are equal to 1 or -1. Hence, two of them will be equal, which is not possible. Since a, b, c, d are distinct integers, $P(e) = 5$ is not possible.

47. (4) $f(x) = x^2 + bx + c$

$f(x)$ is factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$

$\therefore$ $f(x)$ will also be factor of

$3(x^4 + 6x^2 + 25) - (3x^4 + 4x^2 + 28x + 5) = 14(x^2 - 2x + 5)$

$\therefore$ $x^2 - 2x + 5$ and $f(x)$ have both roots common hence least value of

$f(x) = (x-1)^2 + 4 \geq 4$

48. (1) $x^2 + 2x - n = 0 \Rightarrow (x+1)^2 = n + 1$

$\Rightarrow$ $x = -1 \pm \sqrt{n+1}$

Thus, $n + 1$ should be a perfect square. Now,

$n \in [5, 100] \Rightarrow n + 1 \in [6, 101]$

Perfect square values of $n + 1$ are 9, 16, 25, 36, 49, 64, 81, 100. Hence, number of values is 8.

49. (1) $x^2 - (a+1)x + a - 1 = 0$

$\Rightarrow$ $(x - a)(x - 1) = 1$

Now, $a \in I$ and we want x to be an integer. Hence,

$x - a = 1, x - 1 = 1$ or $x - a = -1, x - 1 = -1$

$\Rightarrow$ $a = 1$ in both cases

50. (4) $(x + a)(x + 1991) + 1 = 0$

or $(x + a)(x + 1991) = -1$

$\Rightarrow$ $(x + a) = 1$ and $x + 1991 = -1$

$\Rightarrow$ $a = 1993$

or $x + a = -1$ and $x + 1991 = 1$ $\Rightarrow$ $a = 1989$

51. (2) Let $f(x) = -3 + x - x^2$. Then $f(x) < 0$ for all x, because coefficient of x^2 is negative and $D < 0$. Thus, L.H.S. of the given equation is always positive, whereas the R.H.S. is always less than zero. Hence, there is no solution.

52. (3) We have

$2\sqrt{5 - 4x - x^2} = 16 - |x|$

$\Rightarrow$ $4(5 - 4x - x^2) = 256 - 32|x| + x^2$

$\Rightarrow$ $5x^2 - 32|x| + 16x + 236 = 0$

$\Rightarrow$ $5x^2 - 16x + 236 = 0$, when $x \geq 0$

and $5x^2 + 48x + 236 = 0$, when $x < 0$.

In both cases, equation has non-real roots.

53. (1) $x = \sqrt[3]{7} + \sqrt[3]{49}$

$\Rightarrow$ $x^3 = 7 + 49 + 3\sqrt[3]{7}\ \sqrt[3]{49}(\sqrt[3]{7} + \sqrt[3]{49}) = 56 + 21x$

or $x^3 - 21x - 56 = 0$

Therefore, the product of roots is 56.

54. (3) Since $\alpha, \beta, \gamma, \sigma$ are the roots of the given equation, we have

$x^4 + 4x^3 - 6x^2 + 7x - 9 = (x - \alpha)(x - \beta)(x - \gamma)(x - \sigma)$

Putting $x = i$ and then $x = -i$, we get

$1 - 4i + 6 + 7i - 9 = (i - \alpha)(i - \beta)(i - \gamma)(i - \sigma)$

and $1 + 4i + 6 - 7i - 9 = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \sigma)$

Multiplying these two equations, we get

$(-2 + 3i)(-2 - 3i) = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

or $13 = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$

55. (2) If $[m_r, (1/m_r)]$ satisfy the given equation

$x^2 + y^2 + 2gx + 2fy + c = 0$, then

$m_r^2 + \frac{1}{m_r^2} + 2gm_r + \frac{2f}{m_r} + c = 0$

$\Rightarrow$ $m_r^4 + 2gm_r^3 + cm_r^2 + 2fm_r + 1 = 0$

Now, roots of given equation are m_1, m_2, m_3, m_4. The product of roots

$$m_1 m_2 m_3 m_4 = \frac{\text{Constant term}}{\text{Coefficient of } m_r^4} = \frac{1}{1} = 1$$

56. (1) Clearly,

$x^n - 1 = (x - 1)(x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow$ $\frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow$ $1 + x + x^2 + \cdots + x^{n-1} = (x - a_1)(x - a_2) \cdots (x - a_{n-1})$

$\Rightarrow$ $n = (1 - a_1)(1 - a_2) \cdots (1 - a_{n-1})$ [putting $x = 1$]

57. (2) $\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = (a + 1)$

$\Sigma \tan\theta_1 \tan\theta_2 = (b - a)$

$\tan\theta_1 \tan\theta_2 \tan\theta_3 = b$

$\therefore$ $\tan(\theta_1 + \theta_2 + \theta_3) = \frac{\Sigma \tan\theta_1 - \Pi \tan\theta_1}{1 - \Sigma \tan\theta_1 \tan\theta_2}$

$= \frac{a + 1 - b}{1 - (b - a)} = 1$

$\Rightarrow$ $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{4}$

58. (1) $\Sigma\alpha = 1, \Sigma\alpha\beta = 0, \alpha\beta\gamma = 1$

$$\Sigma \frac{1 + \alpha}{1 - \alpha} = -\Sigma \frac{-\alpha + 1 - 2}{1 - \alpha} = \Sigma\left(\frac{2}{1 - \alpha} - 1\right)$$

$$= 2\Sigma \frac{1}{1 - \alpha} - 3$$

Now, $x^3 - x^2 - 1 = (x - \alpha)(x - \beta)(x - \gamma)$

$\therefore$ $\log(x - \alpha) + \log(x - \beta) + \log(x - \gamma) = \log(x^3 - x^2 - 1)$

Differentiating, we get

$$\frac{1}{(x - \alpha)} + \frac{1}{(x - \beta)} + \frac{1}{(x - \gamma)} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$$

$\Rightarrow$ $\frac{1}{1 - \alpha} + \frac{1}{1 - \beta} + \frac{1}{1 - \gamma} = \frac{3 - 2}{1 - 1 - 1} = -1$

$\Rightarrow$ $\frac{1 + \alpha}{1 - \alpha} = -5$

59. (2) Roots of equation $x^4 - Kx^3 + Kx^2 + Lx + M = 0$ are $\alpha, \beta, \gamma, \delta$

$\Sigma\alpha = K, \Sigma\alpha\beta = K,$

$\Sigma\alpha\beta\gamma = -L, \alpha\beta\gamma\delta = M$

$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha + \beta + \gamma + \delta)^2 - 2\Sigma\alpha\beta$

$= K^2 - 2K$

$= (K - 1)^2 - 1$

$\therefore$ $(\alpha^2 + \beta^2 + \gamma^2 + \delta^2)_{\min} = -1$

60. (1) $f(x) = \frac{(2a - 1)x^2 + 2(a + 1)x + (2a - 1)}{x^2 - 2x + 40}$

Since $x^2 - 2x + 40 > 0$ for all real x,

$(2a - 1)x^2 + 2(a + 1)x + (2a - 1) < 0\ \forall\ x \in R$

$\therefore$ $2a - 1 < 0$ or $a < \frac{1}{2}$ (1)

and $D < 0$

$\Rightarrow$ $4(a + 1)^2 - 4(2a - 1)^2 < 0$

$\Rightarrow$ $a(a - 2) > 0$

$\Rightarrow$ $a < 0$ or $a > 2$ (2)

From (1) and (2), $a < 0$

61. (1) $D = b^2 - 4a < 0 \Rightarrow a > 0$

Therefore the graph is concave upwards.

$f(x) > 0, \forall\ x \in R$

$\Rightarrow$ $f(-1) > 0$

$\Rightarrow$ $a + b + 1 > 0$

(3) We know that $ax^2 + bx + c \geq 0, \forall x \in R$, if $a > 0$ and $b^2 - 4ac \leq 0$. So,

$$mx - 1 + \frac{1}{x} \geq 0 \text{ or } \frac{mx^2 - x + 1}{x} \geq 0$$

or $mx^2 - x + 1 \geq 0$ as $x > 0$.

Now,

$mx^2 - x + 1 \geq 0$ if $m > 0$ and $1 - 4m \leq 0$

$\Rightarrow$ $m > 0$ and $m \geq 1/4$

Thus, the minimum value of m is 1/4.

(2) Let $f(x) = ax^2 + bx + c$ be a quadratic expression such that $f(x) > 0$ for all $x \in R$. Then, $a > 0$ and $b^2 - 4ac < 0$. Now,

$g(x) = f(x) + f'(x) + f''(x)$

$\Rightarrow$ $g(x) = ax^2 + x(b + 2a) + (b + 2a + c)$

Discriminant of $g(x)$ is

$D = (b + 2a)^2 - 4a(b + 2a + c)$

$= b^2 - 4a^2 - 4ac$

$= (b^2 - 4ac) - 4a^2$

< 0 $(\because b^2 - 4ac < 0)$

Therefore, $g(x) > 0$ for all $x \in R$.

(4) Given equation has non-real roots.

$\therefore$ $D < 0$

$\Rightarrow$ $b^2c^2 - 4(b^3 + c^3 - 4abc)a < 0$

$\Rightarrow$ $(b^2c^2 - 4ab^3) + (16a^2bc - 4ac^3) < 0$

$\Rightarrow$ $b^2(c^2 - 4ab) - 4ac(c^2 - 4ab) < 0$

$\Rightarrow$ $(b^2 - 4ac)(c^2 - 4ab) < 0$

$\Rightarrow$ $D_{P(x)} \cdot D_{Q(x)} < 0$

Therefore, exactly one of $P(x)$ or $Q(x)$ is positive for all real x.

(2) Here, $ax^2 - bx + c^2 = 0$ does not have real roots. So,

$D < 0 \Rightarrow b^2 - 4ac^2 < 0 \Rightarrow a > 0$

Therefore, $f(x)$ is always positive. So,

$f(2) > 0 \Rightarrow 4a - 2b + c^2 > 0$

(1) Suppose $a \neq 0$. We rewrite $f(x)$ as follows:

$$f(x) = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$

$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$f\left(-\frac{b}{2a} + i\right) = a\left\{\left(-\frac{b}{2a} + i + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\}$$

$$= a\left\{-1 + \frac{4ac - b^2}{4a^2}\right\}, \text{ which is a real number}$$

This is against the hypothesis. Therefore, $a = 0$.

Let,

$f(x) = ax^2 + x + c - a$

$f(1) = c + 1 > 0$ $(\because c > -1)$

Therefore, given expression is positive $\forall x \in R$. So,

$$f\left(\frac{1}{2}\right) > 0$$

$\Rightarrow$ $\frac{a}{4} + \frac{1}{2} + c - a > 0$

or $4c - 3a + 2 > 0$

or $4c + 2 > 3a$

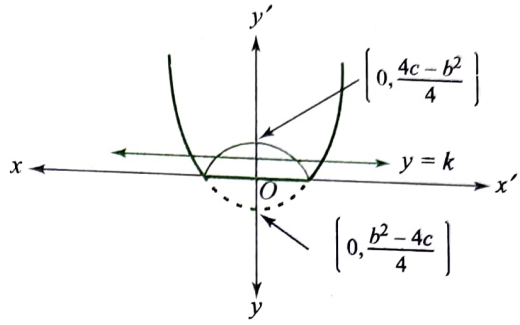
68. (2) $\frac{(b^2 - 4ac)^2}{16a^2} < \frac{4}{1 + 4a^2}$ (1)

Now, $\max(ax^2 + bx + c) = -\frac{b^2 - 4ac}{4a}$

Also, $\frac{-2}{\sqrt{1 + 4a^2}} < -\frac{b^2 - 4ac}{4a} < \frac{2}{\sqrt{1 + 4a^2}}$ [From (1)]

So, maximum value is always less than 2 (when $a \to 0$).

69. (1)



For the equation to have four real roots, the line $y = k$ must intersect $y = |x^2 + bx + c|$ at four points.

$\therefore$ $D > 0$ and $k \in \left(0, \frac{-D}{4}\right)$

70. (3) Given,

$(a - 1)x^2 - (a + 1)x + a - 1 \geq 0$

or $a(x^2 - x + 1) - (x^2 + x + 1) \geq 0$

$$\Rightarrow \quad a \geq \frac{x^2 + x + 1}{x^2 - x + 1} = 1 + \frac{2x}{x^2 - x + 1}$$

$$= 1 + \frac{2}{x + \frac{1}{x} - 1} \quad (1)$$

Let $y = x + 1/x$. Now, y is increasing in $[2, \infty)$. Hence,

$$1 + \frac{2}{x + \frac{1}{x} - 1} \in \left(1, \frac{7}{3}\right]$$

For all $x \geq 2$, Eq. (1) should be true. Hence, $a > 7/3$.

71. (2) Since $ax^2 + bx + c = x$ has no real roots

$\Rightarrow$ $ax^2 + bx + c > x$ or $ax^2 + bx + c < x$ for all real x.

Let $ax^2 + bx + c > x$ and $ax^2 + bx + c = P(x)$

$\therefore$ $P(x) > x$ for all real x.

$\Rightarrow$ $P(P(x)) > P(x)$

$\Rightarrow$ $a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c > ax^2 + bx + c > x$

Hence, given equation has no real solution.

Similarly it can be proved for $ax^2 + bx + c < x$

72. (2) $\left|a\left(x^2 + \frac{y}{a}\right) + (b + 1)x + c\right| = |ax^2 + bx + c| + |x + y|$

$\Rightarrow$ $|(ax^2 + bx + c) + (x + y)| = |ax^2 + bx + c| + |x + y|$ (1)

Now $f(x) = ax^2 + bx + c = 0$ has imaginary roots and

$a - b + c > 0$ or $f(-1) > 0$

$\Rightarrow$ $f(x) = ax^2 + bx + c > 0$ for all real values of x

$\Rightarrow$ $x + y \geq 0$

$\Rightarrow$ (x, y) lies on or above the bisector of II and IV quadrants

73. (2) $\lambda = \dfrac{x^2+y^2}{x^2+xy+4y^2}$

$$= \frac{1+\left(\frac{y}{x}\right)^2}{1+\frac{y}{x}+4\left(\frac{y}{x}\right)^2}$$

$$= \frac{1+z^2}{1+z+4z^2} \qquad \text{(Putting } y/x = z\text{)}$$

or $\lambda + \lambda z + 4\lambda z^2 = 1 + z^2$

$z^2(4\lambda - 1) + z(\lambda) + \lambda - 1 = 0$

Since z is real,

$D \geq 0$

$\therefore \quad \lambda^2 - 4(4\lambda^2 - 5\lambda + 1) \geq 0$

$\therefore \quad 15\lambda^2 - 20\lambda + 4 \leq 0$

$\therefore \quad \lambda \in \left(\dfrac{10-4\sqrt{5}}{15}, \dfrac{10+4\sqrt{5}}{15}\right)$

74. (1) $x_1(x-x_2)^2 + x_2(x-x_1)^2 = 0$

$\Rightarrow \quad x^2(x_1+x_2) - 4x\,x_1x_2 + x_1x_2\,(x_1+x_2) = 0$

$D = 16(x_1x_2)^2 - 4x_1x_2(x_1+x_2)^2 > 0 \ (\because x_1x_2 < 0)$

The product of roots is $x_1x_2 < 0$. Thus, the roots are real and of opposite signs.

75. (4) Given that $a < b < c < d$. Let

$f(x) = (x-a)(x-c) + 2(x-b)(x-d)$

$\Rightarrow \quad f(b) = (b-a)(b-c) < 0$

and $f(d) = (d-a)(d-c) > 0$

Hence, $f(x) = 0$ has one root in (b, d). Also, $f(a)f(c) < 0$. So the other root lies in (a, c). Hence, roots of the equation are real and distinct.

76. (3) $x^2 - (a-3)x + a = 0$

$\Rightarrow \quad D = (a-3)^2 - 4a$

$= a^2 - 10a + 9$

$= (a-1)(a-9)$

Case I:

Both the roots are greater than 2.

$D \geq 0, f(2) > 0, -\dfrac{B}{2A} > 2$

$\Rightarrow \quad (a-1)(a-9) \geq 0;\ 4-(a-3)2+a > 0;\ \dfrac{a-3}{2} > 2$

$\Rightarrow \quad a \in (-\infty, 1] \cup [9, \infty);\ a < 10;\ a > 7$

$\Rightarrow \quad a \in [9, 10) \qquad (1)$

Case II:

One root is greater than 2 and the other root is less than or equal to 2. Hence,

$f(2) \leq 0$

$\Rightarrow \quad 4-(a-3)2+a \leq 0$

$\Rightarrow \quad a \geq 10 \qquad (2)$

From (1) and (2),

$a \in [9, 10) \cup [10, \infty)$

$\Rightarrow \quad a \in [9, \infty)$

77. (3) The given equation is

$x^2 - 2mx + m^2 - 1 = 0$

or $(x-m)^2 - 1 = 0$

or $(x-m+1)(x-m-1) = 0$

or $x = m-1, m+1$

From given condition,

$m - 1 > -2$ and $m + 1 < 4$

$\Rightarrow \quad m > -1$ and $m < 3$

Hence, $-1 < m < 3$.

78. (2) Note that coefficient of x^2 is $(4p - p^2 - 5) < 0$. Therefore, the graph is concave downward. According to the question, 1 must lie between the roots. Hence,

$f(1) > 0$

$\Rightarrow \quad 4p - p^2 - 5 - 2p + 1 + 3p > 0$

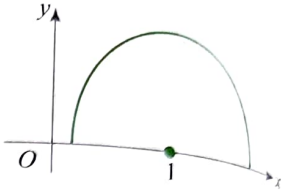
$\Rightarrow \quad -p^2 + 5p - 4 > 0$

$\Rightarrow \quad p^2 - 5p + 4 < 0$

$\Rightarrow \quad (p-4)(p-1) < 0$

$\Rightarrow \quad 1 < p < 4$

$\Rightarrow \quad p \in \{2, 3\}$



79. (2) $\tan x = \dfrac{a-4 \pm \sqrt{(a-4)^2 - 4(4-2a)}}{2}$

$= \dfrac{a-4 \pm a}{2} = a-2, -2$

$\therefore \quad \tan x = a - 2 \qquad (\because \tan x \neq -2)$

$\because \quad x \in \left[0, \dfrac{\pi}{4}\right]$

$\therefore \quad 0 \leq a - 2 \leq 1$

$\Rightarrow \quad 2 \leq a \leq 3$

80. (1) Clearly, $f(-1) > 0, f(2) < 0$.

Now,

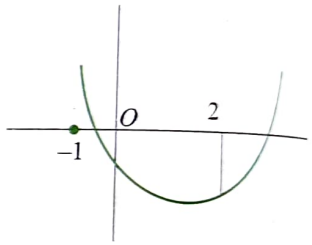
$f(0) = -4 < 0$

$\Rightarrow \quad f(-1) = 1 - a - 4 > 0$

and $f(2) = 4 + 2a - 4 < 0$

$\Rightarrow \quad a < -3$ and $a < 0$

$\Rightarrow \quad a \in (-\infty, -3)$



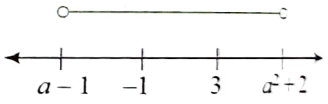
81. (2) We have,

$a - 1 \leq -1$

and $a^2 + 2 \geq 3$

$a \leq 0$ and $a^2 \geq 1$

Hence, $a \leq -1$.



82. (4) $\cot^4 x - 2(1 + \cot^2 x) + a^2 = 0$

or $\cot^4 x - 2\cot^2 x + a^2 - 2 = 0$

or $(\cot^2 x - 1)^2 = 3 - a^2$

Now, for at least one solution

$3 - a^2 \geq 0$

or $a^2 - 3 \leq 0$

$\therefore \quad a \in [-\sqrt{3}, \sqrt{3}]$

Integral values are $-1, 0, 1$.

$\therefore \quad$ Sum $= 0$

83. (1) The equation on simplifying gives

$x(x-b)(x-c) + x(x-c)(x-a) + x(x-a)(x-b) - (x-a)(x-b)(x-c) = 0 \qquad (1)$

Let $f(x) = x(x-b)(x-c) + x(x-c)(x-a) + x(x-a)(x-b) - (x-a)(x-b)(x-c)$

We can assume without loss of generality that $a < b < c$. Now,

$f(a) = a(a-b)(a-c) > 0$

$f(b) = b(b-c)(b-a) < 0$

$f(c) = c(c-a)(c-b) > 0$

So, one root of (1) lies in (a, b) and one root in (b, c). Obviously the third root must also be real.

(4) $a = \dfrac{x^2+4}{|x|} - 3 = |x| + \dfrac{4}{|x|} - 3 = \left(\sqrt{|x|} - \dfrac{2}{\sqrt{|x|}}\right)^2 + 1$

$\Rightarrow \quad a \geq 1$

(4) Here, $f(x) = (2x - a)(2x - c) + (2x - b)$. So,

$$f\left(\frac{a}{2}\right) = a - b,\ f\left(\frac{c}{2}\right) = c - b$$

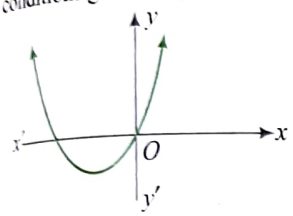
Now,

$$f\left(\frac{a}{2}\right) f\left(\frac{c}{2}\right) = (a - b)(c - b) < 0 \ (a > b > c)$$

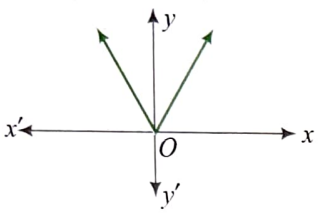
Hence, exactly one of the roots lies between $c/2$ and $a/2$.

(1) Since the equation $x^2 + ax + b = 0$ has distinct real roots and $x^2 + a|x| + b = 0$ has only one real root, so one root of the equation $x^2 + ax + b = 0$ will be zero and other root will be negative. Hence, $b = 0$ and $a > 0$.

Graph of $y = ax^2 + bx + c$ according to conditions given in question



Graph of $y = ax^2 + b|x| + c$



(3) The given equation is

$$2^{2x} + (a - 1)2^{x+1} + a = 0$$

or $\quad t^2 + 2(a - 1)t + a = 0$, where $2^x = t$

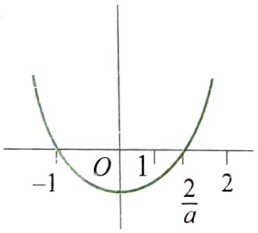
Now, $t = 1$ should lie between the roots of this equation.

$\therefore \quad 1 + 2(a - 1) + a < 0$ or $a < \dfrac{1}{3}$

(2) $f(x) = ax^2 + (a - 2)x - 2 = (ax - 2)(x + 1)$

$f(0) = -2$ and $f(-1) = 0$

If a is negative then expression becomes negative for infinite values of x, therefore it must be positive. Expression to be negative for exactly two integral values of x



So $\quad \dfrac{2}{a} \leq 2$ or $a \geq 1$

and $\quad \dfrac{2}{a} > 1 \Rightarrow a < 2$

$\therefore \quad a \in [1, 2)$

(1) $P(x) = 4a_0 x^3 + 3a_1 x^2 + 2a_2 x + a_3$ is a polynomial, so it is continuous for all x.

$P(x) = 0$ has a root in $(-1, 0)$.

$P(-1)\,P(0) < 0$

Now $P(0) = a_3 > 0$,

$\Rightarrow \quad P(-1) = -4a_0 + 3a_1 - 2a_2 + a_3 < 0$

$\Rightarrow \quad 4a_0 + 2a_2 > 3a_1 + a_3$

Also using Rolle's theorem,

$$Q(x) = \int p(x)dx = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + d$$

For $Q'(x) = p(x) = 0$

for at least one root in $(-1, 0)$

$$Q(-1) = Q(0)$$

$\Rightarrow \quad a_0 - a_1 + a_2 - a_3 + d = d$

or $\quad a_0 + a_2 = a_1 + a_3$

Multiple Correct Answers Type

1. (1), (2)

We can write the given equation as

$$\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$$

or $\quad p(x^2 - c^2) = 2(a + b)x^2 - 2c(a - b)x$

or $\quad (2a + 2b - p)x^2 - 2c(a - b)x + pc^2 = 0$

For this equation to have equal roots,

$$c^2(a - b)^2 - pc^2(2a + 2b - p) = 0$$

or $\quad (a - b)^2 - 2p(a + b) + p^2 = 0 \qquad [\because c^2 \neq 0]$

or $\quad [p - (a + b)]^2 = (a + b)^2 - (a - b)^2 = 4ab$

or $\quad p - (a + b) = \pm 2\sqrt{ab}$

or $\quad p = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

2. (1), (2), (4)

Symmetric functions are those which do not change by interchanging α and β.

3. (1), (4)

One root $\dfrac{\sqrt{a}}{\sqrt{a} + \sqrt{a - b}}$

On rationalizing, we get

$$\frac{\sqrt{a}}{\sqrt{a} + \sqrt{a-b}} \times \frac{\sqrt{a} - \sqrt{a-b}}{\sqrt{a} - \sqrt{a-b}}$$

$$= \frac{\sqrt{a}.(\sqrt{a} - \sqrt{a-b})}{b} = \frac{a - \sqrt{a(a-b)}}{b}$$

$\therefore \quad$ Other root $= \dfrac{a + \sqrt{a(a-b)}}{b}$

Rationalizing, we get

$$\frac{a + \sqrt{a(a-b)}}{b} \times \frac{a - \sqrt{a(a-b)}}{a - \sqrt{a(a-b)}}$$

$$= \frac{a^2 - (a^2 - ab)}{b(a - \sqrt{a(a-b)})} = \frac{\sqrt{a}}{\sqrt{a} - \sqrt{a-b}}$$

4. (1), (3)

Given, $b^2 = ac$

or $\quad \left(\dfrac{b}{a}\right)^2 = \dfrac{c}{a}$

or $\quad \left(-\dfrac{b}{a}\right)^2 = \dfrac{c}{a}$

or $\quad (\alpha + \beta)^2 = \alpha\beta$

or $\quad \alpha^2 + \beta^2 + \alpha\beta = 0$

or $\quad \left(\dfrac{\alpha}{\beta}\right)^2 + \left(\dfrac{\alpha}{\beta}\right) + 1 = 0$

or $\quad \dfrac{\alpha}{\beta} = \dfrac{-1 \pm \sqrt{3}i}{2}$ (where $i = \sqrt{-1}$)

5. (1), (3)

We have $(a + \sqrt{b})^{x^2-15} + \dfrac{1}{(a + \sqrt{b})^{x^2-15}} = 2a$

$y^2 - 2ay + 1 = 0$, where $y = (a + \sqrt{b})^{x^2-15}$

$\therefore \quad y = a \pm \sqrt{a^2 - 1} = a \pm \sqrt{b} = (a + \sqrt{b})^{x^2-15}$

So, $\quad x^2 - 15 = \pm 1$

$\therefore \quad x = \pm 4, \pm\sqrt{14}$

6. (1), (2)

Equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root. Therefore,

$$(q - q')^2 = (pq' - p'q)(p' - p) \quad (1)$$

Subtracting two equations, we have

$$x = \frac{q - q'}{p' - p}$$

Also using (1),

$$x = \frac{q - q'}{p' - p} = \frac{pq' - p'q}{q - q'}$$

7. (1), (2), (3)

$$\sec^2\theta + \text{cosec}^2\theta = \sec^2\theta.\text{cosec}^2\theta$$

Sum of the roots is equal to their product and the roots are real. Hence,

$$-\frac{b}{a} = \frac{c}{a}$$

or $b + c = 0$

Also $b^2 - 4ac \geq 0$

or $c^2 - 4ac \geq 0$

or $c(c - 4a) \geq 0$

or $c - 4a \geq 0 \quad (\because \ c > 0)$

Further

$$b^2 + 4ab \geq 0$$

or $b + 4a \leq 0 \quad (\because \ b < 0)$

8. (1), (4)

Since $\alpha, \beta, \gamma, \delta$ are in H.P., $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$ are in A.P. and they may be taken as $a - 3d, a - d, a + d, a + 3d$. Replacing x by $1/x$, we get the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$. Therefore, equation $x^2 - 4x + A = 0$ has roots $a - 3d, a + d$ and equation $x^2 - 6x + B = 0$ has roots $a - d, a + 3d$. Sum of the roots is

$$2(a - d) = 4, \ 2(a + d) = 6$$

$\therefore \quad a = 5/2, \ d = 1/2$

Product of the roots is

$$(a - 3d)(a + d) = A = 3$$

$$(a - d)(a + 3d) = B = 8$$

9. (1), (2)

We have, $\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\Rightarrow \quad \cos^4\theta - \sin^4\theta = \cos 2\theta$

$\Rightarrow \quad (-2b)^2 - 4b = (-4)^2 - 4 \times 2$

(since L.H.S. is difference of roots of first equation and R.H.S. is difference of roots of second equation)

or $4b^2 - 4b = 16 - 8 = 8$

or $4b^2 - 4b - 8 = 0$

or $b^2 - b - 2 = 0$

or $(b + 1)(b - 2) = 0$

or $b = 2, -1$

10. (2), (4)

Let $f(x) = x^2 + ax + b$. Then,

$$x^2 + (2c + a)x + c^2 + ac + b = f(x + c)$$

Thus, the roots of $f(x + c) = 0$ will be $0, d - c$.

11. (1), (3)

Since each pair has common root, let the roots be α, β for Eq. (1); β, γ for Eq. (2) and γ, α for Eq. (3). Therefore,

$$\alpha + \beta = -a, \ \alpha\beta = bc$$

$$\beta + \gamma = -b, \ \beta\gamma = ca$$

$$\gamma + \alpha = -c, \ \gamma\alpha = ab$$

Adding, we get

$$2(\alpha + \beta + \gamma) = -(a + b + c)$$

$$\Rightarrow \quad \alpha + \beta + \gamma = -\frac{1}{2}(a + b + c)$$

Also by multiplying product of roots, we have

$$\alpha^2\beta^2\gamma^2 = a^2b^2c^2 \text{ or } \alpha\beta\gamma = abc$$

12. (1), (4)

Roots of $4x^2 - x - 1 = 0$ are irrational. So, one root common implies both roots are common. Therefore,

$$\frac{4}{3} = \frac{-1}{\lambda + \mu} = \frac{-1}{\lambda - \mu}$$

$$\Rightarrow \quad \lambda = \frac{-3}{4}, \ \mu = 0$$

13. (2), (3)

$f(x) = x^3 + 3x^2 - 9x + c$ is of the form $(x - \alpha)^2(x - \beta)$, showing that α is a double root so that $f'(x) = 0$ has also one root α, i.e., $3x^2 + 6x - 9 = 0$ has one root α. Hence, $x^2 + 2x - 3 = 0$ or $(x + 3)(x - 1) = 0$ has the root α which can be either -3 or 1. If $\alpha = 1$, then $f(x) = 0$ gives $c - 5 = 0$ or $c = 5$. If $\alpha = -3$, then $f(x) = 0$ gives

$$-27 + 27 + 27 + c = 0$$

$\therefore \quad c = -27$

14. (2), (3), (4)

Given equation is $x^3 - ax^2 + bx - 1 = 0$. If roots of the equation be α, β, γ, then

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= a^2 - 2b$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= b^2 - 2a$$

$$\alpha^2\beta^2\gamma^2 = 1$$

So, the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is given by

$$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$$

It is identical to

$$x^3 - ax^2 + bx - 1 = 0$$

$\Rightarrow \quad a^2 - 2b = a$ and $b^2 - 2a = b$

Eliminating b, we get

$$\frac{(a^2 - a)^2}{4} - 2a = \frac{a^2 - a}{2}$$

or $a\{a(a - 1)^2 - 8 - 2(a - 1)\} = 0$

or $a(a^3 - 2a^2 - a - 6) = 0$

or $a(a - 3)(a^2 + a + 2) = 0$

$\Rightarrow \quad a = 0$ or $a = 3$ or $a^2 + a + 2 = 0$

which gives $b = 0$ or $b = 3$ or $b^2 + b + 2 = 0$. So, $a = b = 0$ or $a = b = 3$ or a, b are roots of $x^2 + x + 2 = 0$.

15. (2), (4)

It is given that $f(x)$ touches x-axis at $(\sqrt{2}, 0)$. So, two roots are $\sqrt{2}, \sqrt{2}$.

Sine coefficients are rational, other roots must be $-\sqrt{2}, -\sqrt{2}$.

16. (1), (2), (3), (4)

Let $x + \frac{1}{x} = t$

Then given equation is

$$(t^3 - 3t) + (t^2 - 2) - 6(t) - 7 = 1$$

or $t^3 + t^2 - 9t - 9 = 0$

$\therefore \quad (t+1)(t^2-9) = 0$

$\therefore \quad t = -1, 3, -3$

If $x + \frac{1}{x} = -1$, then $x^2 + x + 1 = 0$. This equation has no real roots.

If $x + \frac{1}{x} = -3$, then $x^2 + 3x + 1 = 0$. Roots are $\frac{-3 \pm \sqrt{5}}{2}$.

If $x + \frac{1}{x} = 3$, then $x^2 - 3x + 1 = 0$. Roots are $\frac{3 \pm \sqrt{5}}{2}$.

17. (1), (3)

$2x^2 + 6xy + 5y^2 = 1 \qquad (1)$

Equation (1) can be rewritten as

$2x^2 + (6y)x + 5y^2 - 1 = 0$

Since x is real, we have

$36y^2 - 8(5y^2 - 1) \geq 0$

or $y^2 \leq 2$

or $-\sqrt{2} \leq y \leq \sqrt{2}$

Equation (1) can also be rewritten as

$5y^2 + (6x)y + 2x^2 - 1 = 0$

Since y is real, we have

$36x^2 - 20(2x^2 - 1) \geq 0$

or $36x^2 - 40x^2 + 20 \geq 0$

or $-4x^2 \geq -20$

or $x^2 \leq 5$

or $-\sqrt{5} \leq x \leq \sqrt{5}$

18. (1), (2), (3)

(1) If $f(1) = 0 \Rightarrow a + b + c = 0$
So, roots are real.

(2) $D = b^2 - 4ac$
If $ac < 0 \Rightarrow D > 0$
So, roots are real.

(3) $4ac - b^2 < 0 \Rightarrow b^2 - 4ac > 0$
So, roots are real.

(4) This is not the sufficient condition.

19. (2), (3)

Given equation is

$x^2 + 5 - 2x = -4\cos(m + nx)$

$\Rightarrow \quad (x-1)^2 + 4 = -4\cos(m + nx)$

LHS ≥ 4 and RHS ≤ 4

So, solution is possible only if LHS = RHS = 4

$\therefore \quad x = 1$ and $\cos(m + n) = -1$.

$\therefore \quad m + n = (2n + 1)\pi, n \in I$

20. (1), (2), (4)

For each equation, required conditions

(i) Discriminant $\geq 0 \quad \Rightarrow b^2 \geq ac, c^2 \geq ab, a^2 \geq bc$

(ii) $f(0) > 0 \quad \Rightarrow c, a, b > 0$

(iii) Abscissa of vertex $> 0 \Rightarrow \frac{b}{a} > 0, \frac{c}{b} > 0, \frac{a}{c} > 0$.

Now, $b^2 \geq ac, c^2 \geq ab \Rightarrow b^2c^2 \geq a^2bc \Rightarrow bc \geq a^2$.

But $a^2 \geq bc$. So, $a^2 = bc$.

Similarly, $b^2 = ac$ and $c^2 = ab$.

Therefore, $a = b = c$

21. (2), (3), (4)

Given equation is

$x^2 + 2(a+1)x + 9a - 5 = 0$

$D = 4(a+1)^2 - 4(9a - 5) = 4(a-1)(a-6)$

$D \geq 0 \Rightarrow a \leq 1$ or $a \geq 6$

If $a < 0$, then $9a - 5 < 0$. Hence, the products of roots is less than 0. So, the roots are of opposite sign. If $a > 7$, then sum of roots is $-2(a+1) < 0$. Product of roots is greater than 0.

22. (3), (4)

Product of roots is

$\frac{a}{bc} < 0 \qquad [\because abc < 0]$

Hence, roots are real and of opposite sign.

23. (1), (2), (3)

From figure,

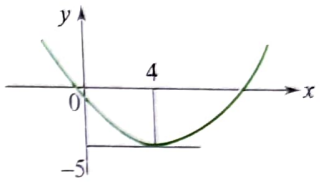
$a > 0$

$-\frac{b}{2a} = 4$ or $-\frac{b}{2a} > 0$

$\therefore \quad b < 0$

$f(0) = c < 0$

Also, $-\frac{b}{2a} = 4$ or $8a + b = 0$



24. (1), (2), (3)

$f(x) = Ax^2 + Bx + C$

$A = a + b - 2c = (a - c) + (b - c) > 0$

$\Rightarrow \quad A > 0$

Hence, the graph is concave upwards. Also, $x = 1$ is obvious solution; therefore, both roots are rational.

$b + c - 2a = \underbrace{(b-a)}_{-ve} + \underbrace{(c-a)}_{-ve} < 0$

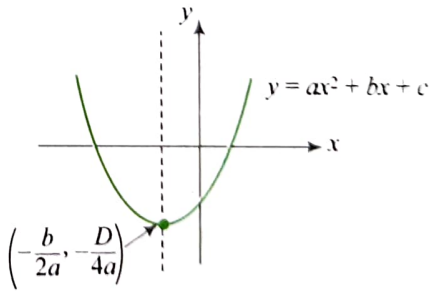
$\Rightarrow \quad B < 0$

$\therefore \quad -\frac{B}{2A} > 0$

Hence, abscissa of the vertex is positive. Option (d) need not be correct as with $a = 5, b = 4, c = 2, P < 0$ and with $a = 6, b = 3, c = 2, P > 0$.

25. (1), (2), (3), (4)

Let $f(x) = ax^2 + bx + c$.



From the diagram, we can see that $a > 0$, $c < 0$ and $-[b(2a)] < 0$. Hence, $b > 0$.

$\therefore \quad a + b - c > 0$

26. (1), (3), (4)

$OR = 5$.

$\therefore \quad f(0) = -5 = c$

$PQ = 9$.

$\therefore \quad -\frac{D}{4a} = -9$

$$\Rightarrow \quad \frac{b^2+20a}{4a}=9$$

$$\Rightarrow \quad b^2=16a \qquad \ldots(1)$$

$OB = 2.5$

So, one root of equation is 2.5.

$$\therefore \quad 25a+10b-20=0$$

or $\quad 5a+2b-4=0 \qquad \ldots(2)$

From (1) and (2),

$$5b^2+32b-64=0$$

$$\Rightarrow \quad (5b-8)(b+8)=0$$

$$\Rightarrow \quad b=-8, \frac{8}{5} \text{ (Not possible. Since } a>0, \text{ we must have } b<0)$$

$$\therefore \quad a=4$$

$$\Rightarrow \quad y=4x^2-8x-5=4x^2-10x+2x-5=(2x-5)(2x+1)$$

So, $x = 2.5, -0.5$ are the roots of $y(x) = 0$.

Clearly, $y(-1) > 0$.

$$y \geq 7$$

$$\Rightarrow \quad 4x^2-8x-5 \geq 7$$

$$\Rightarrow \quad x^2-2x-3 \geq 0$$

$$\Rightarrow \quad (x-3)(x+1) \geq 0$$

$$\Rightarrow \quad x \leq -1 \text{ or } x \geq 3$$

$$ax^2+bx+c=mx$$

$$\Rightarrow \quad 4x^2-(8+m)x-5=0$$

Clearly, above equation has two distinct real roots for any real value of m.

27. (1), (2), (3)

$$f(x)=ax^2+bx+c$$

$$f(0)=c<0, D>0$$

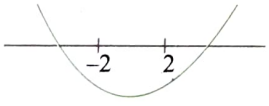
$$\Rightarrow \quad b^2-4ac>0$$

$$f(1)<0 \text{ and } f(-1)<0$$

$$\Rightarrow \quad a-|b|+c<0$$

$$f(2)<0 \text{ and } f(-2)<0$$

$$\Rightarrow \quad 4a-2|b|+c<0$$



Nothing can be said about $f(3)$ or $f(-3)$, whether it is positive or negative.

28. (1), (2), (3)

Since the roots of $ax^2+bx+c=0$ are nonreal, so, $f(x)=ax^2+bx+c$ will have same sign for every value of x. Hence,

$$f(0)=c, f(1)=a+b+c, f(-1)=a-b+c$$

$$f(-2)=4a-2b+c$$

$$\Rightarrow \quad c(a+b+c)>0, c(a-b+c)>0, c(4a-2b+c)>0$$

29. (3), (4)

$$\cos x-y^2-\sqrt{y-x^2-1} \geq 0 \qquad (1)$$

Now, $\sqrt{y-x^2-1}$ is defined when $y-x^2-1 \geq 0$ or $y \geq x^2+1$.

So minimum value of y is 1. From (1),

$$\cos x-y^2 \geq \sqrt{y-x^2-1}$$

where $\cos x - y^2 \leq 0$ [as when $\cos x$ is maximum (=1) and y^2 is minimum (=1), so $\cos x - y^2$ is maximum]. Also,

$$\sqrt{y-x^2-1} \geq 0$$

Hence,

$$\cos x-y^2=\sqrt{y-x^2-1}=0$$

$$\Rightarrow \quad y=1 \text{ and } \cos x=1, y=x^2+1$$

$$\Rightarrow \quad x=0, y=1$$

30. (3), (4)

We have,

$$D=(b-c)^2-4a(a-b-c)>0$$

or $\quad b^2+c^2-2bc-4a^2+4ab+4ac>0$

or $\quad c^2+(4a-2b)c-4a^2+4ab+b^2>0$ for all $c \in R$

Discriminant of the above expression in c must be negative. Hence,

$$(4a-2b)^2-4(-4a^2+4ab+b^2)<0$$

or $\quad 4a^2-4ab+b^2+4a^2-4ab-b^2<0$

or $\quad a(a-b)<0$

$$\Rightarrow \quad a<0 \text{ and } a-b>0 \text{ or } a>0 \text{ and } a-b<0$$

$$\Rightarrow \quad b<a<0 \text{ or } b>a>0$$

31. (1), (4)

Let $\quad \frac{x^2+ax+3}{x^2+x+a}=y$

$$\Rightarrow \quad x^2(1-y)-x(y-a)+3-ay=0$$

$$\because \quad x \in R$$

$$(y-a)^2-4(1-y)(3-ay) \geq 0$$

$$\Rightarrow \quad (1-4a)y^2+(2a+12)y+a^2-12 \geq 0 \qquad (1)$$

Now, (1) is true for all $y \in R$, if $1-4a>0$ and $D \leq 0$. Hence,

$$a<\frac{1}{4} \text{ and } 4(a+6)^2-4(a^2-12)(1-4a) \leq 0$$

$$\Rightarrow \quad a<\frac{1}{4} \text{ and } 4a^3-36a+48 \leq 0$$

$$\Rightarrow \quad a<\frac{1}{4} \text{ and } 4a^3 \leq 36a-48$$

$$\Rightarrow \quad 4a^3<36\left(\frac{1}{4}\right)-48$$

$$\Rightarrow \quad 4a^3+39<0 \qquad \left[\because a<\frac{1}{4}\right]$$

32. (1), (2), (3)

$$y \in [0, 1]$$

where $y=\frac{x+1}{k+x^2}$

$y = 0$ when $x = -1$

If $y \neq 0, yx^2-x+ky-1=0; x \in R$

$$\Rightarrow \quad D \geq 0$$

$$\Rightarrow \quad 1-4y(ky-1) \geq 0$$

$$\Rightarrow \quad 4ky^2-4y-1 \leq 0$$

$$\forall y \in [0, 1] \qquad \text{(Given)}$$

$$f(0) \leq 0 \text{ and } f(1) \leq 0$$

$$\Rightarrow \quad -1 \leq 0 \text{ and } 4k-4-1 \leq 0$$

$$\Rightarrow \quad k \leq \frac{5}{4}$$

$$\therefore \quad k \in \left(-\infty, \frac{5}{4}\right]$$

Also, $k > 0$, otherwise graph of $4ky^2 - 4y - 1$ is concave downward and is negative for unbounded interval of values of y.

Hence, $k \in \left[0, \frac{5}{4}\right]$.

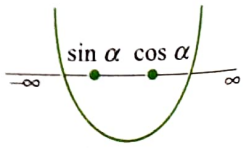
33. (3), (4)

Let, $f(x) = (x - \sin\alpha)(x - \cos\alpha) - 2$.

Then, $f(\sin\alpha) = -2 < 0$ and

$f(\cos\alpha) = -2 < 0$.

So, $\sin\alpha$ and $\cos\alpha$ lie between the roots.

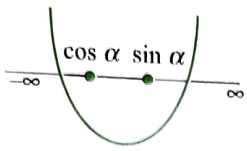


For $0 < \alpha < \frac{\pi}{4}$, $\sin\alpha < \cos\alpha$

Therefore, equation $f(x) = 0$ has one root in $(-\infty, \sin\alpha)$ and other in $(\cos\alpha, \infty)$.

Also, for $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$,

$\cos\alpha < \sin\alpha$



Therefore, equation $f(x) = 0$ has one root in $(-\infty, \cos\alpha)$ and other in $(\sin\alpha, \infty)$.

34. **(1), (3), (4)**

Let the roots be a/r, a, ar, where $a > 0$, $r > 1$. Now,

$a/r + a + ar = -p$ (1)

$a(a/r) + a(ar) + (ar)(a/r) = q$ (2)

$(a/r)(a)(ar) = 1$ (3)

$\Rightarrow a^3 = 1$

$\Rightarrow a = 1$

Hence, (3) is correct. From (1), putting $a = 1$, we get

$-p - 3 > 0$ $\left(\because r + \frac{1}{r} > 2\right)$

$\Rightarrow p < -3$

Hence, (2) is not correct. Also,

$1/r + 1 + r = -p$ (4)

From (2), putting $a = 1$, we get

$1/r + r + 1 = q$ (5)

From (4) and (5), we have

$-p = q \Rightarrow p + q = 0$

Hence, (1) is correct. Now, as $r > 1$

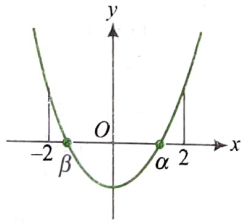
$a/r = 1/r < 1$

and $ar = r > 1$

Hence, (4) is correct.

35. **(1), (2), (3)**

From the given conditions we have following possible graph of $y = f(x)$



From the graph,

$[\alpha] = 0$ or 1, $[\beta] = -2$ or -1

$\therefore$ $[\alpha] + [\beta]$ can be $-1, 0, -2$

36. **(1), (2), (3)**

$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$$

or $$\left(\frac{x}{x+1} + \frac{x}{x-1}\right)^2 - 2\left(\frac{x}{x+1}\right)\left(\frac{x}{x-1}\right) = a(a-1)$$

or $$\left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} - a(a-1) = 0$$

or $t^2 - t - a(a-1) = 0$ where $t = \frac{2x^2}{x^2-1}$

$\Rightarrow t = a$ or $1 - a$

$\Rightarrow \frac{2x^2}{x^2-1} = a$ or $\frac{2x^2}{x^2-1} = 1 - a$

$\Rightarrow x = \pm\sqrt{\frac{a}{a-2}}$ or $x = \pm\sqrt{\frac{a-1}{a+1}}$

When $a < -1 \Rightarrow$ all roots are real

$1 < a < 2 \Rightarrow x = \pm\sqrt{\frac{a}{2-a}}\, i, \pm\sqrt{\frac{a-1}{a+1}}$

$\Rightarrow$ two real roots

When $a > 2 \Rightarrow$ all roots are real

37. **(1), (4)**

Equations $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root.

$(bc - 1)^2 = (c - b^2)(b - c^2)$

$\Rightarrow b^2c^2 + 1 - 2bc = bc - b^3 - c^3 + b^2c^2$

$\Rightarrow 1 + c^3 + b^3 = 3bc$

$\Rightarrow (1 + c + b)(c^2 + b^2 + 1 - b - c - bc) = 0$

$\Rightarrow b + c + 1 = 0$ and $c^2 + b^2 + 1 = c + b + bc$

38. **(1), (2)**

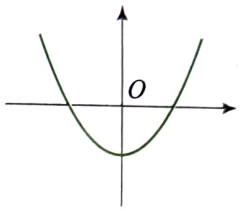
Let $\cot x = t$.

Since $x \in (0, \pi/2)$, $t \in (0, \infty)$

Therefore, $f(t) = t^2 + (k + 1)t - (k - 3) < 0$ for at least one $t \in (0, \infty)$.

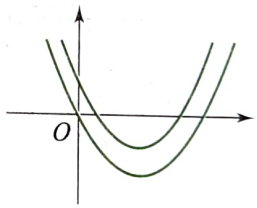
We have following possibilities for the graph of $f(t)$:

Case I:



$f(0) < 0 \Rightarrow k > 3$

Case II:



$D > 0$

$\Rightarrow (k + 1)^2 + 4(k - 3) > 0$

$\Rightarrow k^2 + 6k - 11 > 0$

$\Rightarrow (k + 3)^2 > 20$

$\Rightarrow k + 3 < -2\sqrt{5}$ or $k + 3 > 2\sqrt{5}$

$\Rightarrow k < -3 - 2\sqrt{5}$ or $k > -3 + 2\sqrt{5}$ ...(1)

$\frac{-b}{2a} > 0$

$\Rightarrow -(k + 1) > 0$

$\Rightarrow k < -1$ ...(2)

$f(0) \geq 0$

$\Rightarrow -k + 3 \geq 0$

$\Rightarrow k \leq 3$ ...(3)

From (1), (2) and (3), we get

$k \in (-\infty, -3 - 2\sqrt{5})$

So, from both cases, $k \in (-\infty, 3 - 2\sqrt{5}) \cup (3, \infty)$

Linked Comprehension Type

1. **(4), 2. (3), 3. (3)**

Let unknown polynomial be $P(x)$. Let $Q(x)$ and $R(x)$ be the quotient and remainder, respectively, when it is divided by $(x - 3)(x - 4)$. Then,

$P(x) = (x - 3)(x - 4)Q(x) + R(x)$

Then, we have

$R(x) = ax + b$

$\Rightarrow \quad P(x) = (x-3)(x-4)Q(x) + ax + b$

Given that $P(3) = 2$ and $P(4) = 1$. Hence,

$3a + b = 2$ and $4a + b = 1$

$\Rightarrow \quad a = -1$ and $b = 5$

$\Rightarrow \quad R(x) = 5 - x$

1. (4) $5 - x = x^2 + ax + 1$

$\Rightarrow \quad x^2 + (a+1)x - 4 = 0$

Given that roots are real and distinct. Therefore,

$D > 0 \Rightarrow (a+1)^2 + 16 > 0$

which is true for all real a.

2. (3) $-x + 5 = px^2 + (q-1)x + 6$

$\Rightarrow \quad px^2 + qx + 1 = 0$

Now, $p > 0$ and equation has no distinct real roots or equation has real and equal or imaginary roots. Then,

$px^2 + qx + 1 \geq 0, \forall\, x \in R$

$\Rightarrow \quad f(3) \geq 0 \Rightarrow 9p + 3q + 1 \geq 0 \Rightarrow 3p + q \geq -1/3$

Hence, the least value of $3p + q$ is $-1/3$.

3. (3) $f(x) = y = \dfrac{-x+5}{x^2 - 3x + 2}$

$\Rightarrow \quad yx^2 + (1 - 3y)x + 2y - 5 = 0$

Now, x is real, then

$D \geq 0$

$\Rightarrow \quad (1-3y)^2 - 4y(2y-5) \geq 0$

or $\quad y^2 + 14y + 1 \geq 0$

or $\quad y \in \left(-\infty, \dfrac{-14-\sqrt{192}}{2}\right] \cup \left[\dfrac{-14+\sqrt{192}}{2}, \infty\right)$

$(-\infty, -7-4\sqrt{3}] \cup [-7+4\sqrt{3}, \infty)$

4. (1), 5. (4), 6. (3)

$(\beta - \alpha) = ((\beta + h) - (\alpha + h))$

$(\beta + \alpha)^2 - 4\alpha\beta = [(\beta + h) + (\alpha + h)]^2 - 4(\beta + h)(\alpha + h)$

$(-b_1)^2 - 4c_1 = (-b_2)^2 - 4c_2$

$D_1 = D_2$

The least value of $f(x)$ is

$-\dfrac{D_1}{4} = -\dfrac{1}{4} \Rightarrow D_1 = 1$ and $D_2 = 1$

Therefore, the least value of

$g(x)$ is $-\dfrac{D_2}{4} = -\dfrac{1}{4}$

The least value of $g(x)$ occurs at

$-\dfrac{b_2}{2} = \dfrac{7}{2}$ or $b_2 = -7$

Now, $b_2^2 - 4c_2 = D_2$

or $\quad 49 - 4c_2 = 1$ or $\dfrac{48}{4} = c_2$ or $c_2 = 12$

$\Rightarrow \quad x^2 - 7x + 12 = 0$ or $x = 3, 4$

7. (4) $\because AC = 4\sqrt{2}$

$\therefore \quad AB = BC = \dfrac{4\sqrt{2}}{\sqrt{2}} = 4$ units

$OB = \sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2}$

$\therefore \quad A(-2\sqrt{2}, 0), C(2\sqrt{2}, 0), B(0, -2\sqrt{2})$

Since $y = ax^2 + bx + c$ passes through A, B, and C, we get

$y = \dfrac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

8. (3) Minimum value of $y = x^2/(2\sqrt{2}) - 2\sqrt{2}$ is $-2\sqrt{2}$ at $x = 0$.

9. (3) $f(x) = 0$

$\Rightarrow \quad x^2/(2\sqrt{2}) - 2\sqrt{2} = 0$ or $x = \pm 2\sqrt{2}$

Therefore, number of integral values of k for which k lies in $(-2\sqrt{2}, 2\sqrt{2})$ is 5.

10. (2), 11. (4), 12. (1)

$f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ represents a parabola whose vertex is at $\left(\dfrac{a}{2}, 2 - 2a\right)$.

Case I: $0 < \dfrac{a}{2} < 2$

In this case, $f(x)$ will attain the minimum value at $x = \dfrac{a}{2}$.

Thus, $f\left(\dfrac{a}{2}\right) = 3$.

$\Rightarrow \quad 3 = -2a + 2$

$\Rightarrow \quad a = -\dfrac{1}{2}$ (Rejected)

Case II: $\dfrac{a}{2} \geq 2$

In this case, $f(x)$ attains the global minimum value at $x = 2$.

Thus, $f(2) = 3$.

$\Rightarrow \quad 3 = 16 - 8a + a^2 - 2a + 2$

$\Rightarrow \quad a = 5 \pm \sqrt{10}$

So, $\quad a = 5 + \sqrt{10}$

Case III: $\dfrac{a}{2} \leq 0$

In this case, $f(x)$ attains the global minimum value at $x = 0$.

Thus, $f(0) = 3$.

$\Rightarrow \quad 3 = a^2 - 2a + 2$

$\Rightarrow \quad a = 1 \pm \sqrt{2}$.

So, $\quad a = 1 - \sqrt{2}$.

Hence, permissible values of a are $1 - \sqrt{2}$ and $5 + \sqrt{10}$.

$f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ is monotonic in $[0, 2]$.

Hence, point of minima of function should not lie in $[0, 2]$.

Now, $f'(x) = 0$

$\Rightarrow \quad 8x - 4a = 0$

$\Rightarrow \quad x = a/2$

If $\dfrac{a}{2} \in [0, 2]$, then $a \in [0, 4]$.

For $f(x)$ to be monotonic in $[0, 2]$, $a \notin [0, 4]$. So, $a \leq 0$ or $a \geq 4$.

13. (1), 14. (4), 15. (3)

Given equation is $2 + |x^2 + 4x + 3| = m$

$\therefore \quad |x^2 + 4x + 3| = m - 2$ (1)

Given equation is meaningful if $m \geq 2$

$\therefore \quad x^2 + 4x + 3 = \pm (m - 2)$

$\therefore \quad x^2 + 4x + 5 - m = 0$ or $x^2 + 4x + 1 + m = 0$

$\therefore \quad x = \dfrac{-4 \pm \sqrt{16 - 4(5-m)}}{2}$ or $x = \dfrac{-4 \pm \sqrt{16 - 4(1+m)}}{2}$

$\therefore \quad x = -2 \pm \sqrt{m-1}$ or $x = -2 \pm \sqrt{3-m}$

For four distinct solutions $m-1 \geq 0$ and $3-m \geq 0$

$\therefore \quad m \in (2, 3)$ [using (1)]

For two solutions $m-1 \geq 0$ and $3-m \leq 0 \Rightarrow m \geq 3$

or $m-1 \leq 0$ and $3-m \geq 0$, not possible

or $m = 2$ [using (1)]

$\therefore \quad m \in \{2\} \cup (3, \infty)$

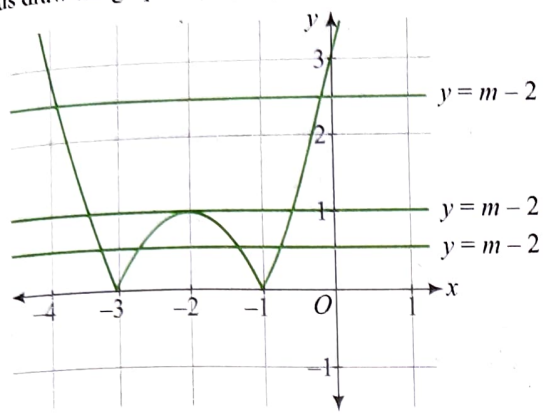
From three solutions, exactly one equation must give equal roots.

$\therefore \quad m = 1$ and $3-m \geq 0 \Rightarrow$ no values of m (as $m \geq 2$)

or $m = 3$ and $m \geq 1 \Rightarrow m = 3$

Alternate method:

Let us draw the graph of $f(x) = |x^2 + 4x + 3|$



Now equation is $|(x+1)(x+3)| = m-2$

From the graph

(1) for three solutions $m-2 = 1 \Rightarrow m = 3$

(2) for four solutions

$0 < m-2 < 1 \quad \Rightarrow \quad 2 < m < 3$

(3) for only two solutions

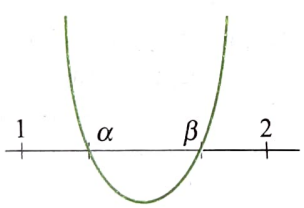
$m-2 = 0$ or $m-2 > 1$

$\Rightarrow \quad m = 2$ or $m > 3$

16. (4), 17. (2), 18. (2)

$$ax^2 - bx + c = 0$$

Let $f(x) = ax^2 - bx + c$ be the corresponding quadratic expression and α, β be the roots of $f(x) = 0$.



Then, $f(x) = a(x-\alpha)(x-\beta)$

Now, $af(1) > 0, af(2) > 0, 1 < \dfrac{b}{2a} < 2, b^2 - 4ac > 0$

$\Rightarrow \quad a(1-\alpha)(1-\beta) > 0, a(2-\alpha)(2-\beta) > 0, 2a < b < 4a,$
$b^2 - 4ac > 0$

$\Rightarrow \quad a^2(1-\alpha)(1-\beta)(2-\alpha)(2-\beta) > 0$

$\Rightarrow \quad a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta) > 0$

As $f(1)$ and $f(2)$ both are integers and $f(1) > 0$, and $f(2) > 0$, so

$f(1)\ f(2) > 0$

$\Rightarrow \quad f(1)\ f(2) \geq 1$

$\Rightarrow \quad 1 \leq a^2(\alpha-1)(2-\alpha)(\beta-1)(2-\beta)$ (1)

Now, $\dfrac{(\alpha-1)+(2-\alpha)}{2} \geq ((\alpha-1)(2-\alpha))^{1/2}$

$\Rightarrow \quad (\alpha-1)(2-\alpha) \leq \dfrac{1}{4}$

Similarly,

$$(\beta-1)(2-\beta) \leq \frac{1}{4}$$

$\Rightarrow \quad (\alpha-1)(2-\alpha)(\beta-1)(2-\beta) < \dfrac{1}{16}$ (Using (1))

As $\alpha \neq \beta$, so $a^2 > 1$

$\Rightarrow \quad a \geq 5$

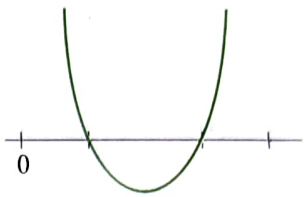
$\Rightarrow \quad b^2 > 20c$ and $b > 10 \Rightarrow b \geq 11$

Also, $b^2 > 100 \Rightarrow c > 5 \Rightarrow c \geq 6$

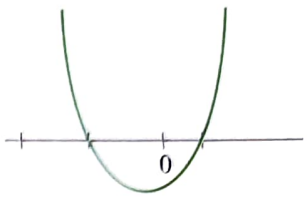
19. (4), 20. (3), 21. (1)

19. (4) Let $f(x) = x^2 + x + a - 9$.

$x^2 + x + a - 9 < 0$ has at least one positive solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-negative or 0 lies between the roots.



(i)

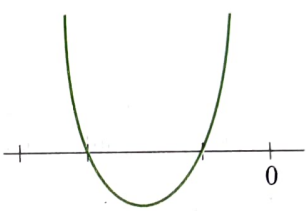


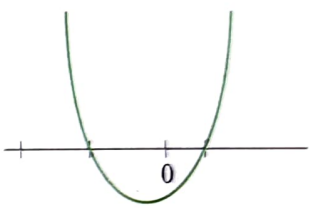
(ii)

Now sum of roots $= -\dfrac{1}{2}$; hence, case I is not possible. For case II,

$f(0) < 0 \Rightarrow a - 9 < 0$ or $a < 9$

20. (3)





If $x^2 + x + a - 9 < 0$ has at least one negative solution, then either both the roots of equation $x^2 + x + a - 9 = 0$ are non-positive or 0 lies between the roots.

For case I, sum of roots is $-1/2 < 0$. Product of roots is $a - 9 > 0$

$\Rightarrow \quad a \geq 9$ and

$D > 0$

$\Rightarrow \quad 1 - 4(a-9) > 0 \quad \Rightarrow a < \dfrac{37}{4}$

Hence, $9 \leq a < 37/4$.

For case II, $f(0) < 0$

$\Rightarrow \quad a < 9 \quad \Rightarrow a \in \left(-\infty, \dfrac{37}{4}\right)$

21. (1) If $x^2 + x + a - 9 < 0$ is true $\forall x \in (-1, 3)$, then $f(-1) \leq 0$ and $f(3) \leq 0$.

$\therefore \quad 1 - 1 + a - 9 \leq 0$ and $9 + 3 + a - 9 \leq 0$

$\Rightarrow \quad a \leq 9$ and $a \leq -3$

$\Rightarrow \quad a \leq -3$

22. (4), 23. (3), 24. (2)

Given that $9^x - a3^x - a + 3 \leq 0$

Let $t = 3^x$. Then,

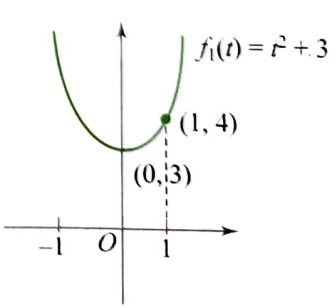
$t^2 - at - a + 3 \leq 0$

or $t^2 + 3 \leq a(t+1)$ (1)

where $t \in R^+, \forall x \in R$

Let $f_1(t) = t^2 + 3$ and

$f_2(t) = a(t+1)$.



22. (4) For $x < 0$, $t \in (0, 1)$. That means (1) should have at least one solution in $t \in (0, 1)$. From (1), it is obvious that $a \in R^+$. Now $f_2(t) = a(t+1)$ represents a straight line. It should meet the

curve $f_1(t) = t^2 + 3$, at least once in $t \in (0, 1)$.

$f_1(0) = 3, f_1(1) = 4, f_2(0) = a, f_2(1) = 2a$

If $f_1(0) = f_2(0)$, Then $a = 3$; if $f_1(1) = f_2(1)$, then $a = 2$. Hence, the required range is $a \in (2, 3)$.

23. (3) For at least one positive solution, $t \in (1, \infty)$. That means graphs of $f_1(t) = t^2 + 3$ and $f_2(t) = a(t + 1)$ should meet at least once in $t \in (1, \infty)$. If $a = 2$, both the curves touch each other at $(1, 4)$. Hence, the required range is $a \in (2, \infty)$.

24. (2) In this case both graphs should meet at least once in $t \in (0, \infty)$. For $a = 2$ both the curves touch, hence, the required range is $a \in [2, \infty)$.

25. (2) $b^2 > (a + c)^2$

$\Rightarrow \quad (a + c - b)(a + c + b) < 0$

$\Rightarrow \quad f(-1) f(1) < 0$

So, there is exactly one root in $(-1, 1)$.

26. (1) $af(1) < 0$ and $f(0) f(1) > 0$

$\Rightarrow \quad af(1) < 0$ and $af(0) < 0$

Hence, both the numbers 0 and 1 lie between the roots.

27. (2) $f(0) f(1) < 0$ and $af(1) > 0$

$\Rightarrow \quad f(0) f(1) < 0$ and $af(0) < 0$

Hence, exactly one root lies in $(0, 1)$ and 0 lies between the roots.

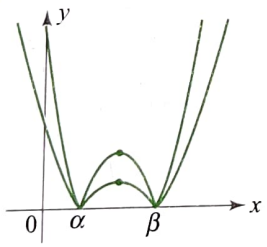
28. (2) 29. (1)

$$|px^2 + qx + r| \le |Px^2 + Qx + R| \ \forall\ x \in R \qquad (1)$$

From the graph we can see that this is possible only when both equations have same roots.

Thus, α and β are roots of $Px^2 + Qx + R = 0$ and also of

$$px^2 + qx + r = 0$$



So from (1),

$$|p|\,|x - \alpha|\,|x - \beta| \le |P|\,|x - \alpha|\,|x - \beta|$$

$\Rightarrow \quad |p| \le |P|$

Also $\left|\dfrac{4pr - q^2}{4p}\right| \le \left|\dfrac{4PR - Q^2}{4P}\right|$

$\Rightarrow \quad |d| \le \left|\dfrac{p}{P}\right| |D|$

$\Rightarrow \quad |d| \le |D| \qquad (\because |p| \le |P|)$

30. (3) 31. (1), 32. (3)

Given equation is

$$x^4 + 2ax^3 + x^2 + 2ax + 1 = 0 \qquad (1)$$

or $\left(x^2 + \dfrac{1}{x^2}\right) + 2a\left(x + \dfrac{1}{x}\right) + 1 = 0$

or $\left(x + \dfrac{1}{x}\right)^2 + 2a\left(x + \dfrac{1}{x}\right) - 1 = 0$

$\Rightarrow \quad t^2 + 2at - 1 = 0 \qquad (2)$

where $t = x + (1/x)$. Now,

$$\left(x + \frac{1}{x}\right) \ge 2 \quad \text{or} \quad \left(x + \frac{1}{x}\right) \le -2$$

$\therefore \quad t \ge 2$ or $t \le -2$

30. (3) Now, Eq. (1) will have at least two positive roots, when at least one root of Eq. (2) will be greater than 2. From Eq. (2),

$$D = 4a^2 - 4(-1) = 4(1 + a^2) > 0, \ \forall\ a \in R \qquad (3)$$

Let the roots of Eq. (2) be α, β. If $\alpha, \beta \le 2$, then

$\Rightarrow \quad f(2) \ge 0$ and $\dfrac{-B}{2A} < 2$

$\Rightarrow \quad 4 + 4a - 1 \ge 0$ and $-\dfrac{2a}{2} < 2$

$\Rightarrow \quad a \ge -\dfrac{3}{4}$ and $a > -2$

$\Rightarrow \quad a \ge -\dfrac{3}{4}$

Therefore, at least one root will be greater than 2. Then,

$$a < -\frac{3}{4} \qquad (4)$$

Combining (3) and (4), we get

$$a < -\frac{3}{4}$$

Hence, at least one root will be positive if $a \in (-\infty, -3/4)$.

31. (1) Now, Eq. (1) will have at least two roots negative, when at least one root of Eq. (2) will be less than -2. If $\alpha, \beta \ge -2$, then

$$f(-2) \ge 0 \text{ and } -\frac{B}{2A} > -2$$

$\therefore \quad 4 - 4a - 1 \ge 0$ and $-\dfrac{2a}{2} > -2$

$\therefore \quad a \le \dfrac{3}{4}$ and $a < 2$

$\therefore \quad a \le \dfrac{3}{4} \qquad (5)$

Combining (3) and (5), at least one root will be less than -2 for Eq. (2) if

$$a > \frac{3}{4}$$

$\therefore \quad a \in \left(\dfrac{3}{4}, \ \infty\right)$

32. (3) If exactly two roots are positive, then other two roots are negative. Then -2 and 2 must lie between the roots. So,

$f(-2) < 0$ and $f(2) < 0$

$\Rightarrow \quad a > 3/4$ and $a < -3/4$

Hence, no such values of a exist.

33. (1) 34. (2)

From the question, the real roots of $x^3 - x^2 + \beta x + \gamma = 0$ are x_1, x_2, x_3 and they are in A.P. As x_1, x_2, x_3 are in A.P., let $x_1 = a - d$, $x_2 = a$, $x_3 = a + d$. Now,

$$x_1 + x_2 + x_3 = -\frac{-1}{1} = 1$$

$\Rightarrow \quad a - d + a + a + d = 1$

$\Rightarrow \quad a = \dfrac{1}{3} \qquad (1)$

$$x_1x_2 + x_2x_3 + x_3x_1 = \frac{\beta}{1} = \beta$$

$\Rightarrow \quad (a - d)\,a + a\,(a + d) + (a + d)(a - d) = \beta \qquad (2)$

$$x_1x_2x_3 = -\frac{\gamma}{1} = -\gamma$$

$\Rightarrow \quad (a - d)\,a\,(a + d) = -\gamma \qquad (3)$

From (1) and (2), we get

$$3a^2 - d^2 = \beta$$

$\Rightarrow \quad 3\dfrac{1}{9} - d^2 = \beta$, so $\beta = \dfrac{1}{3} - d^2 < \dfrac{1}{3}$

From (1) and (3), we get

$$\frac{1}{3}\left(\frac{1}{9}-d^2\right)=-\gamma$$

$$\Rightarrow \quad \gamma=\frac{1}{3}\left(d^2-\frac{1}{9}\right)>\frac{1}{3}\left(-\frac{1}{9}\right)=-\frac{1}{27}$$

$$\gamma\in\left(-\frac{1}{27},+\infty\right)$$

35. (3) $x^4-\lambda x^2+9=0$.

Let $x^2=t\geq 0$

$\therefore \quad f(t)=t^2-\lambda t+9=0$ (1)

If given equation has four real and distinct roots, then both roots of (1) must be positive

$\therefore \quad D>0\Rightarrow \lambda^2-36>0$ (2)

$$\frac{-b}{2a}>0 \quad \Rightarrow \quad \frac{\lambda}{2}>0 \text{ or } \lambda>0 \qquad (3)$$

$f(0)>0$ or $9>0$ (4)

$\therefore \quad \lambda\in(6,\infty)$ [from (2), (3), (4)]

36. (2) If equation has no real roots, then (1) must have both roots negative for which

$$\frac{-b}{2a}<0\Rightarrow\lambda<0 \qquad (5)$$

$f(0)>0\Rightarrow 9>0$

$\therefore \quad \lambda\in(-\infty,-6)$ [from (2), (3), (5)]

Also for no real roots we can have $D<0$

$\Rightarrow \quad \lambda^2-36<0\Rightarrow\lambda\in(-6,6)$

Hence, $\lambda\in(-\infty,6)$

37. (4) From above discussion we have for $\lambda\in(-\infty,6)\cup(6,\infty)$ equation has either four distinct real roots or no real roots.

For $\lambda=6$, both roots are $\pm\sqrt{3}$

Alternate method:

$$x^4-\lambda x^2+9=0$$

or $$x^2+\frac{9}{x^2}=\lambda$$

Now $$\left(x-\frac{3}{x}\right)^2=\lambda-6$$

For above equation

(i) four distinct real roots if $\lambda-6>0$ or $\lambda>6$.

(ii) no real roots if $\lambda-6<0$ or $\lambda<6$

(iii) real and equal repeating roots if $\lambda=6$

Matrix Match Type

1. **a → s; b → r; c → q; d → p**

Obviously when $a\geq 0$, we have no roots as all the terms are followed by +ve sign. Also for $a=-2$, we have

$$x^2-2|x|+1=0$$

or $\quad |x|-1=0\Rightarrow x=\pm 1$

Hence, the equation has two roots.

Also when $a<-2$, for given equation

$$|x|=\frac{-a\pm\sqrt{a^2-4}}{2}>0$$

Hence, the equation has four roots as $|-a|>\sqrt{a^2-4}$. Obviously, the equation has no three real roots for any value of a.

2. **a → r; b → r; c → q; d → p**

(a) $(m-2)x^2-(8-2m)x-(8-3m)=0$ has roots of opposite signs. The product of roots is

$$-\frac{8-3m}{m-2}<0$$

or $$\frac{3m-8}{m-2}<0$$

or $\quad 2<m<8/3$

(b) Exactly one root of equation $x^2-m(2x-8)-15=0$ lies in interval $(0, 1)$.

$$f(0)f(1)<0$$

$\Rightarrow \quad (0-m(-8)-14)(1-m(-6)-15)<0$

$\Rightarrow \quad (8m-15)(6m-14)<0$

$\Rightarrow \quad 15/8<m<7/3$

(c) $x^2+2(m+1)x+9m-5=0$ has both roots negative. Hence, sum of roots is

$$-2(m+1)<0 \text{ or } m>-1 \qquad (1)$$

Product of roots is

$$9m-5>0\Rightarrow m>5/9 \qquad (2)$$

Discriminant,

$$D\geq 0\Rightarrow 4(m+1)^2-4(9m-5)\geq 0$$

$\Rightarrow \quad m^2-7m+6\geq 0$

$\Rightarrow \quad m\leq 1 \text{ or } m\geq 6$ (3)

Hence, for (1), (2), and (3), we get

$$m\in\left(\frac{5}{9},1\right]\cup[6,\infty)$$

(d) $f(x)=x^2+2(m-1)x+m+5=0$ has one root less than 1 and the other root greater than 1. Hence,

$$f(1)<0$$

$\Rightarrow \quad 1+2(m-1)+m+5<0$

$\Rightarrow \quad m<-4/3$

3. **a → s; b → p; c → q; d → r**

(a) $x^2+ax+b=0$ has root α. Hence,

$$\alpha^2+a\alpha+b=0 \qquad (1)$$

$x^2+px+q=0$ has root $-\alpha$. Hence,

$$\alpha^2-p\alpha+q=0 \qquad (2)$$

Eliminating α from (1) and (2), we get

$$(q-b)^2=(aq+bp)(-p-a)$$

or $$(q-b)^2=-(aq+bp)(p+a)$$

(b) $x^2+ax+b=0$ has root α. Hence,

$$\alpha^2+a\alpha+b=0 \qquad (1)$$

$x^2+px+q=0$ has root $1/\alpha$. Hence,

$$q\alpha^2+p\alpha+1=0 \qquad (2)$$

Eliminating α from (1) and (2), we get

$$(1-bq)^2=(a-pb)(p-aq)$$

(c) $x^2+ax+b=0$ has roots α, β. Hence,

$$\alpha^2+a\alpha+b=0 \qquad (1)$$

$x^2+px+q=0$ has roots $-2/\alpha, \gamma$. Hence,

$$q\alpha^2-2p\alpha+4=0 \qquad (2)$$

Eliminating α from (1) and (2), we get

$$(4-bq)^2=(4a+2pb)(-2p-aq)$$

(d) $x^2+ax+b=0$ has roots α, β. Hence,

$$\alpha^2+a\alpha+b=0 \qquad (1)$$

$x^2+px+q=0$ has roots $-1/2\alpha, \gamma$. Hence,

$$4q\alpha^2-2p\alpha+1=0 \qquad (2)$$

Eliminating α from (1) and (2), we get

$$(1-4bq)^2 = (a+2bp)(-2p-4aq)$$

4. (1) $(x^2+x)^2 + a(x^2+x) + 4 = 0$

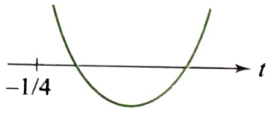
Let $t = x^2 + x = (x+1/2)^2 - 1/4$.

$\Rightarrow \quad t \in \left[-\frac{1}{4}, \infty\right)$

Now, $f(t) = t^2 + at + 4 = 0$ ...(1)

(a) All four real and distinct roots.

So, equation (1) has both roots greater than $-1/4$.



Following conditions are required:

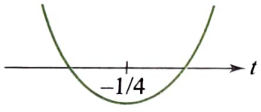
(i) $D > 0 \Rightarrow a^2 - 16 > 0 \Rightarrow |a| > 4$

(ii) $f(-1/4) = \frac{1}{16} - \frac{a}{4} + 4 > 0 \Rightarrow a < 65/4$

(iii) $-\frac{B}{2A} = -\frac{a}{2} > -\frac{1}{4} \Rightarrow a < \frac{1}{2}$

$\Rightarrow \quad a \in (-\infty, -4)$

(b) Two real roots which are distinct.



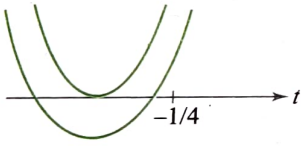
$\Rightarrow \quad f(-1/4) < 0$

$\Rightarrow \quad a > 65/4$

$\Rightarrow \quad a \in (65/4, \infty)$

(c) All four roots are imaginary.

Case I:



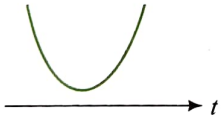
(i) $D \geq 0 \quad \Rightarrow |a| \geq 4$

(ii) $f(-1/4) > 0 \quad \Rightarrow a < \frac{65}{4}$

(iii) $-\frac{B}{2A} < -\frac{1}{4} \quad \Rightarrow a > \frac{1}{2} \quad \Rightarrow a \in \left[4, \frac{65}{4}\right)$

Case II:

$D < 0$

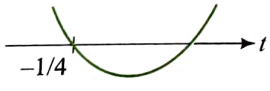


$\Rightarrow \quad a \in (-4, 4)$ ...(2)

From case I and case II,

$$a \in \left(-4, \frac{65}{4}\right)$$

(d) Four real roots in which two are equal.



(i) $D > 0 \quad \Rightarrow |a| > 4$

(ii) $f(-1/4) = 0 \quad \Rightarrow a = 65/4$

(iii) $-\frac{B}{2A} > -\frac{1}{4} \quad \Rightarrow a < \frac{1}{2}$

No common solution.

$\therefore \quad a \in \phi$

5. (1) Equation is satisfied by α, β, α^2 and β^2. So, we have following possibilities:

(i) Let $\alpha^2 = \alpha$ and $\beta^2 = \beta$.

$\Rightarrow \quad (\alpha, \beta) \equiv (1, 1)$

(ii) $\alpha^2 = \beta$ and $\beta^2 = \alpha$

$\Rightarrow \quad \alpha^4 = \alpha$

$\Rightarrow \quad \alpha(\alpha^3 - 1) = 0$

$\Rightarrow \quad \alpha = 0, 1, \frac{-1 \pm \sqrt{3}i}{2}$

$\Rightarrow \quad (\alpha, \beta) \equiv \left(\frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}\right)$

(iii) $\alpha^2 = \beta$ and $\beta^2 = \beta$ (or $\alpha^2 = \alpha$ and $\beta^2 = \alpha$)

$\Rightarrow \quad \alpha^2 = \beta^2 \quad \Rightarrow \alpha = \pm\beta$

$\Rightarrow \quad (\alpha, \beta) \equiv (-1, 1), (1, 1)$

Thus, possible unordered pairs (α, β) such that $\alpha\beta \neq 0$ is

$$(1, 1), (-1, 1) \text{ or } \left(\frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}\right).$$

6. (4) $(x^2 - 2x - 2a)(x^2 - 4x + 2a) = 0$

$\Rightarrow \quad x^2 - 2x - 2a = 0$ ...(1)

or $\quad x^2 - 4x + 2a = 0$ ...(2)

Discriminant of eq. (1) is: $D_1 = 4 + 8a$

Discriminant of eq. (2) is: $D_2 = 16 - 8a$

(a) If equation has four distinct roots then

$D_1 > 0 \Rightarrow 4 + 8a > 0 \Rightarrow a > -1/2$

and $\quad D_2 \Rightarrow 16 - 8a > 0 \Rightarrow a < 2$

$\therefore \quad a \in (-1/2, 2)$

(b) If equation has exactly two distinct roots then

(i) $D_1 > 0$ and $D_2 < 0$

$\Rightarrow \quad a \in (2, \infty)$

(ii) $D_1 < 0$ and $D_2 > 0$

$\Rightarrow \quad a \in (-\infty, -1/2)$

(c) If equation has no real roots then

$D_1 < 0$ and $D_2 < 0$

$\Rightarrow \quad a \in \phi$

(d) Clearly, equation cannot have four roots positive.

Numerical Value Type

1. (2) Given $x - 2 = 2^{2/3} + 2^{1/3}$

Taking cube of both sides, we get

$$(x-2)^3 = 2^2 + 2 + 3 \times 2^{2/3} \times 2^{1/3}(x-2)$$

$\Rightarrow \quad (x-2)^3 = 6 + 6(x-2)$

or $\quad x^3 - 6x^2 + 12x - 8 = -6 + 6x$

$\therefore \quad x^3 - 6x^2 + 6x = 2$

2. (4) $x^{1/8} = (3x^4 + 4)^{1/64}$

$\Rightarrow \quad x^8 = 3x^4 + 4$

$\Rightarrow \quad x^4 = 4$

3. (6) $\sqrt{2x+\sqrt{2x+4}} = 4$

$\Rightarrow \sqrt{2x+4} = 16 - 2x$

$\Rightarrow 2x + 4 = 4(8-x)^2$

$\Rightarrow 2x^2 - 33x + 126 = 0$

$\Rightarrow (2x - 21)(x - 6) = 0$

$\Rightarrow x = 6, 21/2$

But $x = 21/2$ does not satisfy the given equation.

4. (4) Given $a^2 - 4a + 1 = 4 \Rightarrow a^2 + 1 = 4(1 + a)$

$$y = \frac{(a-1)(1+a^2)}{a^2-1} = \frac{a^2+1}{a+1} = \frac{4(a+1)}{a+1} = 4$$

5. (6) $a^2 \geq 8b$ and $4b^2 \geq 4a$

Now $b^2 \geq a$

$\Rightarrow b^4 \geq a^2 \geq 8b \quad (a > 0, b > 0)$

$\therefore b^3 \geq 8 \Rightarrow b \geq 2$ (1)

Again $a^2 \geq 8b$ and $b \geq 2$

$\Rightarrow a^2 \geq 16$

or $a \geq 4$ (2)

From (1) and (2), $(a + b)_{least} = 6$.

6. (6621)

The given equation $x + \frac{1}{x} = 3$

$\therefore x^2 + \frac{1}{x^2} = 7$

$\Rightarrow x^4 + \frac{1}{x^4} = 47$

$\Rightarrow x^8 + \frac{1}{x^8} = (47)^2 - 2$

$\therefore x^8 + x^{-8} = 2207$ (1)

Now $E = x^9 + x^7 + x^{-9} + x^{-7}$

$$= x^8\left(x + \frac{1}{x}\right) + x^{-9} + x^{-7}$$

$$= x^8\left(x + \frac{1}{x}\right) + x^{-8}\left(x + \frac{1}{x}\right)$$

$$= \left(x + \frac{1}{x}\right)(x^8 + x^{-8}) \quad (2)$$

Substitute the value of $x^8 + x^{-8} = 2207$ from (1) and $x + \frac{1}{x} = 3$

$E = (3)(2207) = 6621$

7. (9) We have $x^3 + x^2 + ax + b = 0$

Sum of roots $= -1$

$\Rightarrow 1 - \text{cosec}^2\alpha = -1$

$\Rightarrow \sin^2\alpha = \frac{1}{2}$

Thus, $\cos^2\alpha = \frac{1}{2}$

So, $P(x) = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)(x + 2)$

Hence, $P(2) = \frac{3}{2} \cdot \frac{3}{2} \cdot 4 = 9$

8. (2) Given equation is

$(x-2)^2 - 4 - (3k-1)|x-2| - 2k + 8 = 0$

or $t^2 - (3k-1)t - 2k + 4 = 0$, where $t = |x-2|$

For exactly 3 distinct solutions, one of the roots is zero and other root is positive.

$\therefore k = 2$

9. (−1.25)

$$\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1$$

$$\Rightarrow \cos^2\frac{\pi}{8} = \left(\frac{1}{\sqrt{2}} + 1\right)\frac{1}{2}$$

$$\Rightarrow \cos^4\frac{\pi}{8} = \frac{1}{4}\left(\frac{1}{2} + 1 + \frac{2}{\sqrt{2}}\right) = \left(\frac{3}{2} + \sqrt{2}\right)\frac{1}{4}$$

$$\Rightarrow \frac{1}{4}\left(\frac{3}{2} + \sqrt{2}\right) + \frac{a}{2}\left(\frac{1}{\sqrt{2}} + 1\right) + b = 0$$

$$\Rightarrow \left(\frac{3}{8} + \frac{a}{2} + b\right) + \sqrt{2}\left(\frac{1}{4} + \frac{a}{4}\right) = 0$$

Since a and b are rational,

$$\frac{1}{4} + \frac{a}{4} = 0, \frac{3}{8} + \frac{a}{2} + b = 0$$

$$\Rightarrow a = -1, b = \frac{1}{8}$$

$\therefore x^2 + 8bx + a = x^2 + x - 1 = (x + 1/2)^2 - 5/4$

So, required least value is $-5/4$.

10. (16) Given $\alpha\beta$; $\alpha\beta(\alpha + \beta)$; $\alpha^3 + \beta^3$ are in G.P.

$\alpha + \beta = 4$; $\alpha\beta = k$; $\alpha\beta^2 + \alpha^2\beta = \alpha\beta(\alpha + \beta) = 4k$

$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$= 64 - 3k(4) = 4(16 - 3k)$

$\therefore$ k; $4k$; $4(16 - 3k)$ are in G.P.

$16k^2 = 4k(16 - 3k)$

$4k(4k - 16 + 3k) = 0$

$k = 0; k = \frac{16}{7}$

11. (540) Let $\alpha_1 = A, \beta_1 = AR, \alpha_2 = AR^2, \beta_2 = AR^3$

we have $\alpha_1 + \beta_1 = 6 \Rightarrow A(1 + R) = 6$ (1)

$\alpha_1\beta_1 = p \Rightarrow A^2 R = p$ (2)

Also $\alpha_2 + \beta_2 = 54 \Rightarrow AR^2(1 + R) = 54$ (3)

$\alpha_2\beta_2 = q \Rightarrow A^2R^5 = q$ (4)

Now, on dividing Eq. (3) by Eq. (1), we get

$$\frac{AR^2(1+R)}{A(1+R)} = \frac{54}{6} = 9 \quad \text{or} \quad R^2 = 9$$

$\therefore$ $R = 3$ (as it is an increasing G.P.)

$\therefore$ On putting $R = 3$ in Eq. (1), we get

$$A = \frac{6}{4} = \frac{3}{2}$$

$$\therefore \quad p = A^2R = \frac{9}{4} \times 3 = \frac{27}{4}$$

and $q = A^2R^5 = \frac{9}{4} \times 243 = \frac{2187}{4}$

Hence, $q - p = \frac{2187 - 27}{4} = \frac{2160}{4} = 540$

12. (6) $\alpha + \beta = 1154$ and $\alpha\beta = 1$

$(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta} = 1154 + 2 = 1156 = (34)^2$

$\Rightarrow \sqrt{\alpha} + \sqrt{\beta} = 34$

Again $(\alpha^{1/4} + \beta^{1/4})^2 = \sqrt{\alpha} + \sqrt{\beta} + 2(\alpha\beta)^{1/4} = 34 + 2 = 36$

$\alpha^{1/4} + \beta^{1/4} = 6$

13. (8) $x^2 + mx + n = 0$: roots $2\alpha, 2\beta$ and $x^2 + px + m = 0$: roots α, β

$2(\alpha + \beta) = -m$ (1)

$4\alpha\beta = n$ (2)

and $\alpha + \beta = -p$ (3)

$\alpha\beta = m$ (4)

$\therefore$ (1) and (3) $\Rightarrow$ $2p = m$

and (2) and (4) $\Rightarrow$ $4m = n$

$\Rightarrow \quad \dfrac{n}{p} = \dfrac{4m}{m/2} = 8$

14. (7) Given $a + b + c = 1$ (1)

$ab + bc + ca = 0$ (2)

$abc = 2$ (3)

Now $(a + b + c)^2 = 1$

$a^2 + b^2 + c^2 + 2\Sigma ab = 1$

$\therefore \quad a^2 + b^2 + c^2 = 1$

Now, $a^3 + b^3 + c^3 - 3abc = (a + b + c)\left[\Sigma a^2 - \Sigma ab\right]$

$= 1(1 - 0) = 1$

$\therefore \quad a^3 + b^3 + c^3 = 1 + 3abc = 1 + 3 \times 2 = 7$

15. (4) As $P(x)$ is an odd function

Hence, $P(-x) = -P(x) \Rightarrow P(-3) = -P(3) = -6$

Let $P(x) = Q(x^2 - 9) + ax + b$

(where Q is quotient and $(ax + b) = g(x)$ = remainder)

Now $P(3) = 3a + b = 6$ (1)

$P(-3) = -3a + b = -6$ (2)

Hence, $b = 0$ and $a = 2$

Hence, $g(x) = 2x \Rightarrow g(2) = 4$

16. (12) We have $\alpha^2 - 6\alpha + 12 = 0$

$\therefore \quad 6(\alpha - 2) = \alpha^2$

and $\beta^2 - 6\beta + 12 = 0$

$\therefore \quad \beta(\beta - 6) + 12 = 0$

So, $(\alpha - 2)^{24} - \dfrac{(\beta - 6)^8}{\alpha^8} + 1 = (\alpha - 2)^{24} - \dfrac{12^8}{\alpha^8\beta^8} + 1$

$= (\alpha - 2)^{24} - \dfrac{12^8}{12^8} + 1$

$= (\alpha - 2)^{24}$

$= (3 \pm \sqrt{3}i - 2)^{24} \quad (\alpha = 3 \pm \sqrt{3}i)$

$= (1 \pm \sqrt{3}i)^{24}$

$= \left(2\left(\cos\dfrac{\pi}{3} \pm i\sin\dfrac{\pi}{3}\right)\right)^{24}$

$= 2^{24}(\cos 8\pi + i \sin 8\pi) = 4^{12}$

17. (1210)

Roots of $x^2 - 10cx - 11d = 0$ are a and b.

So, $a + b = 10c$ and $ab = -11d$

c and d are the roots of $x^2 - 10ax - 11b = 0$.

So, $c + d = 10a$ and $cd = -11b$

From these results, we get

$a + b + c + d = 10(a + c)$

$\Rightarrow \quad b + d = 9(a + c)$

and $abcd = 121bd$

$\Rightarrow \quad ac = 121$

Also, we have $a^2 - 10ac - 11d = 0$ and $c^2 - 10ac - 11b = 0$

$\Rightarrow \quad a^2 + c^2 - 20ac - 11(b + d) = 0$

$\Rightarrow \quad (a + c)^2 - 22ac - 99(a + c) = 0$

$\Rightarrow \quad (a + c)^2 - 99(a + c) - 22 \times 121 = 0$

$\Rightarrow \quad 121 - 99(a + c) = 0$

$\Rightarrow \quad a + c = 121$ or -22

For $a + c = -22$, we get $a = c = -11$

Thus, we have $a + c = 121$.

$\therefore \quad a + b + c + d = 10(a + c) = 1210$

18. (12) From the given relations, we can say that a and $\dfrac{1}{b}$ are the roots of the equation $6x^2 + 20x + 15 = 0$.

$\therefore \quad a + \dfrac{1}{b} = -\dfrac{10}{3}$ and $\dfrac{a}{b} = \dfrac{5}{2}$

$$\frac{b^3}{ab^2 - 9(ab+1)^3} = \frac{1}{a\cdot\frac{1}{b} - 9\left(a + \frac{1}{b}\right)^3}$$

$$= \frac{1}{\frac{5}{2} + 9\cdot\frac{1000}{27}} = \frac{6}{2015}$$

19. (1.5) For equation $x^2 + 2x \sin y + 1 = 0$

$D = 4\sin^2 y - 4 < 0$

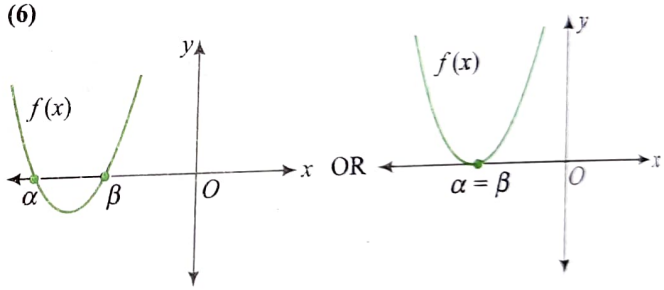
Thus, equations have both roots common.

$\Rightarrow \quad \dfrac{1}{a} = \dfrac{2\sin y}{1} = \dfrac{1}{1}$

$\Rightarrow \quad a = 1, \sin y = \dfrac{1}{2}$

$\therefore \quad a + \sin y = 1.5$

20. (6)



Let $f(x) = x^2 + 2(\lambda + 1)x + \lambda^2 + \lambda + 7$

If both roots of $f(x) = 0$ are negative, then

$D = b^2 - 4ac = 4(\lambda + 1)^2 - 4(\lambda^2 + \lambda + 7) \geq 0$

$\Rightarrow \quad \lambda - 6 \geq 0$

$\Rightarrow \quad \lambda \in [6, \infty)$ (1)

Sum of roots $= -2(\lambda + 1) < 0$

$\Rightarrow \quad \lambda \in (-1, \infty)$ (2)

and product of roots $= \lambda^2 + \lambda + 7 > 0 \Rightarrow \lambda \in R$ (3)

$\therefore$ From (1), (2), (3), we get $\lambda \in [6, \infty)$

(as (1), (2), (3) must be satisfied simultaneously.)

Hence, the least value of $\lambda = 6$.

21. (7) For two distinct roots, $D > 0$ i.e., $k^2 + 8(k^2 + 5) > 0$ which is always true

Also let $f(x) = -2x^2 + kx + k^2 + 5 = 0$

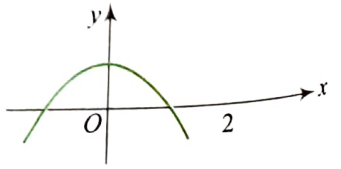
But $f(0) > 0$. So, $f(2) < 0$

$-8 + 2k + k^2 + 5 < 0$

$\Rightarrow \quad k^2 + 2k - 3 < 0$

$\Rightarrow \quad (k + 3)(k - 1) < 0$

$k \in (-3, 1)$

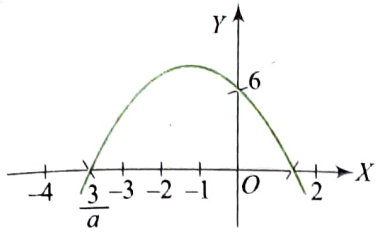


$\Rightarrow \quad a = -3; b = 1$

$\Rightarrow \quad a + 10b = -3 + 10 = 7$

$f(x) = ax^2 - (3 + 2a)x + 6$

$= (ax - 3)(x - 2)$



Here, roots of the equation

$f(x) = 0$ are 2 and $3/a$, and $f(0) = 6$.

$f(x)$ should be positive for exactly three negative integral values of x.

Therefore, graph of $f(x)$ must be a downward parabola passing through $x = 2$ and $x = 3/a$ and $-4 \le \frac{3}{a} < -3$

$\therefore \quad a \in \left(-1, -\frac{3}{4}\right]$

$\therefore \quad c = -1, d = -\frac{3}{4}$

3. (3) We have $(\alpha+\beta)^2 = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)(\alpha^2+\beta^2)$

$\Rightarrow \quad (\alpha+\beta)^2 = \left(\frac{1}{\alpha}+\frac{1}{\beta}\right)[(\alpha+\beta)^2 - 2\alpha\beta]$

Substituting $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ we have

$$\frac{b^2}{a^2} = \frac{-b}{c}\left(\frac{b^2}{a^2} - \frac{2c}{a}\right)$$

$\Rightarrow \quad cb^2 + b(b^2 - 2ac) = 0$

$b \ne 0, \quad \therefore \quad bc + b^2 - 2ac = 0$

a, b, c are in AP, $\quad \therefore b = \frac{a+c}{2}$

Therefore, we have $\frac{(a+c)c}{2} + \left(\frac{a+c}{2}\right)^2 - 2ac = 0$

$\Rightarrow \quad a^2 - 4ac + 3c^2 = 0 \quad$ or $(a-c)(a-3c) = 0$

$a \ne c \quad \therefore a = 3c \quad \therefore \frac{a}{c} = 3$

4. (3) We have $P(x) = \frac{5}{3} - 6x - 9x^2 = -(3x+1)^2 + \frac{8}{3}$

$\Rightarrow \quad P_{max} = \frac{8}{3}$

Similarly, $Q(y) = -4y^2 + 4y + \frac{13}{2} = -(2y-1)^2 + \frac{15}{2}$

$\Rightarrow \quad Q_{max} = \frac{15}{2}$

Now, $P_{max} \times Q_{max} = \frac{8}{3} \times \frac{15}{2} = 20$

So $\quad (x, y) \equiv \left(-\frac{1}{3}, \frac{1}{2}\right)$

Hence, $6x + 10y = 6\left(\frac{-1}{3}\right) + 10\left(\frac{1}{2}\right) = -2 + 5 = 3$

25. (6) Let the roots be $a - 3d, a - d, a + d, a + 3d$

Sum of roots $= 4a = 0 \quad \Rightarrow \quad a = 0$

Hence, roots are $-3d, -d, d, 3d$.

Product of roots $= 9d^4 = m^2 \quad \Rightarrow \quad d^2 = \frac{m}{3}$ (1)

Again $\Sigma x_1 x_2 = 3d^2 - 3d^2 - 9d^2 - d^2 - 3d^2 + 3d^2$

$= -10d^2 = -(3m + 2)$

or $\quad \frac{10m}{3} = 3m + 2$ or $10m = 9m + 6$

or $\quad m = 6$

26. (3) $2x^2 + 4x(y-3) + 7y^2 - 2y + t = 0$

$D = 0$ (for one solution)

$\Rightarrow \quad 16(y-3)^2 - 8(7y^2 - 2y + t) = 0$

or $\quad 2(y-3)^2 - (7y^2 - 2y + t) = 0$

or $\quad 2(y^2 - 6y + 9) - (7y^2 - 2y + t) = 0$

or $\quad -5y^2 - 10y + 18 - t = 0$

or $\quad 5y^2 + 10y + t - 18 = 0$

Again $D = 0$ (for one solution)

$\Rightarrow \quad 100 - 20(t - 18) = 0$

or $\quad 5 - t + 18 = 0$

or $\quad t = 23$

for $\quad t = 23; \; 5y^2 + 10y + 5 = 0$

$(y+1)^2 = 0 \quad$ or $\quad y = -1$

for $\quad y = -1; \; 2x^2 - 16x + 32 = 0$

$x^2 - 8x + 16 = 0$

$x = 4 \quad$ or $\quad x + y = 3$

27. (36) We have $x_1 + x_2 + x_3 = 8$

$x_1 x_2 x_3 = d$

$x_1x_2 + x_2x_3 + x_3x_1 = c$

Possible roots 1, 2, 5 or 1, 3, 4

$\therefore \quad d = 10 \quad$ or $\quad d = 12$

$\Rightarrow \quad c = 2 + 10 + 5 = 17 \quad$ or $\quad 3 + 12 + 4 = 19$

Hence, $d = 10$ and $c = 17 \quad$ or $\quad d = 12$ and $c = 19$

28. (149)

Clearly, $P(x) - x^3 = 0$ has roots 1, 2, 3, 4.

$\therefore \quad P(x) - x^3 = (x-1)(x-2)(x-3)(x-4)$

$\Rightarrow \quad P(x) = (x-1)(x-2)(x-3)(x-4) + x^3$

Hence, $P(5) = 1 \times 2 \times 3 \times 4 + 125 = 149$

29. (6) $x^2 + ax + b \equiv (x+1)(x+b) \quad \Rightarrow \quad b + 1 = a$ (1)

also $x^2 + bx + c \equiv (x+1)(x+c) \quad \Rightarrow \quad c + 1 = b$

or $\quad b + 1 = c + 2$ (2)

Hence $b + 1 = a = c + 2$

Also $(x+1)(x+b)(x+c) \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow \quad x^3 + (1 + b + c)x^2 + (b + bc + c)x + bc \equiv x^3 - 4x^2 + x + 6$

$\Rightarrow \quad 1 + b + c = -4$

$\Rightarrow \quad 2c + 2 = -4$

$\Rightarrow \quad c = -3; \; b = -2$ and $a = -1$

$\Rightarrow \quad a + b + c = -6$

30. (9) We have $x^4 - 4x^3 + 6x^2 - 4x = 69$

$\Rightarrow \quad x^4 - 4x^3 + 6x^2 - 4x + 1 = 70$

$\Rightarrow \quad (x-1)^4 = 70$

$\Rightarrow \quad (x-1)^2 = \pm\sqrt{70}$

For the product of non-real roots, consider

$(x-1)^2 = -\sqrt{70}$

$\Rightarrow \quad x^2 - 2x + 1 + \sqrt{70} = 0$

Therefore, product is $1 + \sqrt{70}$, whose integral part is 9.

31. (–1) Let $\alpha_1 = \frac{\alpha}{1+\alpha}, \beta_1 = \frac{\beta}{1+\beta}$, and $\gamma_1 = \frac{\gamma}{1+\gamma}$.

Also, let $y = \alpha_1, \beta_1$ and γ_1 be roots of an equation in y.

Then $y = \frac{x}{1+x}$

$\therefore \quad x = \frac{y}{1-y}$

Now, $x^3 - 3x^2 + 1 = 0$

$\Rightarrow \quad \left(\frac{y}{1-y}\right)^3 - 3\left(\frac{y}{1-y}\right)^2 + 1 = 0$

$\Rightarrow \quad 3y^3 - 3y + 1 = 0$

$\Rightarrow \quad \alpha_1 + \beta_1 + \gamma_1 = 0$ and $\alpha_1\beta_1\gamma_1 = -\frac{1}{3}$

$\therefore \quad \alpha_1^3 + \beta_1^3 + \gamma_1^3 = 3\alpha_1\beta_1\gamma_1 = 3\left(-\frac{1}{3}\right) = -1$

32. (3) $n, n+1, n+2$

Sum $= 3(n+1) = -a$

$\therefore \quad a^2 = 9(n+1)^2$

Sum of the roots taken two at a time $= +b$

$\therefore \quad n(n+1) + (n+1)(n+2) + (n+2)n + 1 = b + 1$

$n^2 + n + n^2 + 3n + 2 + n^2 + 2n + 1 = b + 1$

$\therefore \quad b + 1 = 3n^2 + 6n + 3$

$b + 1 = 3(n+1)^2 = \frac{a^2}{3}; \quad \therefore \quad \frac{a^2}{b+1} = 3$

33. (5) Let $ax^3 + bx^2 + cx + d = 0$ has roots p, q, r

$$pq + qr + rp = \frac{c}{a} \qquad (1)$$

but $pq + qr + rp \le p^2 + q^2 + r^2$

$= (p + q + r)^2 - 2\Sigma pq$

$\therefore \quad 3(pq + qr + rp) \le (p + q + r)^2 = 16$

$\therefore \quad 3\frac{c}{a} \le 16$

$\Rightarrow \quad \frac{c}{a} \le \frac{16}{3}$

$\Rightarrow$ largest possible integral value of $\frac{c}{a}$ is 5

34. (3) Given equations are

$ax^2 + bx + c = 3x - y$ ...(1)

$ay^2 + by + c = y - 3x$ ...(2)

Now, $b^2 - 4ac \ge 0$

$\therefore \quad ax^2 + bx + c \le 0$ and $ay^2 + by + c \le 0$

$\therefore \quad 3x - y \ge 0$ and $y - 3x \ge 0$

$\Rightarrow \quad 3x = y$

or

$\therefore \quad ax^2 + bx + c \le 0$ and $ay^2 + by + c \le 0$.

$\therefore \quad 3x - y \le 0$ and $y - 3x \le 0$

$\Rightarrow \quad 3x = y$

So, $\frac{y}{x} = 3$

35. (–9) Let $f(x) = (a^2 - 14a + 13)x^2 + (a + 2)x - 2$

Given that the above equation does not have two distinct real roots.

Therefore, either $f(x) \ge 0$ or $f(x) \le 0 \ \forall \ x \in R$.

But $f(0) = -2 < 0$

$\therefore \quad f(0) \le 0 \ \forall \ x \in R$

So, $f(-1) \le 0$

$\Rightarrow \quad (a^2 - 14a + 13) - (a + 2) - 2 \le 0$

$\Rightarrow \quad a^2 - 15a + 9 \le 0$

$\Rightarrow \quad a^2 - 15a \le -9$

So, the maximum value of $a^2 - 15a$ is -9.

36. (–1) Given equation is $px^2 + qx + r = 0$.

Let $f(x) = px^2 + qx + r$

$f(0) = r > 0$

$f(1) = p + q + r < 0$

and $f(-1) = p - q + r < 0$

So, one root lies in $(-1, 0)$ and the other in $(0, 1)$.

$\Rightarrow \quad [\alpha] = -1$ and $[\beta] = 0$

$\Rightarrow \quad [\alpha] + [\beta] = -1$

37. (3) $x^2 + x(y - a) + y^2 - ay + 1 \ge 0 \ x \in R$

$\Rightarrow \quad (y - a)^2 - 4(y^2 - ay + 1) \le 0$

$\Rightarrow \quad -3y^2 + 2ay + a^2 - 4 \le 0 \ \forall \ y \in R$

$\therefore \quad 3y^2 - 2ay + 4 - a^2 \ge 0 \ \forall \ y \in R$

$D \le 0$

$\Rightarrow \quad 4a^2 - 4 \cdot 3(4 - a^2) \le 0$

$\Rightarrow \quad a^2 - 3(4 - a^2) \le 0 \Rightarrow 4a^2 - 12 \le 0$

$\therefore$ Range of $a \in [-\sqrt{3}, \sqrt{3}]$

$\Rightarrow$ number of integer $\{-1, 0, 1\}$

38. (4) $y = \frac{3x^2 + mx + n}{x^2 + 1}$

or $x^2(y - 3) - mx + y - n = 0$

Since $x \in R$

$D \ge 0$

$\Rightarrow \quad m^2 - 4(y - 3)(y - n) \ge 0$

or $m^2 - 4(y^2 - ny - 3y + 3n) \ge 0$

$4y^2 - 4y(n + 3) + 12n - m^2 \le 0$ (1)

Also given $(y + 4)(y - 3) \le 0$

$y^2 + y - 12 \le 0$ (2)

Comparing (1) and (2), we get $\frac{4}{1} = -\frac{4(n+3)}{1} = \frac{12n - m^2}{-12}$

$\Rightarrow \quad m = 0$ and $n = -4$

39. (315) We have $\left(\frac{a^4 + 3a^2 + 1}{a^2}\right)\left(\frac{b^4 + 5b^2 + 1}{b^2}\right)\left(\frac{c^4 + 7c^2 + 1}{c^2}\right)$

$= \left(a^2 + \frac{1}{a^2} + 3\right)\left(b^2 + \frac{1}{b^2} + 5\right)\left(c^2 + \frac{1}{c^2} + 7\right)$

$= \left(\left(a - \frac{1}{a}\right)^2 + 5\right)\left(\left(b - \frac{1}{b}\right)^2 + 7\right)\left(\left(c - \frac{1}{c}\right)^2 + 9\right)$

So, minimum value is $5 \times 7 \times 9 = 315$.

40. (3) Let $a^2 + b^2 = x$

$1 - 2ab = (a + b)^2 - 2ab = a^2 + b^2 = x$

$\therefore \quad ab = \frac{1 - x}{2}$

and $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$ (1)

$a^3 + b^3 = x - ab$

but $ab = \frac{1 - x}{2}$

Hence $a^3 + b^3 = x - \frac{1-x}{2} = \frac{3x-1}{2}$

Hence the equation

$(1 - 2ab)(a^3 + b^3) = 12$, becomes

$$x\left(\frac{3x-1}{2}\right) = 12$$

$\Rightarrow \quad 3x^2 - x - 24 = 0$

$\Rightarrow \quad (x-3)(3x+8) = 0$

$\Rightarrow \quad x = 3$ or $x = -\frac{8}{3}$ (not possible) as $x = a^2 + b^2 \neq 0$

$\therefore \quad x = 3 \quad \Rightarrow \quad a^2 + b^2 = 3$

41. (5) We have $2x^3 - 9x^2 + 12x + k = 0$

Let the roots are α, α, β

$$2\alpha + \beta = \frac{9}{2} \qquad (1)$$

$$\alpha^2 + 2\alpha\beta = \frac{12}{2} = 6 \qquad (2)$$

are $\alpha^2\beta = -\frac{k}{2}$ (3)

putting $\beta = \left(\frac{9}{2} - 2\alpha\right)$ from (1) in (2), we have

$$\alpha^2 + 2\alpha\left(\frac{9}{2} - 2\alpha\right) = 6$$

or $\alpha^2 + 9\alpha - 4\alpha^2 = 6$

or $3\alpha^2 - 9\alpha + 6 = 0$

or $\alpha^2 - 3\alpha + 2 = 0$

or $(\alpha - 2)(\alpha - 1) = 0 \quad \Rightarrow \alpha = 2$ or 1

If $\alpha = 2$, then $\beta = \frac{1}{2}$;

if $\alpha = 1$, then $\beta = \frac{5}{2}$

$\therefore \quad k = -2(2^2)\frac{1}{2} = -4$

or $k = -2(1^2)\left(\frac{5}{2}\right) = -5$

42. (3) Let $\frac{1-a^3}{a} = \frac{1-b^3}{b} = \frac{1-c^3}{c} = k$

$\Rightarrow \quad \frac{1-x^3}{x} = k,$

where x takes 3 values a, b and c.

$\Rightarrow \quad x^3 + kx - 1 = 0$ has roots a, b, c

Now $a + b + c = 0$ (1)

$abc = 1$ (2)

Hence, $a^3 + b^3 + c^3 = 3abc = 3$

43. (8) Given $\frac{x^3 + (a-10)x^2 - x + a - 6}{x^3 + (a-6)x^2 - x + a - 10}$ reduces to a quotient of two linear functions

Let $f(x) = x^3 + (a-10)x^2 - x + a - 6 = 0$ has roots α, β, γ

Then $g(x) = x^3 + (a-6)x^2 - x + a - 10 = 0$ has roots α, β, δ

$\therefore \quad f(x) - g(x) = 0$ has roots α, β.

$\therefore \quad -4x^2 + 4 = 0 \Rightarrow x = \pm 1$

$\therefore \quad \alpha + \beta = 0, \ \alpha\beta = -1$

Now from $f(x) = 0$, $\alpha + \beta + \gamma = -(a - 10) = 10 - a$ (1)

$\Rightarrow \quad \gamma = 10 - a$ (2)

and $\alpha\beta\gamma = 6 - a \Rightarrow \gamma = a - 6$

From (1) and (2), $a = 8$

Archives

JEE Main

Single Correct Answer Type

1. (3) $bx^2 + cx + a = 0$

Since the roots are imaginary,

$$c^2 - 4ab < 0 \quad \Rightarrow \quad c^2 < 4ab \quad \Rightarrow \quad -c^2 > -4ab$$

Now consider $3b^2x^2 + 6bcx + 2c^2$

Since $3b^2 > 0$, the given expression has a minimum value.

So, minimum value $= \frac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)}$

$$= -\frac{12b^2c^2}{12b^2}$$

$$= -c^2 > -4ab$$

2. (2) Let $e^{\sin x} = t$

$\Rightarrow \quad t^2 - 4t - 1 = 0$

$\Rightarrow \quad t = \frac{4 \pm \sqrt{16+4}}{2}$

$\Rightarrow \quad t = e^{\sin x} = 2 \pm \sqrt{5}$

$\Rightarrow \quad e^{\sin x} = 2 - \sqrt{5}, \qquad e^{\sin x} = 2 + \sqrt{5}$

$\Rightarrow \quad e^{\sin x} = 2 - \sqrt{5} < 0$, which is not possible

or $\sin x = \ln(2 + \sqrt{5}) > 1$, which is not possible

Hence no solution.

3. (1) $x^2 + 2x + 3 = 0$

$$D = 2^2 - 4.1.3 < 0$$

i.e., both roots are complex.

Hence, both roots are common of $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$.

$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$

4. (4) We have $\frac{1}{\alpha} + \frac{1}{\beta} = 4$ and $2q = p + r$

$\therefore \quad \frac{2q}{p} = 1 + \frac{r}{p}$

$\Rightarrow \quad -2(\alpha + \beta) = 1 + \alpha\beta$

$\Rightarrow \quad -2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{1}{\alpha\beta} + 1$

$\Rightarrow \quad \frac{1}{\alpha\beta} = -9$

$\therefore \quad \alpha\beta = -\frac{1}{9}$

$\therefore \quad \alpha + \beta = -\frac{4}{9}$

Equation having roots α, β is $9x^2 + 4x - 1 = 0$

$\therefore \quad |\alpha-\beta| = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

$= \sqrt{\frac{16}{81} + \frac{4}{9}}$

$= \frac{2\sqrt{13}}{9}$

5. (4) $(x^2 - 5x + 5)^{x^2+4x-60} = 1$

Case I :

$x^2 + 4x - 60 = 0$

$\therefore \quad (x-6)(x+10) = 0$

$\therefore \quad x = 6, -10$

Case II :

$x^2 - 5x + 5 = 1$

$\therefore \quad x^2 - 5x + 4 = 0$

$\therefore \quad (x-1)(x-4) = 0$

$\therefore \quad x = 1, 4$

Case III :

$x^2 - 5x + 5 = -1$ and $x^2 - 4x - 60$ is even number

$\therefore \quad x^2 - 5x + 6 = 0$

$\therefore \quad (x-2)(x-3) = 0$

$\therefore \quad x = 2, 3$

But for $x = 3$, $x^2 - 4x - 60$ is odd.

Hence $x = 2$ only.

Thus solutions are $6, -10, 1, 4, 2$

Sum of solutions is 3

6. (1) We have

$x(x+1) + (x+1)(x+2) + \ldots + (x+\overline{n-1})(x+n) = 10n$

Roots of above equation are consecutive integers.

Let roots be α and $\alpha + 1$.

$\Rightarrow \quad \alpha(\alpha+1) + (\alpha+1)(\alpha+2) + \ldots + (\alpha+\overline{n-1})(\alpha+n)$

$= 10n \qquad (1)$

and $(\alpha+1)(\alpha+2) + (\alpha+2)(\alpha+3) + \ldots + (\alpha+n)(\alpha+n+1)$

$= 10n \qquad (2)$

Subtracting (2) from (1), we get

$\alpha(\alpha+1) - (\alpha+n)(\alpha+n+1) = 0$

$\Rightarrow \quad \alpha^2 + \alpha - \alpha^2 - (2n+1)\alpha - n(n+1) = 0$

$\Rightarrow \quad \alpha = -\frac{n+1}{2}$

Putting this value in the original equation, i.e.,

$nx^2 + x\sum_{r=1}^{n}(2r-1) + \sum_{r=1}^{n}(r-1)r = 10n$, we get

$n\frac{(n+1)^2}{4} - \frac{(n+1)}{2}n^2 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = 10n$

$\Rightarrow \quad 3(n+1)^2 - 6n(n+1) + 2(n+1)(2n+1) - 6(n+1) = 120$

$\Rightarrow \quad n^2 = 121$

$\Rightarrow \quad n = 11$

7. (4) Given equation is $2|\sqrt{x} - 3| + x - 6\sqrt{x} + 6 = 0$

or $2|\sqrt{x} - 3| + (\sqrt{x} - 3)^2 - 3 = 0$

Let $|\sqrt{x} - 3| = y$

$\therefore \quad y^2 + 2y - 3 = 0$

$\Rightarrow \quad (y-1)(y+3) = 0$

$\Rightarrow \quad y = 1, -3$

$\therefore \quad |\sqrt{x} - 3| = 1$

$\Rightarrow \quad \sqrt{x} - 3 = \pm 1$

$\Rightarrow \quad \sqrt{x} = 2, 4$

$\Rightarrow \quad x = 4, 16$

So, equation has two solutions.

JEE ADVANCED

Single Correct Answer Type

1. (2) $\alpha^3 + \beta^3 = q$

$\Rightarrow \quad (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = q$

$\Rightarrow \quad -p^3 + 3p\alpha\beta = q \Rightarrow \alpha\beta = \frac{q+p^3}{3p}$

Required equation is

$x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 0$

$x^2 - \frac{(\alpha^2+\beta^2)}{\alpha\beta}x + 1 = 0$

$\Rightarrow \quad x^2 - \left(\frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}\right)x + 1 = 0$

$\Rightarrow \quad x^2 - \frac{p^2 - 2\left(\frac{p^3+q}{3p}\right)}{\frac{p^3+q}{3p}}x + 1 = 0$

$\Rightarrow \quad (p^3+q)x^2 - (3p^3 - 2p^3 - 2q)x + (p^3+q) = 0$

$\Rightarrow \quad (p^3+q)x^2 - (p^3 - 2q)x + (p^3+q) = 0.$

2. (2) $x^2 + bx - 1 = 0$

$x^2 + x + b = 0$

Subtracting, we get $(b-1)x - 1 - b = 0$

$\Rightarrow \quad x = \frac{b+1}{b-1}$

This value of x satisfies equation (1)

$\Rightarrow \quad \frac{(b+1)^2}{(b-1)^2} + \frac{b+1}{b-1} + b = 0$

$\Rightarrow \quad b = \sqrt{3}i, -\sqrt{3}i, 0$

3. (3) $a_n = \alpha^n - \beta^n$

Also $\alpha^2 - 6\alpha - 2 = 0$

Multiply with α^8 on both sides

$\Rightarrow \quad \alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0$

similarly $\beta^{10} - 6\beta^9 - 2\beta^8 = 0$

Subtracting (2) from (1) we have

$\alpha^{10} - \beta^{10} - 6(\alpha^9 - \beta^9) = 2(\alpha^8 - \beta^8)$

$\Rightarrow \quad a_{10} - 6a_9 = 2a_8$

$\Rightarrow \quad \frac{a_{10} - 2a_8}{2a_9} = 3.$

4. (4) Since $p(x) = 0$ has purely imaginary roots,

$p(x) = ax^2 + c$, where a and c have same sign.

Also, $p(p(x)) = 0$

$\Rightarrow \quad p(x)$ is purely imaginary

$\Rightarrow \quad ax^2 + c$ is purely imaginary

Hence, x cannot be either purely real or purely imaginary.

5. (3) Solving given equations and from given values of θ which lie in fourth quadrant

$\alpha_1 = \sec\theta - \tan\theta$

$\beta_1 = \sec\theta + \tan\theta$

$\alpha_2 = \sec\theta - \tan\theta$

$\beta_2 = -\sec\theta - \tan\theta$

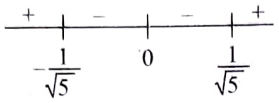
$\alpha_1 + \beta_2 = -2\tan\theta$

Multiple Correct Answers Type

1. **(1), (4)**

$|x_1 - x_2| < 1$

$\therefore \quad (x_1 + x_2)^2 - 4x_1x_2 < 1$



$\Rightarrow \quad \frac{1}{\alpha^2} - 4 < 1$

$\Rightarrow \quad 5 - \frac{1}{\alpha^2} > 0$

$\Rightarrow \quad \frac{5\alpha^2 - 1}{\alpha^2} > 0$

$\Rightarrow \quad \alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right) \qquad (1)$

Also $D > 0$

$\therefore \quad 1 - 4\alpha^2 > 0$

$\therefore \quad \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad (2)$

From (1) and (2)

$$\alpha \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{2}}\right)$$

Linked Comprehension Type

1. (2) Since α, β be the roots of the equation, $x^2 - x - 1 = 0$, ...(i)

$\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$

$a_n + a_{n-1} = p\alpha^n + q\beta^n + p\alpha^{n-1} + q\beta^{n-1}$

$= p\alpha^n + p\alpha^{n-1} + q\beta^n + q\beta^{n-1}$

$= p\alpha^{n-1}(\alpha + 1) + q\beta^{n-1}(\beta + 1)$

$= p\alpha^{n-1} \cdot \alpha^2 + q\beta^{n-1} \cdot \beta^2$ [Using (i)]

$= p\alpha^{n+1} + q\beta^{n+1}$

$= a_{n+1}$

So, $a_{12} = a_{11} + a_{10}$

2. (4) $a_4 = a_3 + a_2$

$= a_2 + a_1 + a_1 + a_0$

$= a_1 + a_0 + 2a_1 + a_0$

$= 2a_0 + 3a_1 = 2(p + q) + 3(p\alpha + q\beta)$

$= 2(p + q) + 3(p\alpha + q(1 - \alpha))$

$= (2p + 5q) + \alpha(3p - 3q)$

$= 28$ (given)

$\therefore \quad 2p + 5q = 28$ and $3p - 3q = 0$ as $p, q \in Q$ and α is irrational.

$\therefore \quad p = q = 4$

$\therefore \quad p + 2q = 12$

Numerical Value Type

1. (2) Let $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$.

$\therefore \quad f'(x) = 4x^3 - 12x^2 + 24x + 1$

and $f''(x) = 12x^2 - 24x + 24 > 0$ for all real x.

So, graph of $f'(x)$ intersects x-axis only once.

Hence, $f(x)$ has only one turning point.

Also, $f(0) = -1$.

So, graph of $f(x)$ cuts x-axis at two points.

Hence, $f(x) = 0$ has two real roots.

Concept Application Exercises

Exercise 3.1

1. The said computation is not correct, because -2 and -3 both are negative and $\sqrt{ab} = \sqrt{a}\sqrt{b}$ is true when at least one of a and b is positive or zero. The correct computation is

$(\sqrt{-2})(\sqrt{-3}) = (i\sqrt{2})(i\sqrt{3}) = i^2\sqrt{6} = -\sqrt{6}$

2. $\dfrac{i^{584}(i^8 + i^6 + i^4 + i^2 + 1)}{i^{574}(i^8 + i^6 + i^4 + i^2 + 1)} - 1 = \dfrac{i^{584}}{i^{574}} - 1$

$= i^{10} - 1 = -1 - 1 = -2$

3. Let $z = i^{[1+3+5+\cdots+(2n+1)]}$.

Now, $1 + 3 + 5 + \cdots + (2n + 1)$ are $n + 1$ terms of an A.P. whose sum is

$$\frac{n+1}{2}[1 + 2n + 1] = (n+1)^2$$

Hence, $z = i^{(n+1)^2}$. Now put $n = 1, 2, 3, 4, 5, \ldots$.

$n = 1, z = i^4 = 1$, $n = 2, z = i^9 = i$,

$n = 3, z = i^{16} = 1$, $n = 4$,

$z = i^{25} = i$, $n = 5, z = i^{12} = 1, \ldots$

Thus, $z = \begin{cases} 1, \text{ if } n \text{ is odd} \\ i, \text{ if } n \text{ is even} \end{cases}$

4. We have,

$x = -5 + 4i$

or $(x + 5)^2 = -16$

or $x^2 + 10x + 41 = 0$ (1)

Now, $x^4 + 9x^3 + 35x^2 - x + 4$

$= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41)$

$+ 4(x^2 + 10x + 41) - 160$

$= 0x^2 - 0x + 4 \times 0 - 160 = -160$ [Using (1)]

Exercise 3.2

1. We have $z = 4 - 3i$

Multiplicative inverse of z is

$$\frac{1}{z} = \frac{1}{4-3i} = \frac{4+3i}{(4-3i)\times(4+3i)} = \frac{4+3i}{16-9i^2} = \frac{4+3i}{25} = \frac{4}{25} + \frac{3}{25}i$$

2. **(a)** $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \dfrac{(6+6)+i(-4+9)}{(2+2)+i(4-1)}$

$= \dfrac{12+5i}{4+3i}$

$= \dfrac{12+5i}{4+3i}\dfrac{4-3i}{4-3i}$

$= \dfrac{(48+15)+i(-36+20)}{16-9i^2}$

$= \dfrac{63}{25} - \dfrac{16}{25}i$

(b) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}} = \dfrac{2-5i}{1-4i}$

$= \dfrac{2-5i}{1-4i} \times \dfrac{1+4i}{1+4i}$

$= \dfrac{(2+20)+i(8-5)}{1-16i^2}$

$= \dfrac{22+3i}{17} = \dfrac{22}{17} + \dfrac{3}{17}i$

3. $\left(\dfrac{2i}{1+i}\right)^2 = \left(\dfrac{2i(1-i)}{(1+i)(1-i)}\right)^n$

$= \left(\dfrac{2(i-i^2)}{2}\right)^n$

$= (i+1)^n$

$= (2i)^{n/2}$

Hence, $n = 8$ is the least positive integer for which the given complex number is a positive integer.

4. $z^2 - az + a - 1 = 0$ (1)

Putting $z = 1 + i$ in the equation, we get

$a = 2 + i$

$\Rightarrow$ $z^2 - (2+i)z + 1 + i = 0$ is the equation

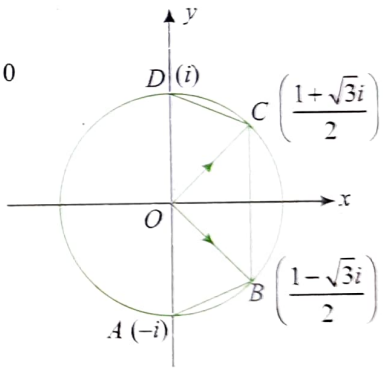
$\Rightarrow$ $z = 1$ is the other root

5. $z^4 - z^3 + 2z^2 - z + 1 = 0$

$\Rightarrow$ $(z^2 + 1)(z^2 - z + 1) = 0$

$\Rightarrow$ $z = \pm i, \dfrac{1 \pm \sqrt{3}i}{2}$

Plotting these points on the Argand plane, we find that $ABCD$ is an equilateral quadrilateral.



6. Let $Z = a + ib$,

$b \neq 0$ where $\text{Im } Z = b$

$\therefore$ $Z^5 = (a + ib)^5$

$= a^5 + 5a^4 bi + 10a^3 b^2 i^2 + 10a^2 b^3 i^3 + 5ab^4 i^4 + i^5 b^5$

$\therefore$ $\text{Im } Z^5 = 5a^4 b - 10a^2 b^3 + b^5$

$$y = \frac{\text{Im } Z^5}{(\text{Im } Z)^5} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1$$

$$= 5\left[\left(\frac{a}{b}\right)^4 - 2\left(\frac{a}{b}\right)^2 + \frac{1}{5}\right]$$

$$= 5\left[\left(\left(\frac{a}{b}\right)^2 - 1\right)^2 + \frac{1}{5} - 1\right]$$

$$= 5\left(\left(\frac{a}{b}\right)^2 - 1\right)^2 - 4$$

So, $y_{\min} = -4$

7. Let $z = (x - iy)(3 + 5i)$
$= 3x + 5xi - 3yi - 5yi^2$
$= (3x + 5y) + i(5x - 3y)$
$\therefore \quad \bar{z} = (3x + 5y) - i(5x - 3y)$
It is given that, $\bar{z} = -6 - 24i$
$\therefore \quad (3x + 5y) - i(5x - 3y) = -6 - 24i$
Equating real and imaginary parts, we get
$3x + 5y = -6$ (1)
$5x - 3y = 24$ (2)
Solving we get
$x = 3, y = -3$

8. $z_3 = (1 - \lambda)z_1 + z_2$
$= \dfrac{(1-\lambda)z_1 + \lambda z_2}{1 - \lambda + \lambda}$
Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda : (1 - \lambda)$. Thus, the given points are collinear.

9. $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$
$= (1+i)^{n_1} + (1-i)^{n_1} + (1+i)^{n_2} + (1-i)^{n_2}$
$= \left[(1+i)^{n_1} + \overline{(1+i)^{n_1}}\right] + \left[(1+i)^{n_2} + \overline{(1+i)^{n_2}}\right]$
= Purely real

Exercise 3.3

1. We have,
$(a + b) - i(3a + 2b) = 5 + 2i$
$\Rightarrow \quad a + b = 5$ and $-(3a + 2b) = 2$
$\Rightarrow \quad a = -12, b = 17$

2. Let $z = x + iy$. Then
$\bar{z} = iz^2$
$\Rightarrow \quad x - iy = i(x^2 - y^2 + 2ixy)$
$\Rightarrow \quad x - iy = i(x^2 - y^2) - 2xy$
$\Rightarrow \quad x(1 + 2y) = 0$ (1)
and $x^2 - y^2 + y = 0$ (2)
From (1), $x = 0$ or $y = -1/2$. From (2), when $x = 0$, $y = 0$, 1 and when $y = -1/2$, $x = \pm(\sqrt{3}/2)$. For nonzero complex number z,

$$z = i, \frac{\sqrt{3}}{2} - \frac{i}{2}, -\frac{\sqrt{3}}{2} - \frac{i}{2}$$

3. Let $i\alpha$ be a root where $\alpha \in R$
$\Rightarrow \quad -a\alpha^2 + bi\alpha + c + i = 0$
$\Rightarrow \quad c = a\alpha^2, 1 + b\alpha = 0$
$\Rightarrow \quad \alpha^2 = \dfrac{c}{a}, \alpha = \dfrac{-1}{b}$
$\Rightarrow \quad \dfrac{1}{b^2} = \dfrac{c}{a} \Rightarrow a = b^2c$

4. Let the roots be α and β.
We have $\alpha + \beta = -(p + iq)$; $\alpha\beta = 3i$
Given: $\alpha^2 + \beta^2 = 8$
or $(\alpha + \beta)^2 - 2\alpha\beta = 8$
or $(p + iq)^2 - 6i = 8$
or $p^2 - q^2 + i(2pq - 6) = 8$
or $p^2 - q^2 = 8$ and $pq = 3$
$\Rightarrow \quad p = 3$ and $q = 1$ or $p = -3$ and $q = -1$

5. Let $\sqrt{9 + 40i} = x + iy$. Then
$(x + iy)^2 = 9 + 40i$ (1)
or $x^2 - y^2 = 9$ (2)
and $xy = 20$
Squaring (1) and adding with 4 times the square of (2), we get
$x^4 + y^4 - 2x^2y^2 + 4x^2y^2 = 81 + 1600$
or $(x^2 + y^2)^2 = 1681$ (3)
or $x^2 + y^2 = 41$
From (1) + (3), we get
$x^2 = 25$ or $x = \pm 5$
and $y = \pm 4$
From Eq. (2), we can see that x and y are of same sign. Therefore,
$x + iy = (5 + 4i)$ or $-(5 + 4i)$

6. $\dfrac{\left(\sqrt{5+12i} + \sqrt{5-12i}\right)\left(\sqrt{5+12i} + \sqrt{5-12i}\right)}{\left(\sqrt{5+12i} - \sqrt{5-12i}\right)\left(\sqrt{5+12i} + \sqrt{5-12i}\right)}$
$= \dfrac{5 + 12i + 5 - 12i + 2\sqrt{5+12i}\,\sqrt{5-12i}}{5 + 12i - 5 + 12i}$
$= \dfrac{10 + 2 \times 13}{24i}$
$= -\dfrac{3}{2}i$

7. $\sqrt{x + iy} = \pm(a + bi)$
or $x + iy = a^2 - b^2 + 2iab$
or $x = a^2 - b^2, y = 2ab$
$\therefore \quad \sqrt{-x - iy} = \sqrt{-(a^2 - b^2) - 2iab}$
$= \sqrt{b^2 + (ia)^2 - 2iab}$
$= \sqrt{(b - ia)^2} = \pm(b - ia)$

Exercise 3.4

1. Complex cube roots of unity are ω, ω^2. Let $\alpha = \omega, \beta = \omega^2$. Then
$\alpha^4 + \beta^4 + \alpha^{-1}\beta^{-1} = \omega^4 + (\omega^2)^4 + (\omega^{-1})(\omega^2)^{-1}$
$= \omega + \omega^2 + 1 = 0$

2. $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \cdots$ up to $2n$ factors
$= (-\omega^2)(-\omega)(1 + \omega)(1 + \omega^2) \cdots$ up to $2n$ factors
$= 1 \times 1 \times 1 \times \cdots$ up to n factors $= 1$

3. **(a)** Here, $-1/2 + (1/2)i\sqrt{3}$ is one of the two imaginary cube roots of unity. If we denote it by ω, then
$\omega^{1000} = \omega^{999}\,\omega = (\omega^3)^{333}\,\omega = \omega = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$

(b) $z = \dfrac{(\sqrt{3} + i)^{17}}{(1 - i)^{50}}$
$= \dfrac{1}{i^{17}} \dfrac{(i\sqrt{3} + i^2)^{17}}{[(1 - i)^2]^{25}}$
$= \dfrac{2^{17}}{i} \dfrac{\left(\dfrac{i\sqrt{3} - 1}{2}\right)^{17}}{(-2i)^{25}}$
$= \dfrac{1}{2^8}(\omega)^{17}$
$= \dfrac{1}{2^8}(\omega)^2$
$= \dfrac{1}{2^8}\left(\dfrac{-1 - i\sqrt{3}}{2}\right)$

(c) $(i+\sqrt{3})^{100}+(i-\sqrt{3})^{100}+2^{100}$

$$=\left(\frac{i^2+i\sqrt{3}}{i}\right)^{100}+\left(\frac{i^2-i\sqrt{3}}{i}\right)^{100}+2^{100}$$

$$=\frac{2^{100}}{i^{100}}\left(\frac{-1+i\sqrt{3}}{2}\right)^{100}+\frac{2^{100}}{i^{100}}\left(\frac{-1-i\sqrt{3}}{2}\right)^{100}+2^{100}$$

$$=2^{100}(\omega)^{100}+2^{100}(\omega^2)^{100}+2^{100}$$

$$=2^{100}(\omega^{100}+\omega^{200}+1)$$

$$=2^{100}(\omega+\omega^2+1)=0$$

4. $z+z^{-1}=1$

or $z^2-z+1=0$

$\Rightarrow$ $z=-\omega$ or $-\omega^2$

For $z=-\omega$,

$$z^{100}+z^{-100}=(-\omega)^{100}+(-\omega)^{-100}$$

$$=\omega+\frac{1}{\omega}=\omega+\omega^2=-1$$

For $z=-\omega^2$,

$$z^{100}+z^{-100}=(-\omega^2)^{100}+(-\omega^2)^{-100}$$

$$=\omega^{200}+\frac{1}{\omega^{200}}$$

$$=\omega^2+\frac{1}{\omega^2}=\omega^2+\omega=-1$$

5. $x^{12}-1=(x^6+1)(x^6-1)=(x^6+1)(x^2-1)(x^4+x^2+1)$

Common roots are given by $x^4+x^2+1=0$

$\therefore$ $x^2=\dfrac{-1\pm i\sqrt{3}}{2}=\omega,\ \omega^2$ or $\omega^4,\ \omega^2$ $\quad(\because\ \omega^3=1)$

or $x=\pm\omega^2,\pm\omega$

6. $x^3-3x^2+3x+7=0$

$\Rightarrow$ $(x-1)^3=-8$

$\Rightarrow$ $\dfrac{x-1}{-2}=(1)^{1/3}=1,\ \omega,\ \omega^2$

$\Rightarrow$ $\alpha=-1,\ \beta=1-2\omega,\ \gamma=1-2\omega^2$

$\therefore$ $E=\dfrac{-2}{-2\omega}+\dfrac{-2\omega}{-2\omega^2}+\dfrac{-2\omega^2}{-2}$

$$=\frac{1}{\omega}+\frac{1}{\omega}+\frac{1}{\omega}=\frac{3}{\omega}=3\omega^2$$

7. Put $t+1=Z$.

Then $t^2+3t+3=(t+1)^2+(t+2)=Z^2+Z+1=(Z-\omega)(Z-\omega^2)$, where ω is a complex cube root of unity.

When $Z=\omega$, the expression $(t+1)^{n+1}+(t+2)^{2n-1}$ becomes

$$\omega^{n+1}+(\omega+1)^{2n-1}=\omega^{n+1}+(-\omega^2)^{2n-1}$$

$$=\omega^{n+1}+(-1)^{2n-1}\,\omega^{4n-2}$$

$$=\omega^{n+1}\{1+(-1)^{2n-1}\,\omega^{3n-3}\}$$

$$=\omega^{n+1}(1-1)=0 \text{ as } \omega^{3n-3}=1$$

$\therefore$ $Z-\omega$ is a factor of the expression.

Similarly, $Z-\omega^2$ is also a factor of the expression,

Exercise 3.5

1. (a) Let $z=-1-i\sqrt{3}$. Then $\alpha=\tan^{-1}\left|\dfrac{b}{a}\right|=\tan^{-1}\left|\dfrac{\sqrt{3}}{1}\right|=\dfrac{\pi}{3}$.

Clearly, z lies in III quadrant.

Therefore, argument $\theta=-(\pi-\alpha)=-(\pi-\pi/3)=\dfrac{-2\pi}{3}$

(b) $z=\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}=\dfrac{(1+\sqrt{3}i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}$

$$=\frac{2\sqrt{3}+2i}{4}=\frac{\sqrt{3}+i}{2}$$

$$\arg(z)=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$$

(c) $z=\sin\alpha+i(1-\cos\alpha)$

$$\Rightarrow\quad \arg(z)=\tan^{-1}\left(\frac{1-\cos\alpha}{\sin\alpha}\right)=\tan^{-1}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right)$$

$$=\tan^{-1}\tan\left(\frac{\alpha}{2}\right)=\frac{\alpha}{2}$$

(d) $z=(1+i\sqrt{3})^2=1-3+2i\sqrt{3}=-2+i\,(2\sqrt{3})$

So, z lies in second quadrant.

$$\therefore\quad \arg(z)=\pi-\tan^{-1}\left|\frac{2\sqrt{3}}{-2}\right|$$

$$=\pi-\tan^{-1}\sqrt{3}=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$$

2. $z=(\tan 1-i)^2=(\tan^2 1-1)-(2\tan 1)i$

$$|z|=\sqrt{(\tan^2 1-1)^2+4\tan^2 1}=\sqrt{(\tan^2 1+1)^2}=\sec^2 1$$

Since $\tan^2 1-1<0$ and $-2\tan 1<0$, so, z lies in the third quadrant.

$$\Rightarrow\quad \arg(z)=-\pi+\tan^{-1}\left|\frac{2\tan 1}{1-\tan^2 1}\right|=-\pi+\tan^{-1}|\tan 2|=2-\pi$$

3. $z=(1-\cos 2\alpha)+i\sin 2\alpha$

$$=2\sin^2\alpha+i(2\sin\alpha\cos\alpha)$$

$$=2\sin\alpha(\sin\alpha+i\cos\alpha)$$

$$=-2\sin\alpha(-\sin\alpha-i\cos\alpha)$$

$$=-2\sin\alpha\left(\cos\left(\frac{3\pi}{2}-\alpha\right)+i\sin\left(\frac{3\pi}{2}-\alpha\right)\right)$$

Thus, $|z|=-2\sin\alpha$.

Also, $\dfrac{3\pi}{2}-\alpha\in\left(\dfrac{-\pi}{2},0\right)$

Therefore, z lies in the fourth quadrant.

$\therefore$ $\arg(z)=\dfrac{3\pi}{2}-\alpha$

4. $Z=\sin\dfrac{6\pi}{5}+i\left(1+\cos\dfrac{6\pi}{5}\right)$

$$=2\cos\frac{3\pi}{5}\left(\sin\frac{3\pi}{5}+i\cos\frac{3\pi}{5}\right)$$

$$=2\cos\frac{3\pi}{5}\left(\cos\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)+i\sin\left(\frac{\pi}{2}-\frac{3\pi}{5}\right)\right)$$

$$=2\cos\frac{3\pi}{5}\left[\cos\left(\frac{-\pi}{10}\right)+i\sin\left(\frac{-\pi}{10}\right)\right]$$

$$=-2\cos\frac{3\pi}{5}\left[-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right]$$

$$=-2\cos\frac{3\pi}{5}\left[\cos\left(\pi-\frac{\pi}{10}\right)+i\sin\left(\pi-\frac{\pi}{10}\right)\right]$$

$$= -2\cos\frac{3\pi}{5}\left[\cos\frac{9\pi}{10} + i\sin\frac{9\pi}{10}\right]$$

$\therefore$ $\arg(Z) = \dfrac{9\pi}{10}$; and $|z| = -2\cos\dfrac{3\pi}{5}$

5. $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $iz = ir(\cos\theta + i\sin\theta)$

$= -r\sin\theta + ir\cos\theta$

$\Rightarrow$ $e^{iz} = e^{(-r\sin\theta + ir\cos\theta)}$

$= e^{-r\sin\theta}\, e^{ri\cos\theta}$

$\Rightarrow$ $|e^{iz}| = |e^{-r\sin\theta}|\,|e^{ri\cos\theta}|$

$= e^{-r\sin\theta}|e^{i\alpha}|$, where $\alpha = r\cos\theta$

$= e^{-r\sin\theta}[\cos^2\alpha + \sin^2\alpha]^{1/2}$

$= e^{-r\sin\theta}$

6. $z = x + iy$

or $z^2 = x^2 - y^2 + 2ixy$

$\Rightarrow$ $\mathrm{Re}(z^2) = x^2 - y^2$

Also, $|z| = \sqrt{x^2 + y^2}$

$\Rightarrow$ $x^2 - y^2 = 0,\ x^2 + y^2 = 3$

$\Rightarrow$ $x^2 = y^2 = \dfrac{3}{2}$

$\Rightarrow$ $x = \pm\sqrt{\dfrac{3}{2}}\ ,\ y = \pm\sqrt{\dfrac{3}{2}}$

$\Rightarrow$ $z = \pm\sqrt{\dfrac{3}{2}} \pm \sqrt{\dfrac{3}{2}}\ i$

Thus, there are four complex numbers.

7. $z = x + iy$

$\Rightarrow$ $\mathrm{Re}(z) = x,\ \mathrm{Im}(z) = y$

$|z - i\mathrm{Re}(z)| = |z - \mathrm{Im}(z)|$

$\Rightarrow$ $|x + iy - ix| = |x + iy - y|$

$\Rightarrow$ $x^2 + (x - y)^2 = (x - y)^2 + y^2$

$\Rightarrow$ $x^2 = y^2$

$\Rightarrow$ $|x| = |y|$

Hence, z lies on the bisectors the quadrants.

8. $|z + 5|^2 - |z - 5|^2 = 10$

or $(z + 5)(\bar{z} + 5) - (z - 5)(\bar{z} - 5) = 10$

or $5(z + \bar{z}) + 25 + 5(z + \bar{z}) - 25 = 10$

or $2 \times 2x = 2$

or $x = \dfrac{1}{2}$

which is the equation of a straight line.

9. $z^2 + |z| = 0$

$\Rightarrow$ $x^2 - y^2 + i(2xy) + \sqrt{x^2 + y^2} = 0$

$\Rightarrow$ $x^2 - y^2 + \sqrt{x^2 + y^2} = 0$ (1)

and $2xy = 0$ (2)

From (2), let $x = 0$. From (1),

$-y^2 + \sqrt{y^2} = 0$

$\Rightarrow$ $-|y|^2 + |y| = 0$

$\Rightarrow$ $|y| = 0$ or 1

$\Rightarrow$ $y = 0$ or $y = \pm 1$

From (2), if $y = 0$, then from (1),

$x^2 + \sqrt{x^2} = 0$

$\Rightarrow$ $|x|^2 + |x| = 0$

$\Rightarrow$ $x = 0$

Hence, complex numbers are $0 + i0,\ 0 + i,\ 0 - i$.

10. $z = x + iy$

$A = \{z : |z| \le 2\}$

$\Rightarrow$ $\sqrt{x^2 + y^2} \le 2$

or $x^2 + y^2 \le 4$

$\Rightarrow$ z lies on or inside the circle $x^2 + y^2 = 4$

$B = \{z : (1 - i)z + (1 + i)\bar{z} \ge 4\}$

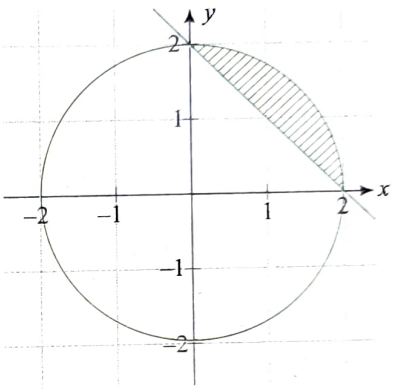
$\Rightarrow$ $(1 - i)(x + iy) + (1 + i)(x - iy) \ge 4$

$\Rightarrow$ $x + iy - ix + y + x - iy + ix + y \ge 4$

$\Rightarrow$ $x + y \ge 2$

Area of region $A \cap B$ is the shaded region shown in the figure.

$$\text{Area} = \frac{\pi(2)^2}{4} - \frac{1}{2}\times 2\times 2 = \pi - 2$$



11. $e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta}\,[e^{i\sin\theta}]$

$= e^{\cos\theta}[\cos(\sin\theta) + i\sin(\sin\theta)]$

Therefore, the real part is $e^{\cos\theta}[\cos(\sin\theta)]$.

12. $z = i^i = \left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)^i = (e^{i\pi/2})^i = e^{-\frac{\pi}{2}}$

$\Rightarrow$ $\mathrm{Re}(z) = e^{-\frac{\pi}{2}}$

Exercise 3.6

1. $|z_1| = \left|\dfrac{1 - i}{1 + i\sqrt{3}}\right|^{\frac{1}{6}} = \left|\dfrac{\sqrt{2}}{2}\right|^{\frac{1}{6}} = 2^{-\frac{1}{12}}$

Similarly,

$|z_2| = 2^{-\frac{1}{12}}\ ;\ |z_3| = 2^{-\frac{1}{12}}$

Hence, the result.

2. We have,

$\sqrt{3} + i = (a + ib)(c + id)$

$\Rightarrow$ $ac - bd = \sqrt{3}$ and $ad + bc = 1$

Now, $\tan^{-1}\left(\dfrac{b}{a}\right) + \tan^{-1}\left(\dfrac{d}{c}\right) = \tan^{-1}\left(\dfrac{\frac{b}{a} + \frac{d}{c}}{1 - \frac{b}{a}\frac{d}{c}}\right)$

$= \tan^{-1}\left(\dfrac{bc + ad}{ac - bd}\right)$

$= \tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) = n\pi + \dfrac{\pi}{6}\ ,\ n \in Z$

3. We have $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$. Therefore,

$z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

Now, $\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1 z_2}{z_4 z_3}\right)$

$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right)$

$= \arg\left(\left|\frac{z_1}{z_3}\right|^2\right) = 0$

4. $z = \dfrac{i-1}{i\left(1-\cos\frac{2\pi}{5}\right)+\sin\frac{2\pi}{5}}$

$= \dfrac{i-1}{i2\sin^2\frac{\pi}{5}+2\sin\frac{\pi}{5}\cos\frac{\pi}{5}}$

$= \dfrac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}$

$|z| = \dfrac{|i-1|}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|}$

$= \dfrac{\sqrt{2}}{\left(2\sin\frac{\pi}{5}\right)\left|\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)\right|}$

$= \frac{1}{\sqrt{2}}\operatorname{cosec}\frac{\pi}{5}$

$\arg z = \arg\left[\dfrac{i-1}{\left(2\sin\frac{\pi}{5}\right)\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)}\right]$

$= \arg(-1+i) - \arg\left(2\sin\frac{\pi}{5}\right) - \arg\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$

$= \frac{3\pi}{4} - 0 - \frac{\pi}{5} = \frac{11\pi}{20}$

5. We have,

$(1+i)(1+2i)(1+3i)\cdots(1+ni) = x+iy$

$\Rightarrow \quad |(1+i)(1+2i)\cdots(1+ni)| = |x+iy|$

$\Rightarrow \quad |1+i|\,|1+2i|\cdots|1+ni| = |x+iy|$

$[\because |z_1 z_2 \cdots z_n| = |z_1|\,|z_2|\cdots|z_n|]$

$\Rightarrow \quad \sqrt{1+1}\sqrt{1+4}\cdots\sqrt{1+n^2} = \sqrt{x^2+y^2}$

$\Rightarrow \quad 2\times 5\times 10\cdots(1+n^2) = (x^2+y^2)$ [On squaring both sides]

6. $a+ib = \dfrac{(x+i)^2}{2x^2+1}$

or $\quad |a+ib| = \left|\dfrac{(x+i)^2}{2x^2+1}\right|$

or $\quad \sqrt{a^2+b^2} = \dfrac{|x+i|^2}{2x^2+1}$

or $\quad a^2+b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$

7. $(z^3+3)^2 = 16i^2$

$z^3+3 = 4i$ or $-4i$

$z^3 = -3+4i$ or $-3-4i$

$|z^3| = |-3+4i| = 5$

$|z|^3 = 5$

$\Rightarrow \quad |z| = 5^{1/3}$

8. $z_1^2 + z_2^2 + 2z_1 z_2\cos\theta = 0$

or $\quad \left(\frac{z_1}{z_2}\right)^2 + 2\left(\frac{z_1}{z_2}\right)\cos\theta + 1 = 0$

or $\quad \left(\frac{z_1}{z_2}+\cos\theta\right)^2 = -(1-\cos^2\theta) = -\sin^2\theta$

or $\quad \frac{z_1}{z_2} = -\cos\theta \pm i\sin\theta$

or $\quad \left|\frac{z_1}{z_2}\right| = \sqrt{(-\cos\theta)^2+\sin^2\theta} = 1$

or $\quad |z_1| = |z_2|$

or $\quad |z_1 - 0| = |z_2 - 0|$

Thus, triangle with vertices O, z_1, z_2 is iscosceles.

9. Here, $\left|\frac{z_2-z_0}{z_0-z_1}\right| = 1$ and $\operatorname{amp}\left(\frac{z_2-z_0}{z_0-z_1}\right) = \frac{\pi}{2}$

$\therefore \quad \frac{z_2-z_0}{z_0-z_1} = 1\left\{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right\} = i$

$\Rightarrow \quad z_2 - z_0 = iz_0 - iz_1$

$\Rightarrow \quad z_2 + iz_1 = (i+1)z_0$

or $\quad z_0 = \dfrac{z_2+iz_1}{1+i}$

$= \dfrac{(z_2+iz_1)(1-i)}{1^2+1^2}$

$= \frac{1}{2}\{(i+1)z_1 + (1-i)z_2\}$

10. $\left|\frac{z-2}{z-3}\right| = 2$

$\Rightarrow \quad |z-2|^2 = 4|z-3|^2$

$\Rightarrow \quad |x-2+iy|^2 = 4|x-3+iy|^2$

$\Rightarrow \quad (x-2)^2 + y^2 = 4[(x-3)^2+y^2]$

$\Rightarrow \quad 3x^2+3y^2-24x+4x+36-4 = 0$

$\Rightarrow \quad x^2+y^2-\frac{20}{3}x+\frac{32}{3} = 0$

$\Rightarrow \quad \left(x-\frac{10}{3}\right)^2 + y^2 = \frac{4}{9}$

Thus, distance of (x, y) from the point $(10/3, 0)$ is $2/3$.

So, centre of the circle is $(10/3, 0)$ and radius is $2/3$.

Exercise 3.7

1. (a) $\dfrac{(\cos\alpha+i\sin\alpha)^4}{(\sin\beta+i\cos\beta)^5} = \dfrac{\cos 4\alpha+i\sin 4\alpha}{i^5(\cos\beta-i\sin\beta)^5}$

$= -i(\cos 4\alpha+i\sin 4\alpha)(\cos\beta-i\sin\beta)^{-5}$

$= -i[\cos 4\alpha+i\sin 4\alpha][\cos 5\beta+i\sin 5\beta]$

$= -i[\cos(4\alpha+5\beta)+i\sin(4\alpha+5\beta)]$

$= \sin(4\alpha+5\beta) - i\cos(4\alpha+5\beta)$

(b) $\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n$

$$=\left[\frac{2\cos^2(\phi/2)+i\sin(\phi/2)\cos(\phi/2)}{2\cos^2(\phi/2)-2i\sin(\phi/2)\cos(\phi/2)}\right]^n$$

$$=\left[\frac{\cos(\phi/2)+i\sin(\phi/2)}{\cos(\phi/2)-i\sin(\phi/2)}\right]^n$$

$$=\left[\frac{(\cos\phi+i\sin\phi)^{1/2}}{(\cos\phi+i\sin\phi)^{-1/2}}\right]^n$$

$$=[\cos\phi+i\sin\phi]^n$$
$$=\cos n\phi+i\sin n\phi$$

(c) $\frac{(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)}{(\cos\gamma+i\sin\gamma)(\cos\delta+i\sin\delta)}$

$$=\cos(\alpha+\beta-\gamma-\delta)+i\sin(\alpha+\beta-\gamma-\delta)$$

Let $\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}=z$

$\Rightarrow \quad \cos\frac{\pi}{10}+i\sin\frac{\pi}{10}=\frac{1}{z}$

$$\Rightarrow \quad \left[\frac{1-\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}}{1-\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}}\right]^{10}=\left(\frac{1-z}{1-\frac{1}{z}}\right)^{10}$$

$$=\left\{\frac{-(z-1)z}{(z-1)}\right\}^{10}$$
$$=(-z)^{10}$$
$$=z^{10}$$
$$=\left(\cos\frac{\pi}{10}-i\sin\frac{\pi}{10}\right)^{10}$$
$$=\cos\pi-i\sin\pi$$
$$=-1$$

$iz^4=-1$

$$z^4=\frac{-1}{i}$$

or $\quad z^4=i$

or $\quad z=(i)^{1/4}$

or $\quad z=(0+i)^{1/4}$

or $\quad z=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{1/4}=\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$

4. $(1+i)^n+(1-i)^n$

$$=(2)^{n/2}\left\{\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}+\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right\}$$

$$=2^{\frac{n}{2}}\,2\cos\frac{n\pi}{4}=2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}=(\sqrt{2})^{n+2}\cos\frac{n\pi}{4}$$

5. Let $a=r\cos\theta$, $b=r\sin\theta$

Then $z=(a+ib)^5+(b+ia)^5$

$$=r^5\{(\cos\theta+i\sin\theta)^5+(\sin\theta+i\cos\theta)^5\}$$

$$=r^5\left[(\cos5\theta+i\sin5\theta)+\left\{\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)\right\}^5\right]$$

$$=r^5\left[(\cos5\theta+i\sin5\theta)+\cos5\left(\frac{\pi}{2}-\theta\right)+i\sin5\left(\frac{\pi}{2}-\theta\right)\right]$$

$$=r^5[(\cos5\theta+\sin5\theta)(1+i)]$$

Clearly $\mathrm{Re}(z)=\mathrm{Im}(z)=r^5(\cos5\theta+\sin5\theta)$

6. Let $z_1=\cos\alpha+i\sin\alpha$, $z_2=\cos\beta+i\sin\beta$,

$z_3=\cos\gamma+i\sin\gamma$

$\therefore \quad z_1+z_2+z_3=(\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)$

$$=0+i\times0=0 \qquad (1)$$

(a) Now, $\frac{1}{z_1}=(\cos\alpha+i\sin\alpha)^{-1}=\cos\alpha-i\sin\alpha$

$$\frac{1}{z_2}=\cos\beta-i\sin\beta$$

$$\frac{1}{z_3}=\cos\gamma-i\sin\gamma$$

$\therefore \quad \frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}$

$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma) \qquad (2)$$
$$=0-i\times0=0$$

$$z_1^2+z_2^2+z_3^2=(z_1+z_2+z_3)^2-2(z_1z_2+z_2z_3+z_3z_1)$$

$$=0-2z_1z_2z_3\left(\frac{1}{z_3}+\frac{1}{z_1}+\frac{1}{z_2}\right)$$

$$=0-2z_1z_2z_3\times0=0 \quad \text{[Using (1) and (2)]}$$

$\Rightarrow (\cos\alpha+i\sin\alpha)^2+(\cos\beta+i\sin\beta)^2+(\cos\gamma+i\sin\gamma)^2=0$

$\Rightarrow (\cos2\alpha+i\sin2\alpha)+(\cos2\beta+i\sin2)+(\cos2\gamma+i\sin2\gamma)$

$$=0+i\times0$$

Equating real and imaginary parts on both sides,

$$\cos2\alpha+\cos2\beta+\cos2\gamma=0$$

and $\quad \sin2\alpha+\sin2\beta+\sin2\gamma=0$

(b) $z_1^3+z_2^3+z_3^3=(z_1+z_2)^3-3z_1z_2(z_1+z_2)+z_3^3$

$$=(-z_3)^3-3z_1z_2(-z_3)+z_3^3 \quad \text{[Using (1)]}$$
$$=3z_1z_2z_3$$

$\Rightarrow \quad (\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$

$\Rightarrow \quad \cos3\alpha+i\sin3\alpha+\cos3\beta+i\sin3\beta+\cos3\gamma+i\sin3\gamma$

$$=3\{\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)\}$$

Equating imaginary parts on both sides,

$$\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma)$$

(c) Equating real parts on both sides,

$$\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma)$$

Exercise 3.8

1. a, b, c are complex numbers on the unit circle $|z|=1$

$\therefore \quad |a|=|b|=|c|=1$

$\therefore \quad a\bar{a}=b\bar{b}=c\bar{c}=1$

$abc=a+b+c$

$\Rightarrow \quad |abc|=|a+b+c|$

$$=|\bar{a}+\bar{b}+\bar{c}|$$
$$=\left|\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right|$$
$$=\left|\frac{ab+bc+ca}{abc}\right|$$

$\therefore \quad |ab+bc+ca|=|abc|\,|abc|=(|a|\,|b|\,|c|)^2=1$

2. $\dfrac{1+z+z^2}{1-z+z^2} = \dfrac{1-z+z^2+2z}{1-z+z^2}$

$= 1 + \dfrac{2}{z-1+(1/z)}$ is real

Hence, $z + \dfrac{1}{z} \in R$

$\Rightarrow \quad z + \dfrac{1}{z} = \bar{z} + \dfrac{1}{\bar{z}}$

$\Rightarrow \quad (z - \bar{z}) = \dfrac{1}{\bar{z}} - \dfrac{1}{z} = \dfrac{z - \bar{z}}{|z|^2}$

$\Rightarrow \quad (z - \bar{z})(|z|^2 - 1) = 0$

$\Rightarrow \quad |z| = 1$ ($z = \bar{z}$ is not possible as z is not real)

3. $\dfrac{a}{|z_1 - z_2|} = \dfrac{b}{|z_2 - z_3|} = \dfrac{c}{|z_3 - z_1|} = \lambda$ (say)

$\Rightarrow \quad a^2 = \lambda^2 (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$b^2 = \lambda^2 (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)$

$c^2 = \lambda^2 (z_3 - z_1)(\bar{z}_3 - \bar{z}_1)$

$\Rightarrow \quad \dfrac{a^2}{z_1 - z_2} + \dfrac{b^2}{z_2 - z_3} + \dfrac{c^2}{z_3 - z_1}$

$= \lambda^2 (\bar{z}_1 - \bar{z}_2 + \bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1) = 0$

4. Given that $|z_1| < 1 < |z_2|$.

Now, $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$

or $|1 - z_1\bar{z}_2| < |z_1 - z_2|$

or $|1 - z_1\bar{z}_2|^2 < |z_1 - z_2|^2$

or $(1 - z_1\bar{z}_2)\left(\overline{1 - z_1\bar{z}_2}\right) < (z_1 - z_2)\left(\overline{z_1 - z_2}\right)$

or $(1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

or $1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_1\bar{z}_1 z_2\bar{z}_2 < z_1\bar{z}_1 - z_1\bar{z}_2 - \bar{z}_1 z_2 + z_2\bar{z}_2$

or $1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$

or $(1 - |z_1|^2)(1 - |z_2|^2) < 0$

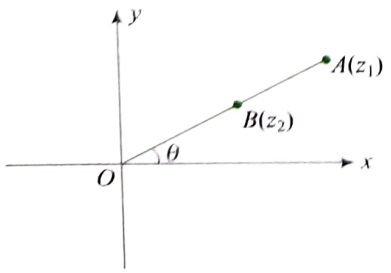
which is obviously true as

$|z_1| < 1 < |z_2|$

$\Rightarrow \quad |z_1|^2 < 1 < |z_2|^2$

$\Rightarrow \quad (1 - |z_1|^2) > 0$ and $(1 - |z_2|^2) < 0$

5. **(a)** $|z_1 - z_2| = |z_1| - |z_2|$



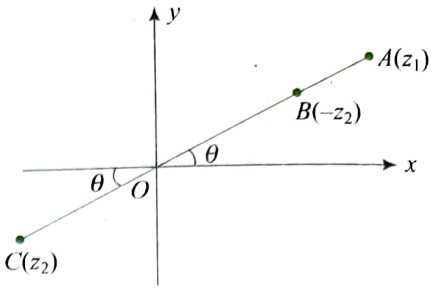
$\Rightarrow \quad AB = OA - OB$ as shown in figure

$\Rightarrow$ Points $A(z_1)$, $O(0)$, $B(z_2)$ are collinear

$\Rightarrow \quad \arg(z_1) = \arg(z_2) = \theta$

(b) $|z_1 + z_2| = |z_1| - |z_2|$

$\Rightarrow$ Points $A(z_1)$, $O(0)$, $B(-z_2)$ are collinear as shown in figure



$\Rightarrow$ Points $A(z_1)$, $O(0)$, $C(z_2)$ are collinear

If $\arg(z_1) = \theta$, then $\arg(z_2) = \theta - \pi$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \pi$

6. We have, for $z \in C$

$|2i| = |z + (2i - z)|$

$\le |z| + |2i - z|$

$\Rightarrow \quad 2 \le |z| + |z - 2i|$

Thus, minimum value of $|z| + |z - 2i|$ is 2.

7. $|z + 1| = |z + 4 - 3|$

$= |(z + 4) + (-3)|$

$\le |z + 4| + |-3|$

$= |z + 4| + 3$

$\le 3 + 3 = 6$ [$\because |z + 4| \le 3$]

Hence, the greatest value of $|z + 1|$ is 6.

8. $\left|Z + \dfrac{1}{Z}\right| = \left|Z - \left(-\dfrac{1}{Z}\right)\right| \ge |Z| \sim \left|-\dfrac{1}{Z}\right| \ge 3 \sim \dfrac{1}{3} = \dfrac{8}{3}$

9. $|a| = |b| = |c| = r$

Again $az^2 + bz = -c$

$\Rightarrow \quad |c| = |-az^2 - bz| \le |a||z^2| + |b||z|$

$\Rightarrow \quad r \le r|z|^2 + r|z|$

$\Rightarrow \quad |z|^2 + |z| - 1 \ge 0$

$\Rightarrow \quad |z| \ge \dfrac{\sqrt{5} - 1}{2}$ (1)

Also from $az^2 = -bz - c$,

$|z|^2 - |z| - 1 \le 0$

$\Rightarrow \quad 0 < |z| \le \dfrac{\sqrt{5} + 1}{2}$ (2)

From (1) and (2),

$\dfrac{\sqrt{5} - 1}{2} \le |z| \le \dfrac{\sqrt{5} + 1}{2}$

10. $|iz + 3 - 4i| \le |iz| + |3 - 4i|$

$= |z| + 5 \le 4 + 5 = 9$

Hence, $|z|_{\max} = 9$.

11. $z\bar{z} = 1$ or $z = \dfrac{1}{\bar{z}}$

Now, $z = (z_1 + z_2 + z_3 + \cdots + z_n)\left(\dfrac{1}{z_1} + \dfrac{1}{z_2} + \cdots + \dfrac{1}{z_n}\right)$

$= (z_1 + z_2 + \cdots + z_n)(\bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n)$

$= (z_1 + z_2 + \cdots + z_n)\overline{(z_1 + z_2 + \cdots + z_n)}$

$= |z_1 + z_2 + \cdots + z_n|^2$ which is real (a)

$\le (|z_1| + |z_2| + \cdots + |z_n|)^2 = n^2$

$\therefore \quad 0 < z \le n^2$

Also, z is real number.

Exercise 3.9

1. $\omega = \dfrac{z}{z - \frac{1}{3}i}$

or $\quad |\omega| = \left|\dfrac{z}{z - \frac{1}{3}i}\right|$.

or $\quad |z| = |\omega| \left|z - \dfrac{1}{3}i\right|$

or $\quad |z| = \left|z - \dfrac{1}{3}i\right|$ $\quad [\because |\omega| = 1]$

Hence, locus of z is perpendicular bisector of the line joining $0 + 0i$ and $0 + (1/3)i$. Hence z lies on a straight line.

2. Let $u = \dfrac{z-1}{e^{\theta i}} \Rightarrow \dfrac{e^{\theta i}}{z-1} = \dfrac{1}{u}$

Now $\left(u + \dfrac{1}{u}\right) - \left(\bar{u} + \dfrac{1}{\bar{u}}\right) = 0$

$\Rightarrow \quad (u - \bar{u})\left(1 - \dfrac{1}{u\bar{u}}\right) = 0$

If u is not purely real, then $u\bar{u} = 1$

$\Rightarrow \quad \left|\dfrac{z-1}{e^{\theta i}}\right| = 1 \quad \Rightarrow \quad |z - 1| = 1$

Hence, z lies on circle having center at $1 + i0$ and radius 1

3. Consider the triangle formed by $A(Z)$, $B(Z_1)$ and $C(Z_2)$.

Let midpoint of BC be D having complex number $\dfrac{Z_1 + Z_2}{2}$.

By Apollonius theorem, we have

$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

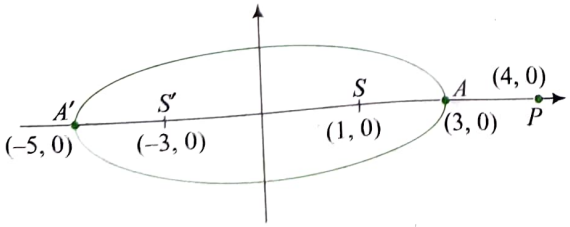
$$\therefore \quad \left|Z - \frac{Z_1 + Z_2}{2}\right|^2 + \left|\frac{Z_1 - Z_2}{2}\right|^2$$

$$= AD^2 + BD^2$$

$$= \frac{1}{2}AB^2 + \frac{1}{2}AC^2$$

$$= \frac{1}{2}|Z - Z_1|^2 + \frac{1}{2}|Z - Z_2|^2$$

4.



Given $|z - 1| + |z + 3| \le 8$. Then z lies inside or on the ellipse whose foci are $(1, 0)$ and $(-3, 0)$ and vertices are $(-5, 0)$ and $(3, 0)$. Clearly, the minimum and maximum values of $|z - 4|$ are 1 and 9, respectively, representing the distances PA and PA'. Thus, $1 \le |z - 4| \le 9$.

5. Given $z = \dfrac{3}{2 + \cos\theta + i\sin\theta}$

$$\Rightarrow \quad \cos\theta + i\sin\theta = \frac{3}{z} - 2 = \frac{3 - 2z}{z}$$

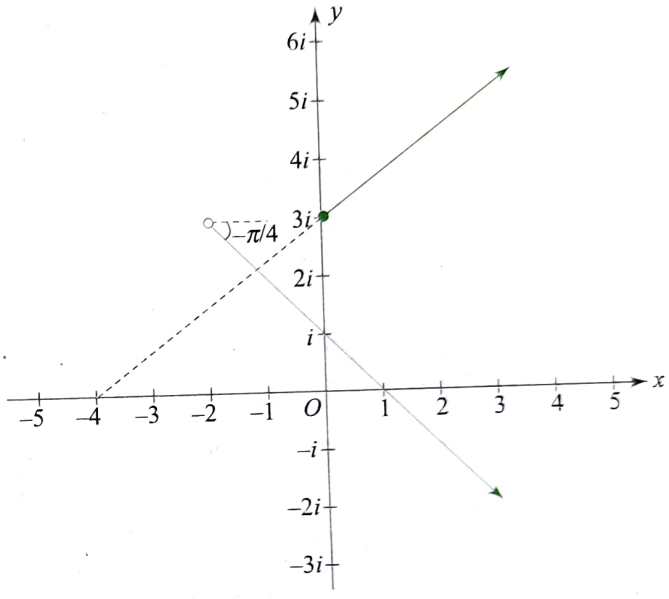
$$\Rightarrow \quad 1 = \frac{|3 - 2z|}{|z|} \quad \text{[taking modulus]}$$

$$\Rightarrow \quad \frac{\left|z - \frac{3}{2}\right|}{|z|} = \frac{1}{2}$$

Therefore, locus of z is circle.

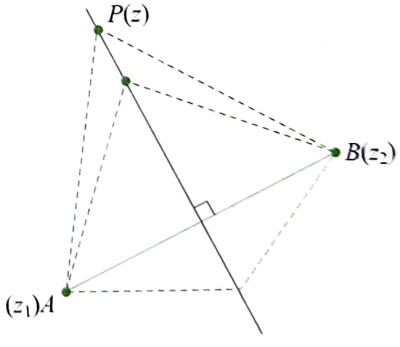
6. Equation $\arg(z - (-2 + 3i)) = -\pi/4$ represents the ray emanating from $(-2 + 3i)$ and making an angle of $-\pi/4$ with horizontal.

Equation $|z + 4| - |z - 3i| = 5$ or $|z - (-4)| - |z - 3i| = |3i + 4|$ represents the ray emanating from '$3i$' and moving to the right of '$3i$'.



Clearly, these two rays do not intersect. So, system of equations has no solution.

7. Let z be the complex number of any point on the perpendicular bisector.



Then $PA = PB$

$\Rightarrow \quad |z - z_1| = |z - z_2|$

$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow \quad |z|^2 + |z_1|^2 - z\bar{z}_1 - z_1\bar{z} = |z|^2 + |z_2|^2 - z\bar{z}_2 - z_2\bar{z}$

$\Rightarrow \quad z(\bar{z}_2 - \bar{z}_1) + \bar{z}(z_2 - z_1) + |z_1|^2 - |z_2|^2 = 0$

8. We have $|z + 1 - i| = 1$ $\quad (1)$

and $\quad w = \dfrac{z + i}{1 - i}$

$\therefore \quad w(1 - i) = z + i$

$\Rightarrow \quad z = w(1 - i) - i$

Putting the value of z in (1), we get

$$|w(1 - i) - i + 1 - i| = 1$$

$$\Rightarrow \quad |1-i|\left|w+\frac{1-2i}{1-i}\right|=1$$

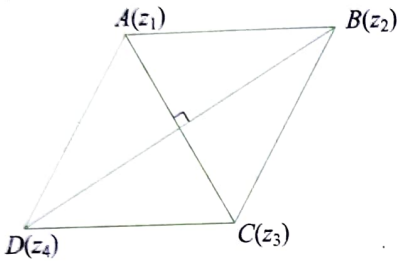
$$\Rightarrow \quad \left|w+\frac{3-i}{2}\right|=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \left|w-\frac{-3+i}{2}\right|=\frac{1}{\sqrt{2}}$$

Therefore, locus of w is a circle having center at (–3/2, 1/2) and radius $\frac{1}{\sqrt{2}}$.

Exercise 3.10

1. Since $ABCD$ is rhombus, $AC \perp BD$.



$$\arg\left(\frac{z_3-z_1}{z_4-z_2}\right)=\pm\frac{\pi}{2}$$

Therefore, $\frac{z_3-z_1}{z_4-z_2}$ is purely imaginary.

$$\therefore \quad \text{Re}\left(\frac{z_3-z_1}{z_4-z_2}\right)=0$$

2. Given that points z, i and iz are collinear.

$$\therefore \quad \arg\left(\frac{z-i}{iz-i}\right)=0,\ \pi$$

So, $\frac{z-i}{iz-i}$ is purely real.

$$\Rightarrow \quad \frac{z-i}{iz-i}=\frac{\bar{z}+i}{-i\bar{z}+i}$$

$$\Rightarrow \quad \frac{z-i}{z-1}=\frac{\bar{z}+i}{-\bar{z}+1}$$

$$\Rightarrow \quad 2z\bar{z}+iz-i\bar{z}-\bar{z}-z=0$$

$$\Rightarrow \quad 2(x^2+y^2)-2x-2y=0$$

$$\Rightarrow \quad x^2+y^2-x-y=0$$

3. For circle $|z|=2$, points $z=2$ and $z=-2$ are end points of diameter.

$$\Rightarrow \quad \arg\left(\frac{z-2}{z+2}\right)=\pm\frac{\pi}{2}$$

$$\Rightarrow \quad \arg\left(\frac{z_1-z_3}{z_2-z_3}\right)=\pm\frac{\pi}{2}$$

Therefore, z_1, z_2, and z_3 are the vertices of a right angled triangle.

4. Since $AD \perp BC$

$$\Rightarrow \quad \arg\left(\frac{\alpha-z}{\gamma-\beta}\right)=\frac{\pi}{2}$$

So, $\frac{z-\alpha}{\beta-\gamma}$ is purely imaginary.

$$\Rightarrow \quad \text{Re}\left(\frac{z-\alpha}{\beta-\gamma}\right)=0$$

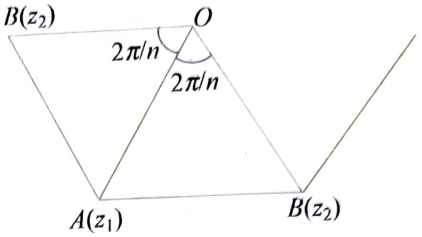
5. We know that if z_1, z_2 and z_3 form an equilateral triangle, then

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Putting $z_3=0$, we get

$$z_1^2+z_2^2-z_1z_2=0$$

6. Let A be the vertex with affix z_1.

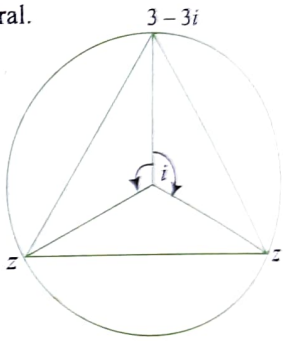


z_2 can be obtained by rotating z_1 through $2\pi/n$ either in clockwise or in anticlockwise direction. Therefore,

$$\frac{z_2}{z_1}=\left|\frac{z_2}{z_1}\right|e^{\pm\frac{i2\pi}{n}}$$

or $\quad z_2=z_1\,e^{\pm\frac{i2\pi}{n}} \qquad [\because |z_2|=|z_1|]$

7. Inscribed triangle is clearly equilateral. Let the other vertex be z.

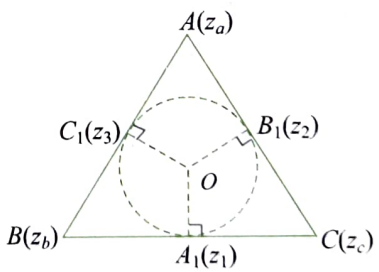


$$\therefore \quad \frac{z-i}{3-3i-i}=e^{\pm i\frac{2\pi}{3}}$$

or $\quad z-i=(3-4i)e^{\pm i\frac{2\pi}{3}}$

or $\quad z=i+(3-4\text{i})\left(-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}\right)$

8. Let $A(z_a)$, $B(z_b)$ and $C(z_c)$ be the vertices of the triangle.



Considering the rotation about points A_1 and B_1, respectively, we get

$$\frac{z_c-z_1}{0-z_1}=\frac{|z_c-z_1|}{r}e^{-i\pi/2} \quad \text{and} \quad \frac{z_c-z_2}{0-z_2}=\frac{|z_c-z_2|}{r}e^{i\pi/2}$$

$$\therefore \quad \frac{(z_c-z_1)z_2}{z_1(z_c-z_2)}=e^{-i\pi}=-1$$

$$\Rightarrow \quad (z_c-z_1)z_2=-z_1(z_c-z_2)$$

or $\quad z_c=\frac{2z_1z_2}{(z_1+z_2)}$

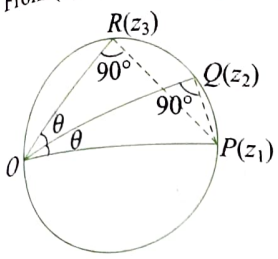
Similarly,

$$z_a=\frac{2z_2z_3}{(z_2+z_3)} \text{ and } z_b=\frac{2z_1z_3}{(z_1+z_3)}$$

9. $z_2=\frac{OQ}{OP}z_1e^{i\theta}=\cos\theta\, z_1e^{i\theta}$ (1)

$z_3=\frac{OR}{OP}z_1e^{i2\theta}=\cos 2\theta\, z_1e^{i2\theta}$ (2)

From (1), $z_2^2 = \cos^2\theta\, z_1^2 e^{2i\theta}$ (3)

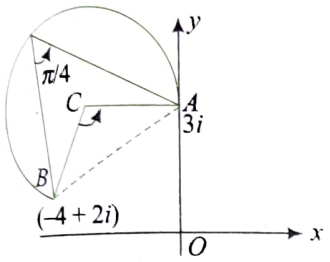


Dividing (3) by (2), we get

$$\frac{z_2^2}{z_3} = \frac{\cos^2\theta\, z_1}{\cos 2\theta}$$

Hence, $z_2^2 \cos 2\theta = z_1 z_3 \cos^2\theta$

10. If C is the center of the arc, then $\angle BCA = \pi/2$.



Let C be z_c. Then,

$$\frac{z_c - 3i}{z_c - 2i + 4} = e^{i\pi/2} = i$$

$$z_c = 3i + i(z_c - 2i + 4)$$

$$\therefore \quad z_c = \frac{7i+2}{(1-i)} = \frac{1}{2}(9i - 5)$$

Exercise 3.11

1. α is a complex fifth roots of unity.

$\therefore \quad 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$

Now, given that

$$(1 + \alpha + \alpha^2 + \alpha^3)^{2005} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow \quad (-\alpha^4)^{2005} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow \quad -\alpha^{8020} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow \quad -(\alpha^5)^{1604} = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow \quad -1 = p + q\alpha + r\alpha^2 + s\alpha^3$$

$$\Rightarrow \quad p = -1, q = r = s = 0$$

$$\therefore \quad p + q + r + s = -1$$

2. We have $z^{15} = 1$

$$z = 1^{\frac{1}{15}} = e^{i\frac{2r\pi}{15}}, \; r = 0, 1, 2, \ldots, 14$$

or $\quad z = e^{\pm i\frac{2r\pi}{15}}, r = 0, 1, 2, \ldots, 7.$

The complex numbers having their arguments between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are $e^0, e^{\pm i\frac{2\pi}{15}}, e^{\pm i\frac{4\pi}{15}}, e^{\pm i\frac{6\pi}{15}}$.

Hence number of solutions is 7.

3. Let $z = \sqrt[7]{-1}$. Then

$$z^7 = -1$$

$$\therefore \quad z^{86} + z^{175} + z^{289}$$

$$= (z^7)^{12} \times z^2 + (z^7)^{25} + (z^7)^{41} \times z^2$$

$$= (-1)^{12} \times z^2 + (-1)^{25} + (-1)^{41} \times z^2$$

$$= z^2 - 1 - z^2 = -1$$

4. As α is the fifth nonreal root of unity, we have

$$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$$

β is the fourth nonreal root of unity. Therefore,

$$\beta^3 + \beta^2 + \beta + 1 = 0$$

Now, $(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \beta)(1 + \beta^2)(1 + \beta^3)$

$$= (1 + \alpha + \alpha^2 + \alpha^3)(1 + \alpha^4)(1 + \beta + \beta^2 + \beta^3)(1 + \beta^3)$$

$$= 0 \qquad (\because \; 1 + \beta + \beta^2 + \beta^3 = 0)$$

5. $x^6 = -64$

$$\Rightarrow \quad \left(\frac{x}{2i}\right)^6 = 1$$

$$\Rightarrow \quad \frac{x}{2i} = \cos\left(\frac{2n\pi}{6}\right) + i\sin\left(\frac{2n\pi}{6}\right), \text{ where } n = 0, 1, 2, 3, 4, 5$$

$$\Rightarrow \quad x = 2i\cos\left(\frac{n\pi}{3}\right) - 2\sin\left(\frac{n\pi}{3}\right), \text{ where } n = 0, 1, 2, 3, 4, 5$$

For $n = 4$ and 5 we have positive real part. Hence, the required product is

$$\left[2i\cos\left(\frac{4\pi}{3}\right) - 2\sin\left(\frac{4\pi}{3}\right)\right]\left[2i\cos\left(\frac{5\pi}{3}\right) - 2\sin\left(\frac{5\pi}{3}\right)\right]$$

$$= 4\left[-i\frac{1}{2} + \frac{\sqrt{3}}{2}\right]\left[i\frac{1}{2} + \frac{\sqrt{3}}{2}\right]$$

$$= 4$$

6. $E = \frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \cdots + \frac{1}{z_{50} - 1}$

where $z_1, z_2, \ldots, z_{50}$ are the roots of the equation $z^{51} - 1 = 0$ other than 1. Let

$$1 + z + z^2 + \cdots + z^{50} = (z - z_1)(z - z_2) \cdots (1 - z_{50})$$

$$\Rightarrow \quad \log(1 + z + z^2 + \cdots + z^{50}) = \log[(z - z_1)(z - z_2) \cdots (1 - z_{50})]$$

Differentiating both sides w.r.t. z and putting $z = 1$,

$$\frac{1 + 2z + 3z^2 + \cdots + 50z^{49}}{1 + z + z^2 + \cdots + z^{50}}$$

$$= \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_{50}}$$

$$\Rightarrow \quad \frac{50 \times 51}{2 \times 51} = -\left[\frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \cdots + \frac{1}{z_{50} - 1}\right]$$

$$\therefore \quad \sum \frac{1}{z_r - 1} = -25$$

Exercises

Single Correct Answer Type

1. **(2)** Verify by selecting particular values of a and b.

Let $a = -9$ and $b = 4$. Then

$$\sqrt{a}\sqrt{b} = \sqrt{-9}\sqrt{4} = (3i)(2) = 6i$$

From option (1), we have

$$-\sqrt{|a|b} = -\sqrt{|-9| \times 4} = -\sqrt{36} = -6$$

From option (2), we have

$$\sqrt{|a|b}\, i = \sqrt{|-9| \times 4}\; i = 6i$$

2. (4) $10z^2 - 3iz - k = 0$

$$\Rightarrow \quad z = \frac{3i \pm \sqrt{-9 + 40k}}{20}$$

Now, $D = -9 + 40k$. If $k = 1$, then $D = 31$. So (1) is false.

If k is a negative real number, then D is a negative real number. So (4) is true.

If $k = i$, then $D = -9 + 40i = 16 + 40i - 25 = (4 + 5i)^2$, and the roots are $(1/5) + (2/5)i$ and $-(1/5 - 1/10)i$. So (3) is false.

If $k = 0$ (which is a complex number), then the roots are 0 and $(3/10)i$. So (2) is false.

3. (4) Let $z = x + iy$, so that $\bar{z} = x - iy$.

$\therefore \quad z^2 + \bar{z} = 0$

$\Rightarrow \quad (x^2 - y^2 + x) + i(2xy - y) = 0$

Equating real and imaginary parts, we get

$$x^2 - y^2 + x = 0 \qquad (1)$$

and $\quad 2xy - y = 0 \quad \Rightarrow y = 0$ or $x = \frac{1}{2}$

If $y = 0$, then (1) gives $x^2 + x = 0$

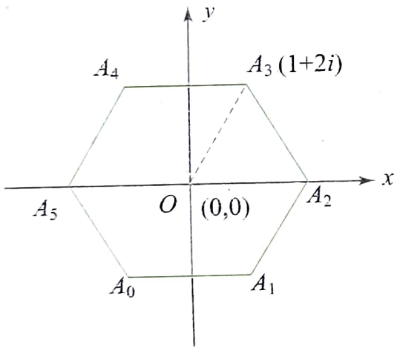
$\Rightarrow \quad x = 0$ or $x = -1$

If $x = 1/2$, then from Eq. (1),

$$y^2 = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \text{ or } y = \pm \frac{\sqrt{3}}{2}$$

Hence, there are four solutions in all.

4. (4)



Let the vertices be $z_0, z_1, \ldots, z_5$ w.r.t. center O at origin and $|z_0| = \sqrt{5}$.

Now ΔOA_2A_3 is equilateral

$\Rightarrow \quad OA_2 = OA_3 = A_2A_3 = \sqrt{5}$

Perimeter $= 6\sqrt{5}$

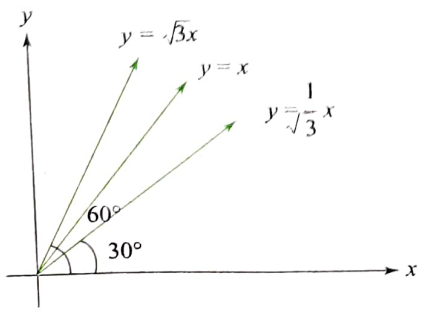
5. (3) Observing carefully the system of equations, we find

$$\frac{1+i}{2i} = \frac{1-i}{2} = \frac{1}{1+i}$$

Hence, there are infinite number of solutions.

6. (2) Note that $z_1 = 3 + \sqrt{3}i$ lies on the line $y = (1/\sqrt{3})x$ and $z_2 = 2\sqrt{3} + 6i$ lies on the line $y = \sqrt{3}x$.

Hence, $z = 5 + 5i$ will only lie on the bisector of z_1 and z_2, i.e., $y = x$.



7. (4) Let $f(x) = x^6 + 4x^5 + 3x^4 + 2x^3 + x + 1$. Hence,

$$\begin{aligned} f(\omega) &= \omega^6 + 4\omega^5 + 3\omega^4 + 2\omega^3 + \omega + 1 \\ &= 1 + 4\omega^2 + 3\omega + 2 + \omega + 1 \\ &= 4(\omega^2 + \omega + 1) \\ &= 0 \end{aligned}$$

Hence, $f(x)$ is divisible by $x - \omega$. Then $f(x)$ is also divisible by $x - \omega^2$ (as complex roots occur in conjugate pairs).

$$\begin{aligned} f(-\omega) &= (-\omega)^6 + 4(-\omega)^5 + 3(-\omega)^4 + 2(-\omega)^3 + (-\omega) + 1 \\ &= \omega^6 - 4\omega^5 + 3\omega^4 - 2\omega^3 - \omega + 1 \\ &= 1 - 4\omega^2 + 3\omega - 2 - \omega + 1 \\ &\neq 0 \end{aligned}$$

8. (2) $f(z) = g(z)(z - i)(z + i) + az + b;\ a, b \in C$

Given, $f(i) = i$

$$\Rightarrow \quad ai + b = i \qquad (1)$$

and $\quad f(-i) = 1 + i$

$$\Rightarrow \quad a(-i) + b = 1 + i \qquad (2)$$

From (1) and (2), we have

$$a = \frac{i}{2}, \; b = \frac{1}{2} + i$$

Hence, the required remainder is $az + b = (1/2)iz + (1/2) + i$.

9. (4) Let $z_1 = \sin x + i \cos 2x$ and $z_2 = \cos x - i \sin 2x$.

Then $\bar{z}_1 = z_2$

$\Rightarrow \quad \sin x - i \cos 2x = \cos x - i \sin 2x$

$\Rightarrow \quad \sin x = \cos x$ and $\cos 2x = \sin 2x$

$\Rightarrow \quad \tan x = 1$ and $\tan 2x = 1$

$$\Rightarrow \quad x = \frac{\pi}{4} \text{ and } x = \frac{\pi}{8}$$

This is not possible.

Hence, there is no value of x.

10. (2) Let xi be the root where $x \neq 0$ and $x \in R$

$$x^4 - a_1x^3i - a_2x^2 + a_3xi + a_4 = 0$$

$$\Rightarrow \quad x^4 - a_2x^2 + a_4 = 0 \qquad (1)$$

and $\quad a_1x^3 - a_3x = 0 \qquad (2)$

From Eq. (2),

$$a_1x^2 - a_3 = 0$$

$$\Rightarrow \quad x^2 = a_3/a_1 \text{ (as } x \neq 0)$$

Putting the value of x^2 in Eq. (1), we get

$$\frac{a_3^2}{a_1^2} - \frac{a_2 a_3}{a_1} + a_4 = 0$$

$$\Rightarrow \quad a_3^2 + a_4a_1^2 = a_1a_2a_3$$

$$\Rightarrow \quad \frac{a_3}{a_1 a_2} + \frac{a_1a_4}{a_2 a_3} = 1 \text{ (dividing by } a_1a_2a_3)$$

11. (3) We have,

$$z_1(z_1^2 - 3z_2^2) = 2 \qquad (1)$$

$$z_2(3z_1^2 - z_2^2) = 11 \qquad (2)$$

Multiplying (2) by i and adding it to (1), we get

$$z_1^3 - 3z_2^2z_1 + i(3z_1^2z_2 - z_2^3) = 2 + 11i \qquad (3)$$

$$\Rightarrow \quad (z_1 + iz_2)^3 = 2 + 11i$$

Multiplying (2) by i and subtracting it from (1), we get

$$z_1^3 - 3z_2^2 z_1 - i(3z_1^2z_2 - z_2^3) = 2 - 11i \qquad (4)$$

$$\Rightarrow \quad (z_1 - iz_2)^3 = 2 - 11i$$

Multiplying (3) and (4), we get

$$(z_1^2 + z_2^2)^3 = 2^2 - 121i^2 = 4 + 121 = 125$$

$$\Rightarrow \quad z_1^2 + z_2^2 = 5$$

12. (3) Given that $a^2 + b^2 = 1$. Therefore,

$$\frac{1+b+ia}{1+b-ia} = \frac{(1+b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)}$$

$$= \frac{(1+b)^2 - a^2 + 2ia(1+b)}{1+b^2+2b+a^2}$$

$$= \frac{(1-a^2)+2b+b^2+2ia(1+b)}{2(1+b)}$$

$$= \frac{2b^2+2b+2ia(1+b)}{2(1+b)} = b + ia$$

13. (1) $$\frac{1+iz}{1-iz} = \frac{1+i(b+ic)/(1+a)}{1-i(b+ic)/(1+a)}$$

$$= \frac{1+a-c+ib}{1+a+c-ib}$$

$$= \frac{(1+a-c+ib)(1+a+c+ib)}{(1+a+c)^2+b^2}$$

$$= \frac{1+2a+a^2-b^2-c^2+2ib+2iab}{1+a^2+c^2+b^2+2ac+2(a+c)}$$

$$= \frac{2a+2a^2+2ib+2iab}{2+2ac+2(a+c)} \qquad (\because a^2+b^2+c^2=1)$$

$$= \frac{a+a^2+ib+iab}{1+ac+(a+c)}$$

$$= \frac{a(a+1)+ib(a+1)}{(a+1)(c+1)} = \frac{a+ib}{c+1}$$

14. (1) Let α be the real root and $i\beta$ be the imaginary root of the given equation. Then

$\alpha + i\beta = -a$

$\Rightarrow \quad \alpha - i\beta = -\bar{a}$

So, $2\alpha = -(a+\bar{a})$ and $2i\beta = -(a-\bar{a})$

Multiplying these, we get

$$4i\alpha\beta = a^2 - (\bar{a})^2$$

$$\therefore \quad 4b = a^2 - (\bar{a})^2$$

15. (2) Let $z = x + iy$. Then,

$x = \lambda + 3$ and $y = -\sqrt{5-\lambda^2}$

$\Rightarrow \quad (x-3)^2 = \lambda^2 \qquad (1)$

and $\quad y^2 = 5 - \lambda^2 \qquad (2)$

From Eqs. (1) and (2),

$(x-3)^2 = 5 - y^2$ or $(x-3)^2 + y^2 = 5$

Obviously it is a semicircle as $y < 0$. Hence, part of the circle lies below the x-axis.

16. (1) $x + iy = 1 - t + i\sqrt{t^2+t+2}$

$\Rightarrow \quad x = 1 - t, y = \sqrt{t^2+t+2}$

Eliminating t,

$$y^2 = t^2 + t + 2 = (1-x)^2 + 1 - x + 2 = \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$$

$$\Rightarrow \quad y^2 - \left(x - \frac{3}{2}\right)^2 = \frac{7}{4}, \text{ which is a hyperbola}$$

17. (4) $(x-3)^3 + 1 = 0$

$$\Rightarrow \quad \left(\frac{x-3}{-1}\right)^3 = 1$$

$$\Rightarrow \quad \frac{x-3}{-1} = 1, \omega, \omega^2$$

$\Rightarrow \quad x = 2, 3 - \omega, 3 - \omega^2$

Hence, the sum of complex roots is $6 - (\omega + \omega^2) = 6 + 1 = 7$.

18. (2) $x = \sqrt[3]{-1}$

$\Rightarrow \quad x^3 = -1$

$\Rightarrow \quad (-x)^3 = 1$

$\Rightarrow \quad -x = 1, \omega, \omega^2$

$\Rightarrow \quad x = -1, -\omega, -\omega^2$

$$= -1, \frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$$

$$= -1, \frac{-\sqrt{3}+i}{2i}, \frac{\sqrt{3}+i}{2i}$$

$$= -1, \frac{-\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}, \frac{\sqrt{3}+\sqrt{-1}}{\sqrt{-4}}$$

19. (4) $x^2 + x + 1 = 0$

$\Rightarrow \quad x = \omega$ or ω^2

Let $x = \omega$. Then,

$$x + \frac{1}{x} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$$

$$x^2 + \frac{1}{x^2} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$$

$$x^3 + \frac{1}{x^3} = \omega^3 + \frac{1}{\omega^3} = 2$$

$$x^4 + \frac{1}{x^4} = \omega^4 + \frac{1}{\omega^4} = \omega + \omega^2 = -1, \text{ etc.}$$

$$\therefore \quad \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^3+\frac{1}{x^3}\right)^2 + \cdots + \left(x^{27}+\frac{1}{x^{27}}\right)^2$$

$$= \left[\left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)^2 + \left(x^4+\frac{1}{x^4}\right)^2 + \cdots + \left(x^{26}+\frac{1}{x^{26}}\right)^2\right]$$

$$+ \left[\left(x^3+\frac{1}{x^3}\right)^2 + \left(x^6+\frac{1}{x^6}\right)^2 + \left(x^9+\frac{1}{x^9}\right)^2 + \cdots + \left(x^{27}+\frac{1}{x^{27}}\right)^2\right]$$

$$= 18 + 9(2^2) = 54$$

20. (1) We have, $z^3 + 2z^2 + 2z + 1 = 0$

$\Rightarrow \quad (z^3 + 1) + 2z(z + 1) = 0$

$\Rightarrow \quad (z + 1)(z^2 + z + 1) = 0$

$\Rightarrow \quad z = -1, \omega, \omega^2$

Since $z = -1$ does not satisfy $z^{1985} + z^{100} + 1 = 0$ while $z = \omega, \omega^2$ satisfy it; hence, sum is $\omega + \omega^2 = -1$.

21. (4) Let $f(x) = 5x^3 + Mx + N$.

Also, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

It is given that $f(x)$ is divisible by $x^2 + x + 1$.

$\therefore \quad f(\omega) = 5 + M\omega + N = 0$

and $\quad f(\omega^2) = 5 + M\omega^2 + N = 0$

So, $M = 0$ and $N = -5$.

$\therefore \quad M + N = -5$

22. (1) Here $x = 4\cos\theta, y = 4\sin\theta$.

$$\therefore \quad ||x| - |y|| = |4|\cos\theta| - 4|\sin\theta||$$
$$= 4||\cos\theta| - |\sin\theta||$$
$$= 4\sqrt{1 - 2|\cos\theta||\sin\theta|}$$
$$= 4\sqrt{1 - |\sin 2\theta|}$$

Hence, the range is $[0, 4]$.

23. (1) Let $z^3 = t$

$$\therefore \quad t^2 - 6t + 25 = 0$$
$$\Rightarrow \quad t = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4i = z^3$$

So, $|z^3| = 5$

$$\Rightarrow \quad |z| = 5^{1/3}$$

24. (1) $8iz^3 + 12z^2 - 18z + 27i = 0$

or $4iz^2(2z - 3i) - 9(2z - 3i) = 0$

or $(2z - 3i)(4iz^2 - 9) = 0$

$$\Rightarrow \quad z = \frac{3i}{2} \text{ and } z^2 = \frac{9}{4i}$$
$$\Rightarrow \quad |z| = \frac{3}{2} \text{ and } |z^2| = \frac{9}{4}$$
$$\Rightarrow \quad |z| = \frac{3}{2}$$

25. (1) Let $z_1 = a + ib$ and $z_2 = c - id$, where $a > 0$ and $d > 0$. Then,

$$|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2 \qquad (1)$$

$$\text{Now, } \frac{z_1 + z_2}{z_1 - z_2} = \frac{(a+ib)+(c-id)}{(a+ib)-(c-id)}$$
$$= \frac{[(a+c)+i(b-d)][(a-c)-i(b+d)]}{[(a-c)+i(b+d)][(a-c)-i(b+d)]}$$
$$= \frac{(a^2+b^2)-(c^2+d^2)-2(ad+bc)i}{a^2+c^2-2ac+b^2+d^2+2bd}$$
$$= \frac{-(ad+bc)i}{a^2+b^2-ac+bd} \qquad \text{[Using (1)]}$$

Hence, $(z_1 + z_2)/(z_1 - z_2)$ is purely imaginary. However, if $ad - bc = 0$, then $(z_1 + z_2)/(z_1 - z_2)$ will be equal to zero. According to the conditions of the equation, we can have $ad + bc = 0$.

26. (1) We have

$$\arg\left(\frac{z_1}{z_2}\right) = \pi$$

or $\arg(z_1) - \arg(z_2) = \pi$

or $\arg(z_1) = \arg(z_2) + \pi$

Let $\arg(z_2) = \theta$. Then $\arg(z_1) = \pi + \theta$.

$$\therefore \quad z_1 = |z_1|[\cos(\pi + \theta) + i\sin(\pi + \theta)]$$
$$= |z_1|(-\cos\theta - i\sin\theta)$$

and $z_2 = |z_2|(\cos\theta + i\sin\theta)$

$$= |z_1|(\cos\theta + i\sin\theta) \qquad (\because |z_1| = |z_2|)$$
$$= -z_1$$
$$\Rightarrow \quad z_1 + z_2 = 0$$

27. (3) We have,

$$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$$

where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$. Given,

$$\arg(z_1) - \arg(z_2) = 0$$
$$\Rightarrow \quad |z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$$
$$= (|z_1| - |z_2|)^2$$
$$\Rightarrow \quad |z_1 - z_2| = ||z_1| - |z_2||$$

28. (2) Let $z_1 = |z_1|(\cos\theta_1 + i\sin\theta_1)$.

$$\text{Now, } \left|\frac{z_1}{z_2}\right| = 1 \Rightarrow |z_1| = |z_2|$$

Also, $\arg(z_1 z_2) = 0$ or $\arg(z_1) + \arg(z_2) = 0$

$$\Rightarrow \quad \arg(z_2) = -\theta_1$$
$$\Rightarrow \quad z_2 = |z_2|(\cos(-\theta_1) + i\sin(-\theta_1))$$
$$= |z_1|(\cos\theta_1 - i\sin\theta_1) = \bar{z}_1$$
$$\Rightarrow \quad \bar{z}_2 = (\bar{z}_1) = z_1$$
$$\Rightarrow \quad |z_2|^2 = z_1 z_2$$

29. (2) Let $A = x + iy$. Given,

$$|A| = 1 \Rightarrow x^2 + y^2 = 1$$

and $|A + 1| = 1 \Rightarrow (x + 1)^2 + y^2 = 1$

$$\Rightarrow \quad x = -\frac{1}{2} \text{ and } y = \pm\frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad A = \omega \text{ or } \omega^2$$
$$\Rightarrow \quad (\omega)^n = (1 + \omega)^n = (-\omega^2)^n$$

Therefore, n must be even and divisible by 3.

30. (3) Let $z = \cos x + i\sin x, x \in [0, 2\pi)$. Then,

$$1 = \left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right|$$
$$= \frac{|z^2 + \bar{z}^2|}{|z|^2}$$
$$= |\cos 2x + i\sin 2x + \cos 2x - i\sin 2x|$$
$$= 2|\cos 2x|$$
$$\Rightarrow \quad \cos 2x = \pm 1/2$$

Now, $\cos 2x = 1/2$

$$\Rightarrow \quad x_1 = \frac{\pi}{6}, x_2 = \frac{5\pi}{6}, x_3 = \frac{7\pi}{6}, x_4 = \frac{11\pi}{6}$$

$$\cos 2x = -\frac{1}{2}$$

$$\Rightarrow \quad x_5 = \frac{\pi}{3}, x_6 = \frac{2\pi}{3}, x_7 = \frac{4\pi}{3}, x_8 = \frac{5\pi}{3}$$

31. (3) $\bar{z} + i\bar{w} = 0$

$$\Rightarrow \quad z - iw = 0 \qquad (1)$$
$$\Rightarrow \quad z = iw$$
$$\arg zw = \pi$$
$$\text{or} \quad \arg z + \arg w = \pi$$
$$\text{or} \quad \arg z + \arg\frac{z}{i} = \pi \qquad \text{[Using (1)]}$$
$$\text{or} \quad \arg z + \arg z - \arg i = \pi$$
$$\text{or} \quad 2\arg z - \frac{\pi}{2} = \pi$$
$$\text{or} \quad 2\arg z = \frac{3\pi}{2}$$
$$\text{or} \quad \arg z = \frac{3\pi}{4}$$

32. (3) $z = (3a - 7b) + i(3b + 7a)$

For z to be purely imaginary, we have

$3a = 7b$

$\Rightarrow$ $a = 7$ or $b = 3$ (for least value of $|z|$)

$\Rightarrow$ $|z| = |3 + 7i|$

33. (3) We have,

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \cdots \times (\cos n\theta + i\sin n\theta) = 1$

or $\cos(\theta + 2\theta + 3\theta + \cdots + n\theta) + i\sin(\theta + 2\theta + \cdots + n\theta) = 1$

or $\cos\left(\frac{n(n+1)}{2}\theta\right) + i\sin\left(\frac{n(n+1)}{2}\theta\right) = 1$

or $\cos\left(\frac{n(n+1)}{2}\theta\right) = 1$ and $\sin\left(\frac{n(n+1)}{2}\theta\right) = 0$

$\Rightarrow$ $\frac{n(n+1)}{2}\theta = 2m\pi \Rightarrow \theta = \frac{4m\pi}{n(n+1)}$, where $m \in Z$

34. (3) $z = (1 + i\sqrt{3})^{100} = 2^{100}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{100}$

$= 2^{100}\left(\cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3}\right)$

$= 2^{100}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$= 2^{100}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$\Rightarrow$ $\frac{\text{Re}(z)}{\text{Im}(z)} = \frac{-1/2}{-\sqrt{3}/2} = \frac{1}{\sqrt{3}}$

35. (2) Let $\sin\frac{\pi}{8} + i\cos\frac{\pi}{8} = z$

$\Rightarrow$ $\left[\frac{1 + \sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1 + \sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}\right]^8$

$= \left(\frac{1+z}{1+\frac{1}{z}}\right)^8 = z^8$

$= \left(\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}\right)^8$

$= \left(\cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{8}\right)\right)^8$

$= \left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)^8 = \cos 3\pi = -1$

36. (1) Let $z = x + iy$. Then,

$|z - 3 - i| = |z - 9 - i|$

$\Rightarrow$ $\sqrt{(x-3)^2 + (y-1)^2} = \sqrt{(x-9)^2 + (y-1)^2}$

$\Rightarrow$ $x = 6$

$|z - 3 + 3i| = 3$

$\Rightarrow$ $\sqrt{(x-3)^2 + (y+3)^2} = 3$

For $x = 6$, $y = -3$.

$\therefore$ $z = 6 - 3i$

37. (1) When $|z - 1| < |z + 1|$ (or $x > 0$)

$|z| = |z - 1|$

$\Rightarrow$ $x^2 + y^2 = (x-1)^2 + y^2$

$\Rightarrow$ $x = 1/2$

$\Rightarrow$ $z + \bar{z} = 1$

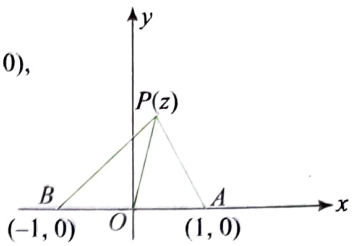
When $|z - 1| > |z + 1|$ (or $x < 0$),

$|z| = |z + 1|$

$\Rightarrow$ $x^2 + y^2 = (x+1)^2 + y^2$

$\Rightarrow$ $x = -1/2$

$\Rightarrow$ $z + \bar{z} = -1$



38. (4) Let $z = x + iy$. Then,

$|z^2 - 1| = |z|^2 + 1$

$\Rightarrow$ $|(x^2 - y^2 - 1) + 2ixy| = x^2 + y^2 + 1$

$\Rightarrow$ $(x^2 - y^2 - 1)^2 + 4x^2y^2 = (x^2 + y^2 + 1)^2$

$\Rightarrow$ $x = 0$

Hence, z lies on imaginary axis.

39. (2) The given equation is

$|z|^n = (z^2 + z)|z|^{n-2} + 1$

$\Rightarrow$ $z^2 + z$ is real

$\Rightarrow$ $z^2 + z = \bar{z}^2 + \bar{z}$

$\Rightarrow$ $(z - \bar{z})(z + \bar{z} + 1) = 0$

$\Rightarrow$ $z = \bar{z} = x$ as $z + \bar{z} + 1 \neq 0$ $(x \neq -1/2)$

Hence, the given equation reduces to

$x^n = x^n + x|x|^{n-2} + 1$

$\Rightarrow$ $x|x|^{n-2} = -1$

$\Rightarrow$ $x = -1$

So number of solutions is 1.

40. (4) Given, $z^3 + \frac{3(\bar{z})^2}{|z|} = 0$

Let $z = re^{i\theta}$

$\Rightarrow$ $r^3e^{i3\theta} + 3re^{-i2\theta} = 0$

Since r cannot be zero, so

$re^{i5\theta} = -3$

which will hold for $r = 3$ and five distinct values of θ. Thus, there are five solutions.

41. (3) Let $z = a + ib$

$\Rightarrow$ $\bar{z} = a - ib$

So, we have

$z^{2008} = \bar{z}$

$\therefore$ $|z|^{2008} = |\bar{z}| = |z|$

$\Rightarrow$ $|z|[|z|^{2007} - 1] = 0$

$\Rightarrow$ $|z| = 0$ or $|z| = 1$

If $|z| = 0$, then $z = 0$.

If $|z| = 1$, then $z^{2009} = z\bar{z} = |z|^2 = 1$.

So, there are 2009 values of z.

Therefore, total number of ordered pairs is 2010.

42. (2) We have $az^3 + bz^2 + \bar{b}z + \bar{a} = 0$ ...(1)

Taking conjugate of both sides, we get

$\Rightarrow$ $\bar{a}\bar{z}^3 + \bar{b}\bar{z}^2 + bz + a = 0$

Dividing this equation by $\bar{z}^3$ and writing the terms in reverse order, we get

$\frac{a}{\bar{z}^3} + \frac{b}{\bar{z}^2} + \frac{\bar{b}}{\bar{z}} + \bar{a} = 0$ [For $\bar{z} \neq 0$] ...(2)

Since equations (1) and (2) are identical,

$$z^3 = \frac{1}{\bar{z}^3} \quad \Rightarrow |z|^6 = 1 \quad \Rightarrow |z| = 1$$

Therefore, if α is a root of the given equation, then $|\alpha| = 1$.

43. (1) $\alpha = \dfrac{z - \bar{w}}{k^2 + z\bar{w}} \Rightarrow \bar{\alpha} = \dfrac{\bar{z} - w}{k^2 + \bar{z}w}$

But $z\bar{z} = w\bar{w} = k^2$. Hence,

$$\Rightarrow \quad \bar{\alpha} = \frac{\frac{k^2}{z} - \frac{k^2}{\bar{w}}}{k^2 + \frac{k^2}{z}\frac{k^2}{\bar{w}}} = \frac{\bar{w} - z}{z\bar{w} + k^2} = -\alpha$$

$$\Rightarrow \quad \alpha + \bar{\alpha} = 0$$

$$\Rightarrow \quad \text{Re}(\alpha) = 0$$

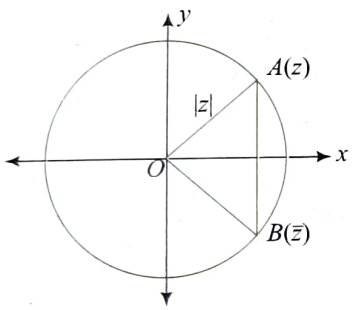
44. (1) Assuming $\arg z_1 = \theta$ and $\arg z_2 = \theta + \alpha$,

$$\frac{az_1}{bz_2} + \frac{bz_2}{az_1} = \frac{a\,|\,z_1\,|\,e^{i\theta}}{b\,|\,z_2\,|\,e^{i(\theta+\alpha)}} + \frac{b\,|\,z_2\,|\,e^{i(\theta+\alpha)}}{a\,|\,z_1\,|\,e^{i\theta}}$$

$$= e^{-i\alpha} + e^{i\alpha} = 2\cos\alpha$$

Hence, the point lies on the line segment $[-2, 2]$ of the real axis.

45. (1)



$$|z - \bar{z}| = AB$$

while $|z|(\arg z - \arg \bar{z}) = \text{Arc } AB$

$$\therefore \quad |z - \bar{z}| \le |z|(\arg z - \arg \bar{z})$$

46. (3) Let $a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

$c = \cos\gamma + i\sin\gamma$

Then, $a + 2b + 3c = (\cos\alpha + 2\cos\beta + 3\cos\gamma)$

$$+ i(\sin\alpha + 2\sin\beta + 3\sin\gamma) = 0$$

$$\Rightarrow \quad a^3 + 8b^3 + 27c^3 = 18\,abc$$

$$\Rightarrow \quad \cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$$

and $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma = 18\sin(\alpha + \beta + \gamma)$

47. (1) $u^2 - 2u + 2 = 0 \Rightarrow u = 1 \pm i$

$$\Rightarrow \quad \frac{(x+\alpha)^n - (x+\beta)^n}{\alpha - \beta}$$

$$= \frac{[(\cot\theta - 1) + (1+i)]^n - [(\cot\theta - 1) + (1-i)]^n}{2i}$$

$$(\because \cot\theta - 1 = x)$$

$$= \frac{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n}{\sin^n\theta\, 2i}$$

$$= \frac{2i\sin n\theta}{\sin^n\theta\, 2i}$$

$$= \frac{\sin n\theta}{\sin^n\theta}$$

48. (1) $i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = e^{\frac{i\pi}{2}}$

$$\Rightarrow \quad i^i = \left(e^{\frac{i\pi}{2}}\right)^i = e^{-\frac{\pi}{2}}$$

$$\Rightarrow \quad z = (i)^{(i)^i} = i^{e^{-\frac{\pi}{2}}}$$

$$\Rightarrow \quad |z| = 1$$

49. (4) $z = i\log(2 - \sqrt{3})$

$$\Rightarrow \quad e^{iz} = e^{i^2\log(2-\sqrt{3})} = e^{-\log(2-\sqrt{3})}$$

$$= e^{\log(2-\sqrt{3})^{-1}} = e^{\log(2+\sqrt{3})} = (2+\sqrt{3})$$

$$\Rightarrow \quad \cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{(2+\sqrt{3}) + (2-\sqrt{3})}{2} = 2$$

50. (1) $|z| = 1$, let $\alpha = -1 + 3z$

$$\Rightarrow \quad \alpha + 1 = 3z$$

$$\Rightarrow \quad |\alpha + 1| = 3|z| = 3$$

Hence, α lies on a circle centered at -1 and radius equal to 3.

51. (2) Let $z = x + iy$. Then,

$$\text{Re}\left(\frac{1}{z}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{1}{x+iy}\right) = k$$

$$\Rightarrow \quad \text{Re}\left(\frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}\right) = k$$

$$\Rightarrow \quad \frac{x}{x^2+y^2} = k$$

$$\Rightarrow \quad x^2 + y^2 - \frac{1}{k}x = 0$$

which is a circle.

52. (2) $2\left|z - \dfrac{1}{2}\right| = |z - 1|$

$$\therefore \quad \frac{|z-1|}{\left|z - \frac{1}{2}\right|} = 2$$

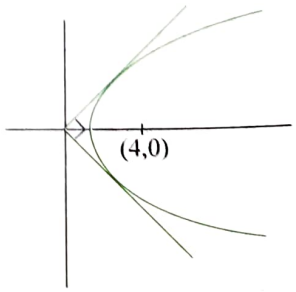
So, locus of z is a circle.

53. (4) $|z - 4| = \text{Re}(z)$

$$\Rightarrow \quad \sqrt{(x-4)^2 + y^2} = x$$

or $\quad x^2 - 8x + 16 + y^2 = x^2$

or $\quad y^2 = 8(x - 2)$



The given relation represents the parabola with focus (4, 0) and the imaginary axis as the directirx. The two tangents from directrix are at right angle. Hence, greatest positive argument of z is $\pi/4$.

54. (3) $z=\dfrac{at+b}{t-c} \Rightarrow t=\dfrac{b+cz}{z-a}$

Now, $|t|=1$

$\Rightarrow \left|\dfrac{b+cz}{z-a}\right|=1$

$\Rightarrow \left|\dfrac{z+\frac{b}{c}}{z-a}\right|=\dfrac{1}{|c|}$ ($\neq 1$ as $|c| \neq |t|$)

$\Rightarrow$ locus of z is a circle

55. (3) $z^2+z|z|+|z|^2=0$

$\Rightarrow \left(\dfrac{z}{|z|}\right)^2+\dfrac{z}{|z|}+1=0$

$\Rightarrow \dfrac{z}{|z|}=\omega, \omega^2$

$\Rightarrow z=\omega|z|$ or $z=\omega^2|z|$

$\Rightarrow x+iy=|z|\left(\dfrac{-1}{2}+\dfrac{i\sqrt{3}}{2}\right)$ or $x+iy=|z|\left(\dfrac{-1}{2}-\dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow x=-\dfrac{1}{2}|z|, y=|z|\dfrac{\sqrt{3}}{2}$ or $x=-\dfrac{|z|}{2}, y=-\dfrac{|z|\sqrt{3}}{2}$

$\Rightarrow y+\sqrt{3}x=0$ or $y-\sqrt{3}x=0 \Rightarrow y^2-3x^2=0$

56. (1) $\log_{1/3}\left(\dfrac{|z-3|^2+2}{11|z-3|-2}\right)>1,\ 11|z-3|-2|>0$

$\Rightarrow \dfrac{|z-3|^2+2}{11|z-3|-2}<\dfrac{1}{3}$

$\Rightarrow (3t-8)(t-1)<0$ (where $|z-3|=t$)

$\Rightarrow 1<|z-3|<8/3$

Hence, z lies between the two concentric circles.

57. (3) We have,

$$|(x-2)+i(y-1)|=|z|\left|\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right|$$

where $\theta=\arg z$.

$$\sqrt{(x-2)^2+(y-1)^2}=\frac{1}{\sqrt{2}}|x-y|,$$

which represents a parabola

58. (4) $|\omega z-1-\omega^2|=a$

$\Rightarrow |z+1|=a \Rightarrow |z-1+2|=a$

$\Rightarrow |z-1|+2\geq a \Rightarrow 0\leq a\leq 4$

59. (2) $|z^2-3|\geq|z|^2-3$

$\Rightarrow 3|z|\geq|z|^2-3$

$\Rightarrow |z|^2-3|z|-3\leq 0$

$\Rightarrow 0<|z|\leq\dfrac{3+\sqrt{21}}{2}$

60. (2) $|2z-1|=|z-2|$

or $|2z-1|^2=|z-2|^2$

or $(2z-1)(2\bar{z}-1)=(z-2)(\bar{z}-2)$

or $4z\bar{z}-2\bar{z}-2z+1=z\bar{z}-2\bar{z}-2z+4$

or $3|z|^2=3$

or $|z|=1$

Again,

$$\begin{aligned}|z_1+z_2| &= |z_1-\alpha+z_2-\beta+\alpha+\beta| \\ &\leq |z_1-\alpha|+|z_2-\beta|+|\alpha+\beta| \\ &< \alpha+\beta+|\alpha+\beta| \\ &= 2|\alpha+\beta| \qquad [\because \alpha,\beta>0]\end{aligned}$$

$\therefore \left|\dfrac{z_1+z_2}{\alpha+\beta}\right|<2$

or $\left|\dfrac{z_1+z_2}{\alpha+\beta}\right|<2|z|$

61. (1) $a_0z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_{n-1}z+a_n=3$

or $|3|=|a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n|$

or $3\leq|a_0||z|^n+|a_1||z|^{n-1}+\cdots+|a_{n-1}||z|+|a_n|$

or $3<2(|z|^n+|z|^{n-1}+\cdots+|z|+1)$ $(\because |a_i|<2)$

or $1+|z|+|z|^2+\cdots+|z|^n>\dfrac{3}{2}$

If $|z|\geq 1$, the inequality is clearly satisfied. For $|z|<1$, we have,

$$\frac{1-|z|^{n+1}}{1-|z|}>\frac{3}{2}$$

or $2-2|z|^{n+1}>3-3|z|$

or $2|z|^{n+1}<3|z|-1$

or $3|z|-1>0$

or $|z|>\dfrac{1}{3}$

62. (1) $|z^2+2z\cos\alpha|\leq|z^2|+|2z\cos\alpha|$

$=|z|^2+2|z||\cos\alpha|$

$\leq|z|^2+2|z|$ $(\because |\cos\alpha|\leq 1)$

$<(\sqrt{2}-1)^2+2(\sqrt{2}-1)=1$

63. (3) $\left|\sum_{r=1}^{n}z_r\right|\leq\sum_{r=1}^{n}|z_r|\leq\sum_{r=1}^{n}|z_r-r|+\sum_{r=1}^{n}r\leq 2\sum_{r=1}^{n}r$

64. (2) $11z^{10}+10iz^9+10iz-11=0$

or $z^9(11z+10i)=11-10iz$

or $z^9=\dfrac{11-10iz}{11z+10i}$

or $|z^9|=\dfrac{|11i-10z|}{|11z+10i|}$

Now $|11i-10z|^2-|11z+10i|^2=21(1-|z|^2)$

For $|z|<1$

$|11i-10z|^2-|11z+10i|^2>0$

$\Rightarrow |z^9|=\dfrac{|11i-10z|}{|11z+10i|}>1$

i.e. $|z^9|>1$ which contradicts with $|z|<1$

For $|z|>1$ we get $|z^9|<1$

$\Rightarrow |z|=1$

65. (2) $2z^2+2z+\lambda=0$

Let the roots be z_1, z_2. Then,

$$z_1+z_2=-1 \text{ and } z_1z_2=\frac{\lambda}{2}$$

$0, z_1, z_2$ form an equilateral triangle.

$\therefore z_1^2+z_2^2=z_1z_2$

$\Rightarrow (z_1+z_2)^2=3z_1z_2$

$\Rightarrow \quad 1 = 3\frac{\lambda}{2}$

or $\quad \lambda = \frac{2}{3}$

66. (3) $S_1 = \Sigma z_1 = -3a,\ S_2 = \Sigma z_1 z_2 = 3b$

Since the triangle is equilateral, we have

$\Sigma z_1^2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 - 2\Sigma z_1 z_2 = \Sigma z_1 z_2$

$\Rightarrow \quad (\Sigma z_1)^2 = 3\Sigma z_1 z_2$

$\Rightarrow \quad (-3a)^2 = 3(3b)$

$\Rightarrow \quad 9a^2 = 9b$

$\Rightarrow \quad a^2 = b$

67. (2) Taking cube roots of both sides, we get

$z + ab = a(1)^{1/3} = a, a\omega, a\omega^2$

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2},\ \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

$\therefore \quad z_1 = a - ab,\ z_2 = a\omega - ab,\ z_3 = a\omega^2 - ab$

$|z_1 - z_2| = |a(1 - \omega)|$

$= |a| \left|1 - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right|$

$= |a| \left|\frac{3}{2} - i\frac{\sqrt{3}}{2}\right|$

$= |a| \left(\frac{9}{4} + \frac{3}{4}\right)^{1/2} = \sqrt{3}\,|a|$

Similarly, $|z_2 - z_3| = |z_3 - z_1| = \sqrt{3}\,|a|$

68. (1) $|z_1| = |z_2| = |z_3| = 1$

Hence, the circumcenter of triangle is origin. Also, centroid $(z_1 + z_2 + z_3)/3 = 0$, which coincides with the circumcenter. So the triangle is equilateral. Since radius is 1, length of side is $a = \sqrt{3}$. Therefore, the area of the triangle is $(\sqrt{3}/4)a^2 = (3\sqrt{3}/4)$.

69. (3) We have $2 = |z + i\omega| \le |z| + |\omega| \quad (\because |z_1 + z_2| \le |z_1| + |z_2|)$

$\therefore \quad |z| + |\omega| \le 2 \quad \ldots(1)$

But it is given that $|z| \le 1$ and $|\omega| \le 1$.

$\Rightarrow \quad |z| + |\omega| \le 2 \quad \ldots(2)$

From (1) and (2),

$|z| = |\omega| = 1$

Also, $|z + i\omega| = |z - i\bar{\omega}|$

$\Rightarrow \quad |z - (-i\omega)| = |z - (i\bar{\omega})|$

This means that z lies on perpendicular bisector of the line segment joining $(-i\omega)$ and $(i\bar{\omega})$, which is real axis, as $(-i\omega)$ and $(i\bar{\omega})$ are conjugate to each other.

For z, $\text{Im}(z) = 0$

If $\quad z = x$, then $|z| \le 1$

$\Rightarrow \quad x^2 \le 1$

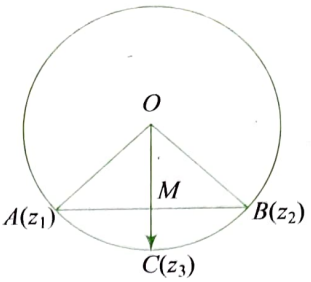
$\Rightarrow \quad -1 \le x \le 1$

Therefore, (3) is the correct option.

70. (4) Given that $4z_3 = 3(z_1 + z_2)$

$\therefore \quad \frac{z_1 + z_2}{2} = \frac{2}{3}z_3$

Therefore, mid-point of AB is the point which divides OM in the ratio 2 : 1.



or $\quad OM = \frac{2}{3}OC = \frac{2}{3}|z_3| = \frac{2}{3}$

$\therefore \quad |z_1 - z_2| = AB = 2AM = 2\sqrt{1 - \frac{4}{9}} = \frac{2\sqrt{5}}{3}$

71. (1)

The first condition implies that $(z_1 + z_3)/2 = (z_2 + z_4)/2$, i.e., diagonals AC and BD bisect each other. Hence, quadrilateral is a parallelogram. The second condition implies that the angle between AD and AB is 90°. Hence, the parallelogram is a rectangle.

72. (3) Given that $|k + z^2| = |z^2| - k = |z^2| + |k|$

$\Rightarrow \quad k, z^2$ and $0 + i0$ are collinear

$\Rightarrow \quad \arg(z^2) = \arg(k)$

$\Rightarrow \quad 2\arg(z) = \pi$

$\Rightarrow \quad \arg(z) = \frac{\pi}{2}$

73. (3) $|z_1 - i| = |z_2 - i| = |z_3 - i|$

Hence, z_1, z_2, z_3 lie on the circle whose center is i. Also given that the triangle is equilateral. Hence, centroid and cirucmcenter coincides.

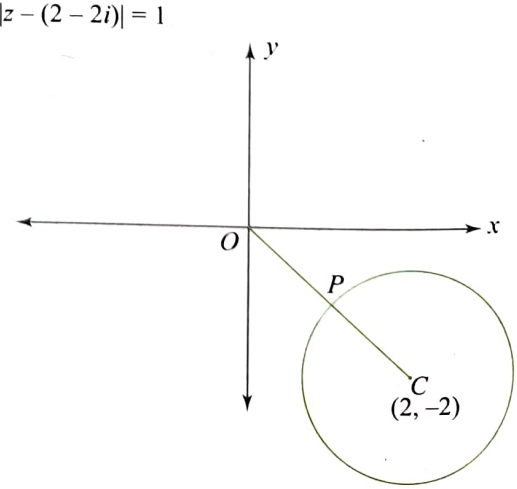
$\therefore \quad \frac{z_1 + z_2 + z_3}{3} = i$

$\Rightarrow \quad |z_1 + z_2 + z_3| = 3$

74. (1) We have,

$|z - 2 + 2i| = 1$

$\Rightarrow \quad |z - (2 - 2i)| = 1$



Hence, z lies on a circle having center at (2, –2) and radius 1. It is evident from the figure that the required complex number z is given by the point P.

We find that OP makes an angle $\pi/4$ with OX and

$$OP = OC - CP$$

$$= \sqrt{2^2+2^2} - 1 = 2\sqrt{2} - 1$$

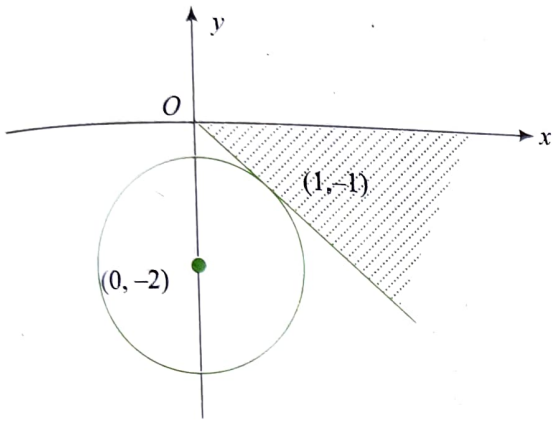
So, coordinates of P are $[(2\sqrt{2}-1)\cos(\pi/4),$

$-(2\sqrt{2}-1)\sin(\pi/4)]$, i.e., $((2-1/\sqrt{2}), -(2-1/\sqrt{2}))$.

Hence,

$$z = \left(2-\frac{1}{\sqrt{2}}\right) + \left\{-\left(2-\frac{1}{\sqrt{2}}\right)\right\} i = \left(2-\frac{1}{\sqrt{2}}\right)(1-i)$$

75. (3) $kz/(k+1)$ represents any point lying on the line joining origin and z.



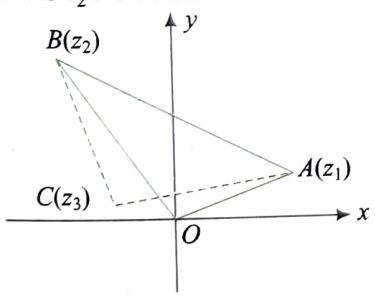
Given, $\left|\frac{kz}{k+1} + 2i\right| > \sqrt{2}$

Hence, $kz/(k+1)$ should lie outside the circle $|z + 2i| > \sqrt{2}$. So, z should lie in the shaded region.

$$\therefore \quad -\frac{\pi}{4} < \arg(z) < 0$$

76. (4) $|z_2 + iz_1| = |z_1| + |z_2| = |iz_1| + |z_2|$

$\Rightarrow$ iz_1, $0 + i0$ and z_2 are collinear



$\Rightarrow$ $\arg(iz_1) = \arg(z_2)$

$\Rightarrow$ $\arg(z_2) - \arg(z_1) = \frac{\pi}{2}$

Let $z_3 = \frac{z_2 - iz_1}{1-i}$

or $(1-i)\,z_3 = z_2 - iz_1$

or $z_2 - z_3 = i(z_1 - z_3)$

$\therefore$ $\angle ACB = \frac{\pi}{2}$

and $|z_1 - z_3| = |z_2 - z_3|$

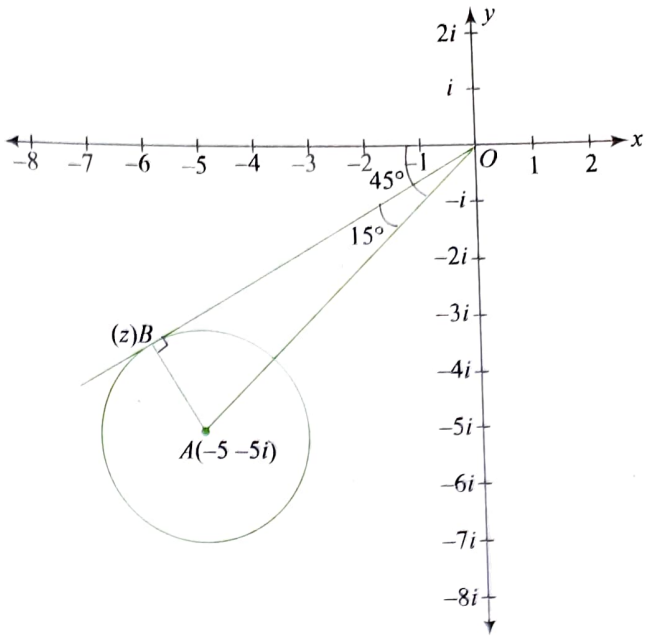
$\Rightarrow$ $AC = BC$

$\because$ $AB^2 = AC^2 + BC^2$

$\Rightarrow$ $AC = \frac{5}{\sqrt{2}}$ $(\because AB = 5)$

Therefore, area of ΔABC is $(1/2)AC \times BC = AC^2/2 = 25/4$ sq. units

77. (1)



$$|2z + 10 + 10i| \le \left(5\sqrt{3} - 5\right)$$

$$\Rightarrow \quad |z + 5 + 5i| \le \frac{5\sqrt{3}-5}{2}$$

Point B has the least principal argument.

Now, $AB = \frac{5\left(\sqrt{3}-1\right)}{2}$

$OA = 5\sqrt{2}$

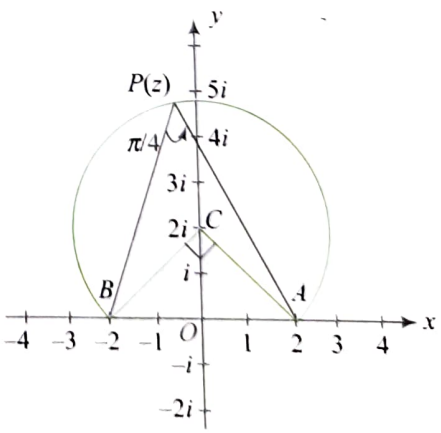
$$\therefore \quad \angle AOB = \frac{\pi}{12}$$

$$\therefore \quad \text{Arg}\,(z) = -\frac{5\pi}{6}$$

78. (2) $CA = CB = 2\sqrt{2}$, $OC = 2$

$\Rightarrow$ $OA = OB = 2$

$\Rightarrow$ $A \equiv 2 + 0i$, $B = -2 + 0i$



Clearly,

$\angle BCA = \pi/2$

$\Rightarrow$ $\angle BPA = \pi/4$

$$\Rightarrow \quad \arg\left(\frac{z-2}{z+2}\right) = \frac{\pi}{4}$$

79. (2) As ΔOAC is a right-angled triangle with right angle at A, so

$$|z_1|^2 + |z_3 - z_1|^2 = |z_3|^2$$

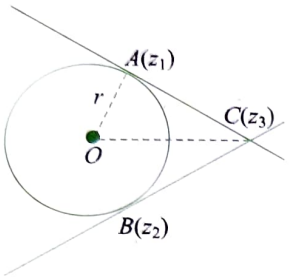
$$\Rightarrow \quad 2|z_1|^2 - \bar{z}_3 z_1 - \bar{z}_1 z_3 = 0$$

$$\Rightarrow \quad 2\bar{z}_1 - \bar{z}_3 - \frac{\bar{z}_1}{z_1} z_3 = 0 \qquad (1)$$

Similarly,

$$2\bar{z}_2 - \bar{z}_3 - \frac{\bar{z}_2}{z_2} z_3 = 0 \qquad (2)$$

Subtracting (2) from (1), we get



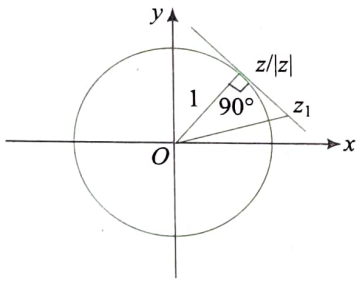
$$2(\bar{z}_2 - \bar{z}_1) = z_3\left(\frac{\bar{z}_1}{z_1} - \frac{\bar{z}_2}{z_2}\right)$$

$$\Rightarrow \quad \frac{2r^2(z_1 - z_2)}{z_1 z_2} = z_3 r^2\left(\frac{z_2^2 - z_1^2}{z_1^2 z_2^2}\right) \qquad [\because |z_1|^2 = |z_2|^2 = r^2]$$

$$\Rightarrow \quad z_3 = \frac{2z_1 z_2}{z_2 + z_1}$$

80. (2) Given that

$$\arg\left(\frac{z_1 - \frac{z}{|z|}}{\frac{z}{|z|}}\right) = \frac{\pi}{2}$$



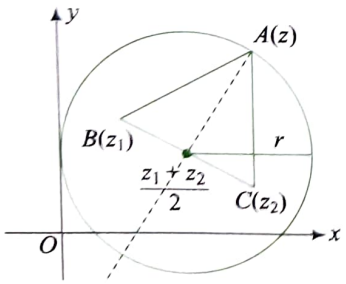
and $\left|\frac{z}{|z|} - z_1\right| = 3$

from which we can establish the above geometry.

From the diagram,

$$|z_1| = \sqrt{9+1} = \sqrt{10}$$

81. (2)



By the given conditions, the area of the triangle ABC is given by

$$(1/2)|z_1 - z_2|r.$$

82. (1) Given $|z| = 1$.

and $|z^2 - az + 1| = 1$

$$\Rightarrow \quad |z|\left|z - a + \frac{1}{z}\right| = 1$$

$$\Rightarrow \quad \left|a - \left(z + \frac{1}{z}\right)\right| = 1 \qquad (1)$$

Now, $z = \cos\theta + i\sin\theta$

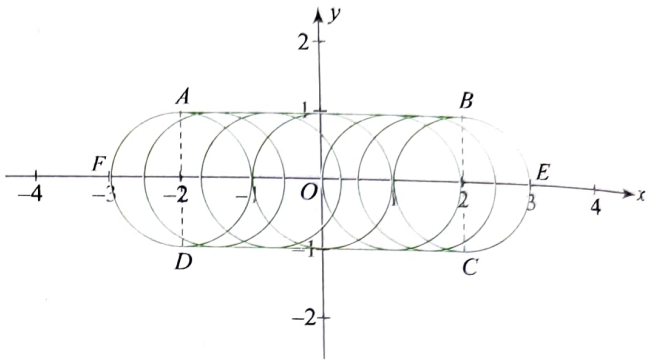
$$\Rightarrow \quad z + \frac{1}{z} = 2\cos\theta$$

So, (1) reduces to

$$|a - 2\cos\theta| = 1$$

So, locus of a is circle whose center is $(2\cos\theta, 0)$ and radius is 1.

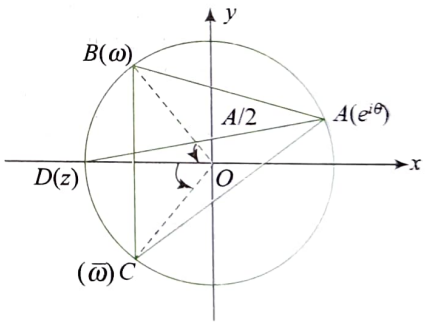
Since, $-2 \le 2\cos\theta \le 2$, we have following region traced by the locus of a.



As shown in the figure the set S consists of the locus of units circles centered at a point on the line segment with end points at –2 and 2 on the Argand plane. Therefore, the area S consists of the area of two semi circles and area of rectangle $ABCD$.

So, the required area is $\pi + 8$.

83. (4) Clearly,



$$\angle DOB = \angle COD = A$$

$$\Rightarrow \quad z = \omega e^{iA} \text{ and } \bar{\omega} = ze^{iA} \qquad \text{(Applying rotation about } O\text{)}$$

$$\Rightarrow \quad z^2 = \omega\bar{\omega} = 1$$

$$\Rightarrow \quad z = -1$$

(As A and D are on opposite sides of BC)

84. (4) We have,

$$e^{\frac{2\pi r i}{p}} = e^{\frac{2\pi m i}{q}}$$

$$r = 0, 1, \ldots, p-1$$

$$m = 0, 1, \ldots, q-1$$

This is possible iff $r = m = 0$ but for $r = m = 0$ we get 1 which is not an imaginary number.

85. (1) $\left(\frac{1+ia}{1-ia}\right)^4 = z$

or $\left|\frac{1+ia}{1-ia}\right|^4 = |z|$

or $\left|\frac{a-i}{a+i}\right|^4 = 1$

or $|a-i| = |a+i|$

Therefore, a lies on the perpendicular bisector of i and $-i$, which is the real axis. Hence, all the roots are real. Obviously roots are distinct.

10. (1) We have

$$\log z + \log z^2 + \log z^3 + \cdots + \log z^n = 0$$

$$\Rightarrow \log(z\, z^2\, z^3 \cdots z^n) = 0$$

$$\Rightarrow \log\left(z^{\frac{n(n+1)}{2}}\right) = 0$$

or $$z^{\frac{n(n+1)}{2}} = 1$$

or $$z = 1^{\frac{2}{n(n+1)}}$$

$$= (\cos 0° + i \sin 0°)^{\frac{2}{n(n+1)}}$$

$$= (\cos 2m\pi + i \sin 2m\pi)^{\frac{2}{n(n+1)}},\ m = 0, 1, 2, 3, \ldots$$

$$= \cos\frac{4m\pi}{n(n+1)} + i\sin\frac{4m\pi}{n(n+1)},\ m = 0, 1, 2, \ldots$$

11. (4) The equation $z^n = (z+1)^n$ will have exactly $n-1$ roots. We have,

$$\left(\frac{z+1}{z}\right)^n = 1 \quad \text{or} \quad \left|\frac{z+1}{z}\right| = 1$$

or $|z+1| = |z|$

Therefore, z lies on the right bisector of the segment connecting the points $(0, 0)$ and $(-1, 0)$. Thus, $\text{Re}(z) = -1/2$. Hence, roots are collinear and will have their real parts equal to $-1/2$.

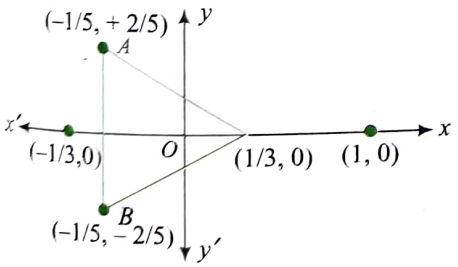
Hence, sum of the real parts of roots is $(-1/2)(n-1)$.

12. (3) $$\left(\frac{z+1}{z}\right)^4 = 16$$

$$\Rightarrow \frac{z+1}{z} = \pm 2, \pm 2i$$

The roots are $1,\ -1/3, (-1/5 - (2/5)i)$, and $(-1/5 + (2/5)i)$.

Note that $(-1/3, 0)$ and $(1, 0)$ are equidistant from $(1/3, 0)$ and since it lies on the perpendicular bisector of AB, it will be equidistant from A and B also.



13. (1) Given,

$$z_k = 1 + a + a^2 + \cdots + a^{k-1} = \frac{1-a^k}{1-a}$$

$$\Rightarrow z_k - \frac{1}{1-a} = -\frac{a^k}{1-a}$$

$$\Rightarrow \left|z_k - \frac{1}{1-a}\right| = \frac{|a|^k}{|1-a|} < \frac{1}{|1-a|} \qquad [\because |a| < 1]$$

Hence, z_k lies within the circle.

$$\therefore \left|z - \frac{1}{1-a}\right| = \frac{1}{|1-a|}$$

Multiple Correct Answers Type

1. **(1), (3)**

Clearly, we have to find it for real z. Let $z = x$. Then,

$$|x-w| = |x-w^2| = |w-w^2|$$

$$\Rightarrow \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \left|\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right|^2 = 3 \Rightarrow x + \frac{1}{2} = \pm\frac{3}{2}$$

$$\Rightarrow x = 1, -2$$

2. **(2), (3)**

$$\text{amp}(z_1 z_2) = 0$$

$$\Rightarrow \text{amp}\, z_1 + \text{amp}\, z_2 = 0$$

$$\therefore \text{amp}\, z_1 = -\text{amp}\, z_2 = \text{amp}\, \bar{z}_2$$

Since $|z_1| = |z_2|$, we get $|z_1| = |\bar{z}_2|$.

So, $z_1 = \bar{z}_2$. Also, $z_1 z_2 = \bar{z}_2 z_2 = |z_2|^2 = 1$ because $|z_2| = 1$.

3. **(1), (2), (3), (4)**

$$\sqrt{5-12i} = \sqrt{(3-2i)^2} = \pm(3-2i)$$

$$\sqrt{-5-12i} = \sqrt{(2-3i)^2} = \pm(2-3i)$$

$$\Rightarrow z = \sqrt{5-12i} + \sqrt{-5-12i}$$

$$= -1-i, -5+5i, 5-5i, 1+i$$

Therefore, principal values of arg z are $-3\pi/4, 3\pi/4, -\pi/4, \pi/4$.

4. **(1), (3)**

$$(-i)^{1/3} = (i^3)^{1/3} = i, i\omega, i\omega^2$$

where $\omega = \dfrac{-1+\sqrt{3}i}{2}$

Hence, values are $i,\ (-\sqrt{3}-i)/2, (\sqrt{3}-i)/2$.

5. **(1), (3), (4)**

We have $a^3 + b^3 - 8c^3 + 6abc = 0$

$$\Rightarrow a^3 + b^3 + (-2c)^3 - 3ab(-2c) = 0$$

$$\Rightarrow (a+b-2c)(a+b\omega-2c\omega^2)(a+b\omega^2-2c\omega) = 0$$

6. **(3), (4)**

$$|z_1 + z_2| = |z_1| = |z_2|$$

Thus, angle between z_1 and z_2 is $120°$.

$$\therefore \frac{z_1}{z_2} = \omega \text{ or } \omega^2$$

7. **(1), (3)**

$$p + q + r = a + b\omega + c\omega^2 + b + c\omega + a\omega^2 + c + a\omega + b\omega^2$$

$$\therefore p + q + r = (a+b+c)(1+\omega+\omega^2) = 0 \qquad (1)$$

p, q, r lie on the circle $|z| = 2$, whose circumcenter is origin. Also, $(p+q+r)/3 = 0$. Hence the centroid coincides with circumcenter. So, the triangle is equilateral.

Now, $(p+q+r)^2 = 0$.

$$\Rightarrow p^2 + q^2 + r^2 = -2pqr\left[\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right]$$

$$= -2pqr\left[\frac{1}{a+b\omega+c\omega^2} + \frac{1}{b+c\omega+a\omega^2} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= -2pqr\left[\frac{1}{\omega^2(a\omega+b\omega^2+c)} + \frac{1}{\omega(b\omega^2+c+a\omega)} + \frac{1}{c+a\omega+b\omega^2}\right]$$

$$= \frac{-2pqr}{a\omega + b\omega^2 + c}\left[\frac{1}{\omega^2} + \frac{1}{\omega} + \frac{1}{1}\right] = 0 \qquad (2)$$

Hence, $p^2 + q^2 + r^2 = 2(pq + qr + rp)$

8. (3), (4)

We have,

$$x^2 + x + 1 = (x - \omega)(x - \omega^2)$$

Since $f(x)$ is divisible by $x^2 + x + 1$, $f(\omega) = 0$, $f(\omega^2) = 0$, so

$$P(\omega^3) + \omega Q(\omega^3) = 0 \Rightarrow P(1) + \omega Q(1) = 0 \quad (1)$$

$$P(\omega^6) + \omega^2 Q(\omega^6) = 0 \Rightarrow P(1) + \omega^2 Q(1) = 0 \quad (2)$$

Solving Eqs. (1) and (2), we obtain

$$P(1) = 0 \text{ and } Q(1) = 0$$

Therefore, both $P(x)$ and $Q(x)$ are divisible by $x - 1$. Hence, $P(x^3)$ and $Q(x^3)$ are divisible by $x^3 - 1$ and so by $x - 1$ Since $f(x) = P(x^3) + xQ(x^3)$, we get $f(x)$ is divisible by $x - 1$.

9. (1), (3), (4)

Let $z = c$ be a real root. Then,

$$\alpha c^2 + c + \bar{\alpha} = 0 \quad (1)$$

Putting $\alpha = p + iq$, we have

$$(p + iq)c^2 + c + p - iq = 0$$

$$\Rightarrow \quad pc^2 + c + p = 0 \text{ and } qc^2 - q = 0$$

$$\Rightarrow \quad c = \pm 1 \qquad (\because q \neq 0)$$

$$\therefore \quad (1) \Rightarrow \alpha \pm 1 + \bar{\alpha} = 0$$

Also, $|c| = 1$

10. (1), (2), (4)

Let $z = \alpha$ be a real root. Then,

$$\alpha^3 + (3 + 2i)\alpha + (-1 + ia) = 0$$

$$\Rightarrow \quad (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$$

$$\Rightarrow \quad \alpha^3 + 3\alpha - 1 = 0 \text{ and } \alpha = -a/2$$

$$\Rightarrow \quad -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$$

$$\Rightarrow \quad a^3 + 12a + 8 = 0$$

Let $f(a) = a^3 + 12a + 8$.

$f'(a) = 3a^2 + 12 > 0$.

So, $f(a)$ cuts x-axis only once.

Now, $f(-1) < 0, f(0) > 0$,

$f(-2) < 0, f(1) > 0$ and $f(3) > 0$

Hence, $a \in (-1, 0)$ or $a \in (-2, 1)$ or $a \in (-2, 3)$.

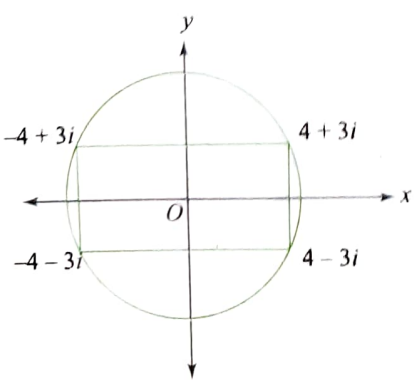
11. (1), (2), (3)

Let $z = x + iy$ where x, y satisfy the given equation. Hence,

$$(x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow \quad x^2 + y^2 = 25 \text{ and } x^2 - y^2 = 7 \text{ (as all other possibilities will give non-integral solutions)}$$

Hence, possible values of z will be $4 + 3i$, $4 - 3i$, $-4 + 3i$, and $-4 - 3i$. Clearly, it will form a rectangle having length of the diagonal 10.



From the diagram, options (1), (2), (3) are correct.

12. (1), (3)

Triangle ABC is equilateral. Hence,

$$z^2 + (-z)^2 + (1 - z)^2 = z(-z) + z(1 - z) + (-z)(1 - z)$$

or $\quad 3z^2 - 2z + 1 = -z^2$

or $\quad 4z^2 - 2z + 1 = 0$

Sum of roots $= \dfrac{1}{2}$

and Product of roots $= \dfrac{1}{4}$

13. (1), (4)

If $|z - 3| = |z - 1|$, then z lies on perpendicular bisector of line segment joining '1' and '3'. So, $\text{Re}(z) = 2$.

If $|z - 3| = |z - 5|$, then z lies on perpendicular bisector of line segment joining '3' and '4'. So, $\text{Re}(z) = 4$.

14. (1), (4)

$$\left|z_1^2 - z_2^2\right| = \left|\bar{z}_1^2 + \bar{z}_1^2 - 2\bar{z}_1\bar{z}_2\right|$$

or $\quad |z_1 - z_2||z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|^2$

or $\quad |z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|$

$$|z_1 + z_2| = |z_1 - z_2|$$

$$\Rightarrow \quad \left|\frac{z_1}{z_2} + 1\right| = \left|\frac{z_1}{z_2} - 1\right|$$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ lies on $\perp$ bisector of 1 and -1

$\Rightarrow \quad \dfrac{z_1}{z_2}$ lies on imaginary axis

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely imaginary

$$\Rightarrow \quad \arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$$

$$|\arg(z_1) - \arg(z_2)| = \frac{\pi}{2}$$

15. (1), (2), (3)

$$|z_1| = |z_2| = 1$$

$$\Rightarrow \quad a^2 + b^2 = c^2 + d^2 = 1 \quad (1)$$

and $\quad \text{Re}(z_1\bar{z}_2) = 0$

$$\Rightarrow \quad \text{Re}\{(a + ib)(c + id)\} = 0$$

$$\Rightarrow \quad ac + bd = 0 \quad (2)$$

Now, from Eqs. (1) and (2), we get

$$a^2 + b^2 = 1$$

$$\Rightarrow \quad a^2 + \frac{a^2c^2}{d^2} = 1$$

$$\Rightarrow \quad a^2 = d^2 \quad (3)$$

Also, $c^2 + d^2 = 1$

$$\Rightarrow \quad c^2 + \frac{a^2c^2}{b^2} = 1$$

$$\Rightarrow \quad b^2 = c^2 \quad (4)$$

$$|\omega_1| = \sqrt{a^2 + c^2} = \sqrt{a^2 + b^2} = 1 \quad \text{[From (1) and (4)]}$$

and $\quad |\omega_2| = \sqrt{b^2 + d^2} = \sqrt{c^2 + d^2} = 1 \quad$ [from (1) and (4)]

Further $\text{Re}(\omega_1\bar{\omega}_2) = \text{Re}\{(a + ic)(b - id)\}$

$$= ab + cd$$

$$= ab - \frac{ac^2}{b} \quad \text{[From (2)]}$$

$$= \frac{ab^2 - ac^2}{b} = 0 \qquad \text{[from (4)]}$$

Also, $\text{Im}(\omega_1 \bar{\omega}_2) = bc - ad$

$$= bc - a\left(-\frac{ac}{b}\right) = \frac{(a^2+b^2)c}{b} = \frac{c}{b} = \pm 1 \neq 0$$

$\therefore \quad |\omega_1| = 1, |\omega_2| = 1$ and $\text{Re}(\omega_1 \bar{\omega}_2) = 0$

16. (1), (4)

Let $z_1 = a + ib$, $a > 0$ and $b \in R$: $z_2 = c + id$, $d < 0$, $c \in R$

Given $|z_1| = |z_2|$

$\Rightarrow \quad a^2 + b^2 = c^2 + d^2$

$\Rightarrow \quad a^2 - c^2 = d^2 - b^2 \qquad \ldots(1)$

Now, $\dfrac{z_1 + z_2}{z_1 - z_2} = \dfrac{(a+c) + i(b+d)}{(a-c) + i(b-d)}$

$$= \frac{[(a^2 - c^2) + (b^2 - d^2)] + i\left[(a-c)(b+d) - (a+c)(b-d)\right]}{(a-c)^2 + (b-d)^2}$$

This values will be purely imaginary as $a^2 - c^2 = d^2 - b^2$ or the value will be zero in case $a + c = b + d = 0$.

17. (1), (3)

$$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$$

$\Rightarrow \quad 5 - 2\sqrt{3} = 2 + 3 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$,

where $\arg(z_1) = \theta_1$ and $\arg(z_2) = \theta_2$

$\Rightarrow \quad -2\sqrt{3} = 2\sqrt{2}\sqrt{3}\cos(\theta_1 - \theta_2)$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = -\dfrac{1}{\sqrt{2}}$

$\Rightarrow \quad \theta_1 - \theta_2 = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

$\Rightarrow \quad \arg(z_1) - \arg(z_2) = \arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$

18. (2), (3)

$$z_3 = \frac{z_2(z_1 - z_4)}{\bar{z}_1 z_4 - 1}$$

$$\Rightarrow \quad |z_3| = \frac{|z_2||z_1 - z_4|}{|\bar{z}_1 z_4 - 1|} = \frac{|z_1 - z_4|}{|\bar{z}_1 z_4 - 1|} \leq 1$$

If and only if $|z_1 - z_4|^2 \leq |\bar{z}_1 z_4 - 1|^2$

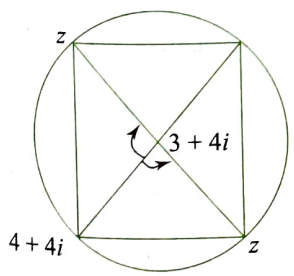
$\Rightarrow \quad |z_1|^2 + |z_2|^2 - z_1\bar{z}_4 - \bar{z}_1 z_4 \leq |\bar{z}_1 z_4|^2 + 1 - z_1\bar{z}_4 - \bar{z}_1 z_4$

$\Rightarrow \quad |z_1|^2 + |z_4|^2 - |z_1|^2|z_4|^2 - 1 \leq 0$

$\Rightarrow \quad (|z_1|^2 - 1)(|z_4|^2 - 1) \geq 0$

Since $|z_1| \leq 1$, we must have $|z_4| \leq 1$

19. (2), (3)



Clearly, the inscribed rectangle is a square. Let the adjacent vertex be z. Then,

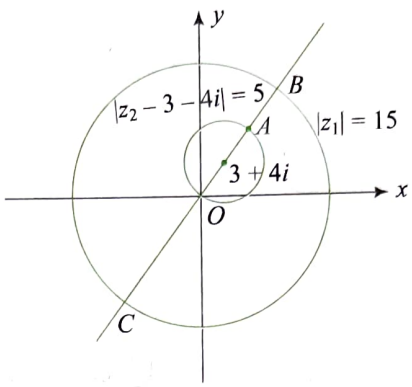
$$\frac{3 + 4i - (z)}{(3+4i) - (4+4i)} = e^{\pm i\pi/2} \qquad \text{(by rotation about center)}$$

$\Rightarrow \quad 3 + 4i - z = \pm i(-1)$

$\Rightarrow \quad z = 3 + 4i \pm i = 3 + 5i$ or $3 + 3i$

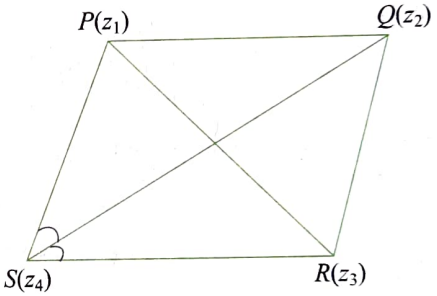
20. (1), (4)

We have, $|z_1| = 15$, $|z_2 - 3 - 4i| = 5$



Minimum value of $|z_1 - z_2|$ is $AB = OB - OA = 15 - 10 = 5$.
Maximum value of $|z_1 - z_2|$ is $CA = OA + OC = 10 + 15 = 25$.

21. (1), (2), (3), (4)



(1) $PS \parallel QR \Rightarrow \arg\left(\dfrac{z_1 - z_4}{z_2 - z_3}\right) = 0 \Rightarrow \dfrac{z_1 - z_4}{z_2 - z_3}$ is purely real

(2) Since diagonal bisect the angle

$$\Rightarrow \text{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \text{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

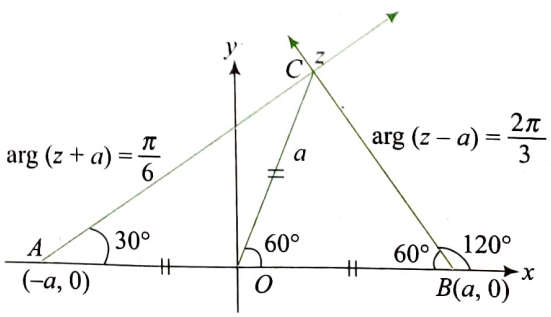
(3) Diagonals of rhombus are perpendicular. Hence, $(z_1 - z_3)/(z_2 - z_4)$ is purely imaginary.

(4) Diagonals of rhombus are not necessarily equal.
Hence, $|z_1 - z_3| \neq |z_2 - z_4|$ not necessarily.

22. (1), (4)

Refer to figure, z lies on the point of intersection of the rays from A and B. $\angle ACB$ is a right angle and OBC is an equilateral triangle. Hence,

$$OC = a \Rightarrow z = a\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$



23. (1), (2), (4)

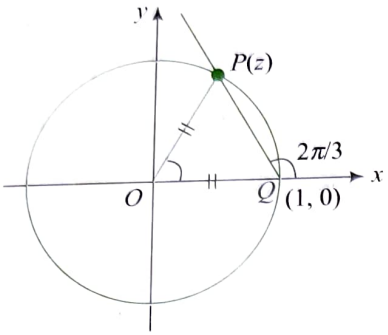
We have

$$|z| = 1$$

This means that z lies on a circle whose centre is at origin and radius is 1.

Also, $\arg(z-1) = \dfrac{2\pi}{3}$

This implies that z lies on a ray emanating from (1, 0) making an angle of $\dfrac{2\pi}{3}$ with positive real axis.



$$\Rightarrow \quad \angle PQX = \frac{2\pi}{3}$$

$$\Rightarrow \quad \angle PQO = \frac{\pi}{3}$$

and $OP = OQ$

Thus, triangle OPQ is equilateral.

$$\Rightarrow \quad OP = OQ = PQ$$

$$\Rightarrow \quad |z| = |z-1| = 1$$

Also, $\arg(z) = 60°$

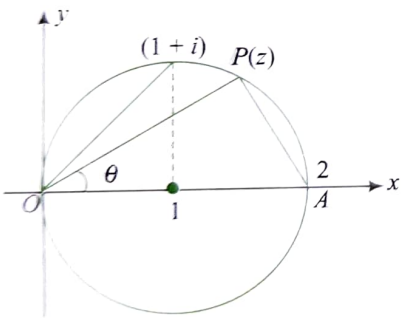
$$\therefore \quad z = \cos 60° + i \sin 60° = \frac{1+i\sqrt{3}}{2} = -\omega^2$$

$$z^2 = \omega^4 = \omega$$

$$\Rightarrow \quad z^2 + z = \omega - \omega^2 = \left(\frac{-1+i\sqrt{3}}{2}\right) - \left(\frac{-1-i\sqrt{3}}{2}\right) = i\sqrt{3}$$

Therefore, $z^2 + z$ is purely imaginary number.

24. (1), (2), (4)



Since $\arg\left((z-1-i)/z\right)$ is the angle subtended by the chord joining the points O and $1 + i$ at the circumference of the circle $|z-1| = 1$, so it is equal to $-\pi/4$. The line joining the points $z = 0$ and $z = 2 + 0i$ is the diameter.

$$\arg\frac{z-2}{z} = \pm\frac{\pi}{2}$$

$$\Rightarrow \quad \frac{z-2}{z-0} \text{ is purely imaginary}$$

We have

$$\angle OPA = \frac{\pi}{2}$$

$$\Rightarrow \quad \arg\left(\frac{2-z}{0-z}\right) = \frac{\pi}{2} \Rightarrow \frac{z-2}{z} = \frac{AP}{OP}i$$

Now in ΔOAP,

$$\tan\theta = \frac{AP}{OP}$$

Thus, $\dfrac{z-2}{z} = i\tan\theta$

25. (1), (2), (4)

$z_1 = 5 + 12i$, $|z_2| = 4$

$$|z_1 + iz_2| \le |z_1| + |z_2| = 13 + 4 = 17$$

$$\therefore \quad |z_1 + (1+i)z_2| \ge ||z_1| - |1+i|\,|z_2|| = 13 - 4\sqrt{2}$$

$$\therefore \quad \min(|z_1 + (1+i)z_2|) = 13 - 4\sqrt{2}$$

$$\left|z_2 + \frac{4}{z_2}\right| \le |z_2| + \frac{4}{|z_2|} = 4 + 1 = 5$$

$$\left|z_2 + \frac{4}{z_2}\right| \ge |z_2| - \frac{4}{|z_2|} = 4 - 1 = 3$$

$$\therefore \quad \max\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{3} \text{ and } \min\left|\frac{z_1}{z_2 + \frac{4}{z_2}}\right| = \frac{13}{5}$$

26. (1), (2), (4)

Given, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$

$$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc = 0$$

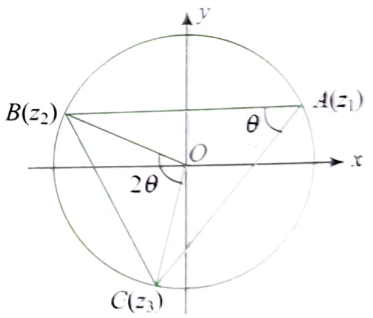
$$\Rightarrow \quad (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow \quad \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow \quad (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow \quad a = b = c \quad [\because a+b+c \ne 0,\ \because z_1 \ne 0,\ \therefore |z_1| = a \ne 0 \text{ etc.}]$$

Hence, $OA = OB = OC$, where O is the origin and A, B, C are the points representing z_1, z_2 and z_3, respectively. Therefore, O is circumcenter of ΔABC. Now,



$$\arg\left(\frac{z_3}{z_2}\right) = \angle BOC \qquad (1)$$

$$= 2\angle BAC = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) \qquad (2)$$

$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2 \qquad [\because \angle BOC = 2\angle BAC]$$

Hence, $\arg\left(\dfrac{z_3}{z_2}\right) = \arg\left(\dfrac{z_3 - z_1}{z_2 - z_1}\right)^2$

Also, centroid is $(z_1 + z_2 + z_3)/3$. Since $HG:GO \equiv 2:1$ (where H is orthocenter and G is centroid), then orthocenter is $z_1 + z_2 + z_3$ (by section formula). When triangle is equilateral centroid coincides with circumcenter; hence $z_1 + z_2 + z_3 = 0$.

Also, the area for equilateral triangle is $(\sqrt{3}/4)L^2$, where L is length of side. Since radius is $|z_1|$, $L = \sqrt{3}|z_1|$, the area is $(3\sqrt{3}/4)|z_1|^2$.

27. (2), (3)

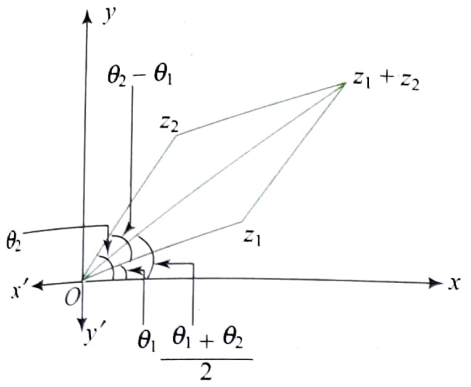
z_1 and z_2 are the roots of the equation $z^2 - az + b = 0$. Hence,

$z_1 + z_2 = a, z_1 z_2 = b$

Now, $|z_1 + z_2| \le |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2| = |a| \le 1 + 1 = 2 \qquad (\because |z_1| = |z_2| = 1)$

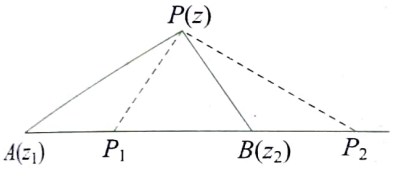
$\Rightarrow \quad \arg(a) = \frac{1}{2}[\arg(z_2) + \arg(z_1)]$



Also, $\arg(b) = \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\Rightarrow \quad 2\arg(a) = \arg(b)$

28. (2), (3)



Let internal and external bisectors of $\angle APB$ meet the line joining A and B at P_1 and P_2, respectively. Hence,

$AP_1 : P_1B \equiv PA : PB \equiv 3:1$ (internal division)

$AP_2 : P_2B \equiv PA : PB \equiv 3:1$ (external division)

Thus, P_1 and P_2 are fixed points. Also,

$$\angle P_1 PP_2 = \frac{\pi}{2}$$

Thus, P lies on a circle having P_1P_2 as its diameter. Clearly, $B(z_2)$ lies inside this circle.

29. (1), (2), (4)

$$\left|\frac{2z-i}{z+1}\right| = m$$

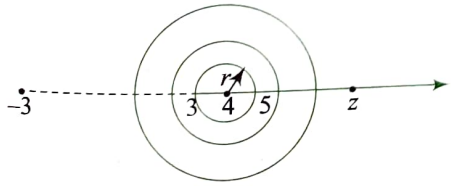
$$\Rightarrow \quad \left|z - \frac{i}{2}\right| = \frac{m}{2}|z+1|$$

This shows that the given equation will represent a circle, if $m/2 \neq 0, 1$, i.e., $m \neq 0, 2$.

30. (1), (3), (4)

Equation $|z + 3| - |z - 3| = 6$ represents a ray to the right of complex number '3'.

$|z - 4| = r$ represents a circle whose center is 4 and radius r.



When radius of circle r = 1, it cuts ray at two points '3' and '5'.

For radius $r < 1$, circle cuts ray at two points.

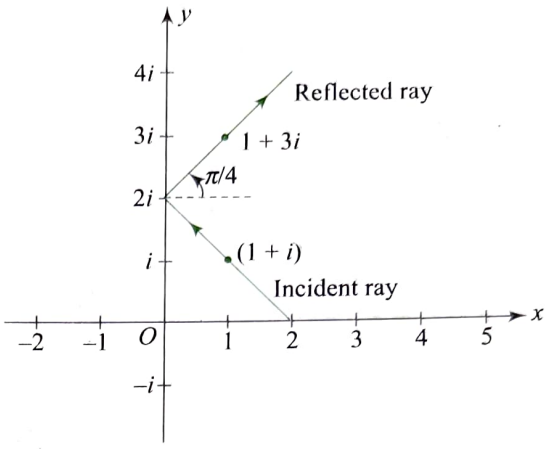
When radius of circle $r > 1$, it cuts ray at only one point.

31. (1), (2)

Incident ray is $|z - 2| - |z - 1 - i| = \sqrt{2} = |2 - (1 + i)|$

This ray is emanating from the point '2' and moving to the left of '2.

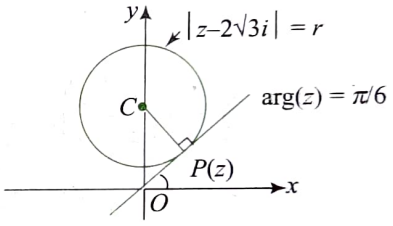
It strikes y-axis at point $2i$.



Then equation of reflected ray is $\arg(z - 2i) = \frac{\pi}{4}$

or $\quad |z - 2i| - |z - 3 - i| = \sqrt{2}$.

32. (1), (4)



$CP = r, OC = 2\sqrt{3}, \angle COP = \pi/3$

$$\Rightarrow \quad CP = OC \sin\frac{\pi}{3} = 2\sqrt{3}\,\frac{\sqrt{3}}{2} = 3$$

Thus, when $r = 3$, the circle touches the line. Hence, for two distinct points of intersection, $3 < r < 2\sqrt{3}$.

33. (1), (2), (4)

We have $z_1 + 2 = 3z$ and $-z_2 - 2 = 3z$

$\Rightarrow \quad |z_1 + 2| = 3|z| = 3$

and $\quad z_2 + 2 = -3z$

$\Rightarrow \quad |z_2 + 2| = |-3z| = 3$

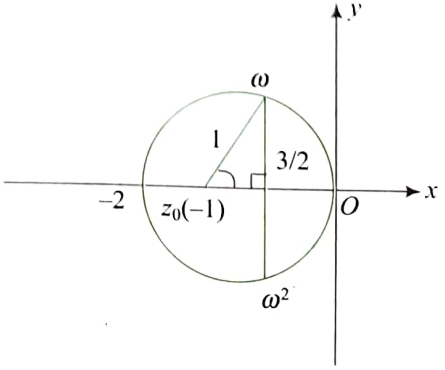
Thus, z_1 and z_2 describe the same circle having center at '−2' and radius '3'.

Now, $z_1 - z_2 = 6z$

$\Rightarrow \quad |z_1 - z_2| = 6|z| = 6$

So, $(z_1 - z_2)$ moves on a circle with centre as origin and radius 6.

34. (1), (4)



Given $|z_0| = |z_0 - \omega| = |z_0 - \omega^2|$

So, the distance of z_0 from 0, ω and ω^2 is same.

So, z_0 is circumcenter of triangle joining ω and ω^2.

$\therefore \quad z_0 = -1$

Hence, set $S = \{z : x = x + iy, y \geq 0, |z + 1| \leq 1\}$

Therefore, minimum value of $|z|$ is 0.

Also, $\arg(\omega - z_0)$ = angle made by line joining $z_0 (= -1)$ to ω with real axis $= \dfrac{\pi}{3}$

35. (2), (3)

We have

$$\left|\frac{1}{z_2}+\frac{1}{z_1}\right|=\left|\frac{1}{z_2}-\frac{1}{z_1}\right|$$

$\Rightarrow \quad |z_1 + z_2| = |z_1 - z_2|$

Squaring both sides, we have

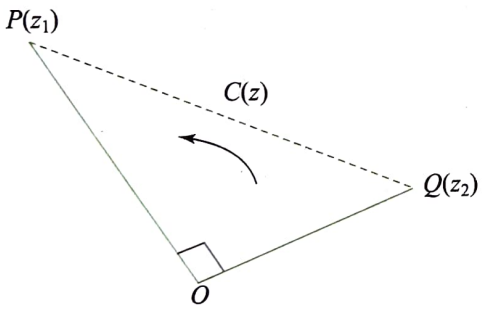
$$|z_1|^2 + |z_2|^2 + 2\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)$$

$$= |z_1|^2 + |z_2|^2 - 2\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right)$$

$\Rightarrow \quad 4\left(z_1\bar{z}_2 + \bar{z}_1 z_2\right) = 0$

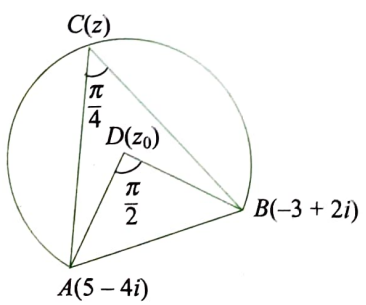
$\Rightarrow \quad \dfrac{z_1}{z_2} + \dfrac{\bar{z}_1}{\bar{z}_2} = 0$

$\Rightarrow \quad \arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{2} = \arg\left(\dfrac{z_1 - 0}{z_2 - 0}\right)$



The angle between z_2, O, and z_1 is a right angle, taken in order, as shown in the above diagram. Now, the circumcenter of the above diagram will lie on the line PQ as diameter and is represented by C which is the center of PQ, such that $z = (z_1 + z_2)/2$, where z is the affix of circumcenter.

36. (1), (3), (4)



$$\frac{z_0 - (-3 + 2i)}{z_0 - (5 - 4i)} = \frac{BD}{AD} e^{i\pi/2} = i$$

$\Rightarrow \quad z_0 + 3 - 2i = iz_0 - 5i - 4$

$\Rightarrow \quad z_0 = -2 - 5i$

$\Rightarrow \quad$ Radius $AD = |5 - 4i - (-2 - 5i)|$

$= |7 + i|$

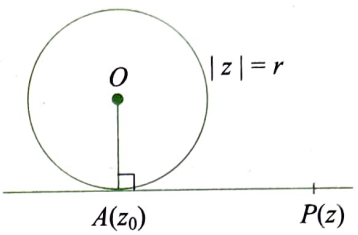
$= \sqrt{50} = 5\sqrt{2}$

Length of arc $= \dfrac{3}{4}$(Perimeter of circle)

$= \dfrac{3}{4}(2\pi \times 5\sqrt{2})$

$= \dfrac{15\pi}{\sqrt{2}}$

37. (1), (2)



$$\angle OAP = \frac{\pi}{2}$$

$\Rightarrow \quad \dfrac{z - z_0}{z_0}$ is purely imaginary

$\Rightarrow \quad \dfrac{z - z_0}{z_0} + \dfrac{\bar{z} - \bar{z}_0}{\bar{z}_0} = 0$

or $\quad \dfrac{z}{z_0} + \dfrac{\bar{z}}{\bar{z}_0} = 2 \qquad (1)$

$\Rightarrow \quad \text{Re}\left(\dfrac{z}{z_0}\right) = 1$

From (1),

$$z\bar{z}_0 + z_0\bar{z} = 2|z_0|^2 = 2r^2$$

38. (1), (3)

Given,

$z^n = (z + 1)^n \quad \Rightarrow |z^n| = |(z + 1)^n|$

$\therefore \quad |z|^n = |z + 1|^n \quad \Rightarrow |z| = |z + 1|$

$\Rightarrow \quad |z|^2 = |z + 1|^2$

$\Rightarrow \quad x^2 + y^2 = (x + 1)^2 + y^2$, where $z = x + iy$

$\Rightarrow \quad x = -\dfrac{1}{2}$

Hence, z lies on the line $x = -1/2$. Hence, sum of real parts of the roots is $-(n-1)/2$ (since equation has $n - 1$ roots).

39. (1), (2)

$$\left|z - \frac{1}{z}\right| = 1$$

$\Rightarrow \quad 1 \geq \left||z| - \dfrac{1}{|z|}\right|$

$\Rightarrow \quad -1 \leq |z| - \dfrac{1}{|z|} \leq 1$

$\Rightarrow \quad -|z| \leq |z|^2 - 1 \leq |z|$

From $|z|^2 - 1 \geq -|z|$, we get

$$|z|^2 + |z| - 1 \geq 0$$

$\Rightarrow \quad |z| \geq \dfrac{-1 + \sqrt{5}}{2} \qquad (1)$

From $|z|^2 - 1 \le |z|$, we get

$$|z|^2 - |z| - 1 \le 0$$

$$\Rightarrow \quad \frac{1-\sqrt{5}}{2} \le |z| \le \frac{1+\sqrt{5}}{2} \qquad (2)$$

From (1) and (2), we get

$$\Rightarrow \quad \frac{-1+\sqrt{5}}{2} \le |z| \le \frac{1+\sqrt{5}}{2}$$

$$\Rightarrow \quad |z|_{\min} = \frac{\sqrt{5}-1}{2}, \; |z|_{\max} = \frac{1+\sqrt{5}}{2}$$

40. (1), (2), (4)

$$x^n - 1 = (x-1)(x-z_1)(x-z_2)\cdots(x-z_{n-1})$$

$$\Rightarrow \quad \frac{x^n-1}{x-1} = (x-z_1)(x-z_2)\cdots(x-z_{n-1})$$

Putting $x = \omega$, we have

$$\prod_{r=1}^{n-1}(\omega - z_r) = \frac{\omega^n - 1}{\omega - 1} = \begin{cases} 0, & \text{if } n = 3k,\ k \in Z \\ 1, & \text{if } n = 3k+1,\ k \in Z \\ 1+\omega, & \text{if } n = 3k+2,\ k \in Z \end{cases}$$

41. (1), (3)

If $p = q$, then equation becomes $z^p = \bar{z}^q$ and it has infinite number of solutions because any $z \in R$ will satisfy it. If $p \ne q$, let $p > q$, then $z^p = \bar{z}^q$.

$\therefore \quad |z|^p = |z|^q$

$\Rightarrow \quad |z|^p (|z|^{p-q} - 1) = 0$

$\Rightarrow \quad |z| = 0$ or $|z| = 1$

$|z| = 0 \Rightarrow z = 0 + i0$

$|z| = 1 \Rightarrow z = e^{i\theta}$

$\Rightarrow \quad e^{(p+q)\theta i} = 1$

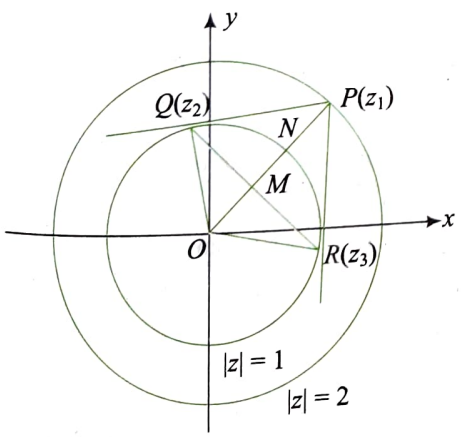
$\Rightarrow \quad z = 1^{1/(p+q)}$

Therefore, the number of solutions is $p + q + 1$.

42. (1), (2), (3), (4)

Number of roots common to $z^{n_1} = 1$ and $z^{n_2} = 1$ is HCF (n_1, n_2)

43. (1), (2), (3), (4)



Since $OQ = 1$ and $OP = 2$, so $\sin(\angle OPQ) = 1/2$ and hence $\angle QPR = \pi/3$. Then $\angle PQR$ is equilateral. Also, $OM \perp QR$. Then from ΔOMQ, $OM = 1/2$. Hence, $MN = 1/2$. Then centroid of ΔPQR lies on $|z| = 1$.

As PQR is an equilateral triangle, so orthocenter, circumcenter, and centroid will coincide. Now,

$$\left|\frac{z_1+z_2+z_3}{3}\right| = 1$$

or $\quad |z_1 + z_2 + z_3|^2 = 9$

or $\quad (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 9$

or $\quad \left(\frac{4}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3}\right)\left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = 9$

and $\quad \angle QOR = 120°$

44. (1), (3), (4)

$$z' = ze^{i\alpha} \qquad (1)$$

$$z'' = ze^{-i\alpha} \qquad (2)$$

$\therefore \quad z'z'' = z^2$

$\Rightarrow \quad z', z, z''$ are in G.P.

Also, $\left(\frac{z'}{z}\right)^2 + \left(\frac{z''}{z}\right)^2 = 2\cos 2\alpha$

$\Rightarrow \quad z'^2 + z''^2 = 2z^2 \cos 2\alpha$

$= 2z^2(2\cos^2\alpha - 1)$

or $\quad z'^2 + z''^2 + 2z^2 = 4z^2\cos^2\alpha$

or $\quad z'^2 + z''^2 + 2z'z'' = 4z^2\cos^2\alpha$

or $\quad (z' + z'')^2 = 4z^2\cos^2\alpha$

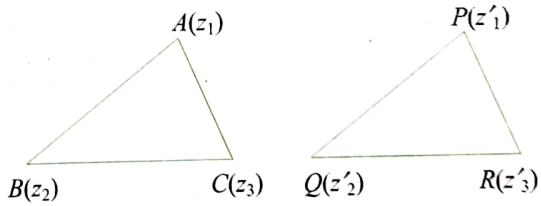
or $\quad z' + z'' = 2z\cos\alpha$

45. (1), (2), (3), (4)

$$z_3 = (1-\lambda)z_1 + z_2 = \frac{(1-\lambda)z_1 + \lambda z_2}{1-\lambda+\lambda}$$

Hence, z_3 divides the line joining $A(z_1)$ and $B(z_2)$ in the ratio $\lambda:(1-\lambda)$. That means the given points are collinear. Also, the ratio $\lambda/(1-\lambda) > 0$ (or $0 < \lambda < 1$) if z_3 divides the line joining z_1 and z_2 internally and $\mu/(1-\mu) < 0$ (or $\mu < 0$ or $\mu > 1$) if z_3' divides the line joining z_1', z_2' externally.

When λ, μ are complex numbers, where $\lambda = \mu$, we have $z_3 = (1-\lambda)z_1 + \lambda z_2$ and $z_3' = (1-\lambda)z_1' + \lambda z_2'$.



Comparing the value of λ, we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{z_3' - z_1'}{z_2' - z_1'}$$

$$\Rightarrow \quad \left|\frac{z_3 - z_1}{z_2 - z_1}\right| = \left|\frac{z_3' - z_1'}{z_2' - z_1'}\right| \text{ and } \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \arg\left(\frac{z_3' - z_1'}{z_2' - z_1'}\right)$$

$$\Rightarrow \quad \frac{AC}{AB} = \frac{PR}{PQ} \text{ and } \angle BAC = \angle QPR$$

Hence, triangles ABC and PQR are similar.

46. (1), (3), (4)

Choice (a) on simplification gives

$$z = \frac{1+x}{1+x^2} + i\frac{1+x}{1+x^2}$$

For $x = 0.5, f'(0.5) > 1$ which is out of range, Hence, (1) is not correct. From choice (2),

$$z = \frac{1-x}{1+x^2} + i\frac{1-x}{1+x^2}$$

$f(x)$ and $g(x) \in (0, 1)$ if $x \in (0, 1)$. Hence, (2) is correct. From choice (3),

$$z = \frac{1+x}{1+x^2} + \frac{1-x}{1+x^2}i$$

Hence, (3) is not correct. From choice (4),

$$z = \frac{1-x}{1+x^2} + \frac{1+x}{1+x^2}i$$

Hence, (4) is not correct.

47. (1), (2), (4)

It is given that z_1 and z_2 are the roots of the equation $az^2 + bz + c = 0$.

So, $|z_1 + z_2| = \left|-\frac{b}{a}\right| = 1$

And $|z_1 z_2| = \left|\frac{c}{a}\right| = 1$

$\therefore \quad |z_2| = 1$

$|z_1 + z_2|^2 = 1$

$\therefore \quad 2 + z_1\bar{z}_2 + z_2\bar{z}_1 = 1$

Now, $z_2 = z_1 e^{i\theta}$

$\therefore \quad |z_1 + z_1 e^{i\theta}| = |z_1||1 + e^{i\theta}| = 1$

$\therefore \quad 2\cos\frac{\theta}{2} = 1$

$\therefore \quad \theta = \frac{2\pi}{3}$

Now, $\frac{(z_1+z_2)^2}{z_1 z_2} = 1$

$\Rightarrow \quad \frac{b^2}{a^2} = \frac{c}{a}$

$\Rightarrow \quad b^2 = ac$

$PQ = |z_1 - z_2| = \sqrt{3}$

48. (1), (2), (3), (4)

Let the roots of $az^3 + bz^2 + cz + d = 0$ be $z_1 = x_1$, $z_2 = x_2 + iy_2$ and $z_3 = x_2 - iy_2$.

$\therefore \quad z_1 + z_2 + z_3 = -\frac{b}{a}$

$\Rightarrow \quad x_1 + 2x_2 = -\frac{b}{a} < 0$

$\Rightarrow \quad ab > 0$

$z_1 z_2 + z_2 z_3 + z_1 z_3 = \frac{c}{a}$

$x_1(x_2 + iy_2) + (x_2^2 + y_2^2) + x_1(x_2 - iy_2)$
$= 2x_1 x_2 + x_2^2 + y_2^2 > 0$

$\Rightarrow \quad \frac{c}{a} > 0$

$\Rightarrow \quad ac > 0$

$\Rightarrow \quad a^2 bc > 0$

$\Rightarrow \quad bc > 0$

Also, $z_1 z_2 z_3 = x_1(x_2^2 + y_2^2) = -\frac{d}{a}$

$\Rightarrow \quad ad > 0$

Further, $-\frac{bc}{a^2} = \left(-\frac{b}{a}\right)\left(\frac{c}{a}\right)$

$= (x_1 + 2x_2)(2x_1 x_2 + x_2^2 + y_2^2)$
$= x_1(x_2^2 + y_2^2) + 2x_1^2 x_2 + 2x_1 x_2^2 + 2x_2(x_2^2 + y_2^2)$
$< x_1(x_2^2 + y_2^2)$

$\Rightarrow \quad -\frac{bc}{a^2} < -\frac{d}{a}$

$\Rightarrow \quad bc > ad$

49. (1), (2), (3)

$$\frac{3}{2 + e^{i\theta}} = ax + iby$$

$\Rightarrow \quad \frac{3}{ax + iby} - 2 = e^{i\theta}$

$\Rightarrow \quad \frac{3 - 2ax - 2iby}{ax + iby} = e^{i\theta}$

$\Rightarrow \quad \left|\frac{3 - 2ax - 2iby}{ax + iby}\right| = |e^{i\theta}| = 1$

$\Rightarrow \quad (3 - 2ax)^2 + 4b^2y^2 = a^2x^2 + b^2y^2$

When $a = 1$ and $b = 2$, we have

$(3 - 2x)^2 + 16y^2 = x^2 + 4y^2$

$\Rightarrow \quad 3x^2 + 12y^2 - 12x + 9 = 0$

$\Rightarrow \quad x^2 - 4x + 4y^2 + 3 = 0$; which is ellipse

When $a = b = 1$, we have

$(3 - 2x)^2 + 4y^2 = x^2 + y^2$.

$\Rightarrow \quad 3x^2 + 3y^2 - 12x + 9 = 0$

or $\quad x^2 + y^2 - 4x + 3 = 0$; which is circle

When $a = 1$ and $b = 0$, we have

$(3 - 2x)^2 = x^2$; which is pair of straight lines

Linked Comprehension Type

1. (1), 2. (4), 3. (3), 4. (4)

$$z = \frac{1 - i\sin\theta}{1 + i\cos\theta} = \frac{(1 - i\sin\theta)(1 - i\cos\theta)}{(1 + i\cos\theta)(1 - i\cos\theta)}$$
$$= \frac{(1 - \sin\theta\cos\theta) - i(\cos\theta + \sin\theta)}{(1 + \cos^2\theta)}$$

If z is purely real, then

$\cos\theta + \sin\theta = 0$

or $\quad \tan\theta = -1$

$\Rightarrow \quad \theta = n\pi - \frac{\pi}{4}, n \in I$

If z is purely imaginary, $1 - \sin\theta\cos\theta = 0$ or $\sin\theta\cos\theta = 1$, which is not possible.

$$|z| = \left|\frac{1 - i\sin\theta}{1 + i\cos\theta}\right| = \frac{\sqrt{1 + \sin^2\theta}}{\sqrt{1 + \cos^2\theta}}$$

If $|z| = 1$, then

$\cos^2\theta = \sin^2\theta$

$\Rightarrow \quad \tan^2\theta = 1$

$\Rightarrow \quad \theta = n\pi \pm \frac{\pi}{4}, n \in I$

We have,

$$\arg(z) = \tan^{-1}\left(\frac{-(\cos\theta + \sin\theta)}{(1 - \sin\theta\cos\theta)}\right)$$

Now, $\arg(z) = \pi/4$

$$\Rightarrow \frac{-(\cos\theta+\sin\theta)}{(1-\sin\theta\cos\theta)} = 1$$

$$\Rightarrow \cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta = 1+\sin^2\theta\cos^2\theta-2\sin\theta\cos\theta$$

$$\Rightarrow 1+4\sin\theta\cos\theta = 1+\sin^2\theta\cos^2\theta$$

$$\Rightarrow \sin^2\theta\cos^2\theta-4\sin\theta\cos\theta = 0$$

$$\Rightarrow \sin\theta\cos\theta(\sin\theta\cos\theta-4) = 0$$

$$\Rightarrow \sin\theta\cos\theta = 0 \quad (\because \sin\theta\cos\theta = 4 \text{ is not possible})$$

$$\Rightarrow \theta = (2n+1)\pi \text{ or } \theta = (4n-1)\pi/2,\ n \in I$$

$(\because -\cos\theta-\sin\theta > 0)$

5. (2), 6. (2), 7. (3), 8. (3)

Given that

$$|z_1+z_2|^2 = |z_1|^2+|z_2|^2$$

$$\Rightarrow |z_1|^2+|z_2|^2+z_1\bar{z}_2+\bar{z}_1z_2 = |z_1|^2+|z_2|^2$$

$$\Rightarrow z_1\bar{z}_2+\bar{z}_1z_2 = 0 \quad (1)$$

$$\Rightarrow \frac{z_1}{z_2}+\frac{\bar{z}_1}{\bar{z}_2} = 0 \quad \text{(dividing by } z_2\bar{z}_2\text{)}$$

$$\Rightarrow \frac{z_1}{z_2}+\overline{\left(\frac{z_1}{z_2}\right)} = 0 \quad (2)$$

From (1), $z_2\bar{z}_2$ is purely imaginary. From (2), z_1/z_2 is purely imaginary. Hence,

$$\arg\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2} \text{ or } \arg(z_1)-\arg(z_2) = \pm\frac{\pi}{2}$$

Also, $i(z_1/z_2)$ is purely real. Hence its possible arguments are 0 and π.

9. (1), 10. (2), 11. (3)

$$z = -\lambda \pm \sqrt{\lambda^2-1}$$

Case I:

When $-1 < \lambda < 1$, we have

$$\lambda^2 < 1 \Rightarrow \lambda^2-1 < 0$$

$$z = -\lambda \pm i\sqrt{1-\lambda^2}$$

or $x = -\lambda,\ y = \pm i\sqrt{1-\lambda^2}$

$\Rightarrow y^2 = 1-x^2$ or $x^2+y^2 = 1$

Case II:

$$\lambda > 1 \Rightarrow \lambda^2-1 > 0$$

$$z = -\lambda \pm \sqrt{\lambda^2-1}$$

or $x = -\lambda \pm \sqrt{\lambda^2-1},\ y = 0$

Roots are $(-\lambda+\sqrt{\lambda^2-1}, 0)$, $(-\lambda-\sqrt{\lambda^2-1}, 0)$. One root lies inside the unit circle and the other root lies outside the unit circle.

Case III:

When λ is very large, then

$$z = -\lambda-\sqrt{\lambda^2-1} \approx -2\lambda$$

$$z = -\lambda+\sqrt{\lambda^2-1} = \frac{(-\lambda+\sqrt{\lambda^2-1})(-\lambda-\sqrt{\lambda^2-1})}{(-\lambda-\sqrt{\lambda^2-1})}$$

$$= \frac{1}{-\lambda-\sqrt{\lambda^2-1}} = -\frac{1}{2\lambda}$$

12. (2), 13. (3)

Given equation is $z(z^3+az^2+(12+9i)z+b) = 0$

So, either $z = 0$

or $z^3+az^2+(12+9i)z+b = 0 \quad \ldots(1)$

So, one of the vertices of the square is origin.

Therefore, other three vertices are z_1, iz_1 and (z_1+iz_1) which are the roots equation (1).

$$\therefore z_1(iz_1)+(iz_1)(z_1+iz_1)+z_1(z_1+iz_1) = 12+9i$$

$$\Rightarrow z_1^2(i+i+i^2+1+i) = 12+9i$$

$$\Rightarrow 3iz_1^2 = 12+9i$$

$$\Rightarrow z_1^2 = 3-4i$$

$$\Rightarrow z_1 = \sqrt{3-4i} = \pm(2-i)$$

$\therefore z_1 = 2-i$ or $-2+i$ (both values will give the same result)

Also, $-a = z_1+iz_1+z_1(1+i) = 2z_1(1+i)$

$$= 2(2-i)(1+i) = 6+2i$$

$$\therefore a = -6-2i$$

Further, $(z_1)(iz_1)(1+i)z_1 = -b$

$$\therefore -b = z_1^3(-1+i)$$

or $b = (2-i)^3(1-i) = -9-13i$

$$b = -9-13i$$

$$|a-b| = |3+11i| = \sqrt{130}$$

$$|z_1| = \sqrt{5}$$

Therefore, area of square is 5 sq. units.

14. (1), 15. (4), 16. (4).

Let z_1 (purely imaginary) be a root of the given equation. Then,

$z_1 = -\bar{z}_1$

and $az_1^2+bz_1+c = 0 \quad (1)$

$$\Rightarrow \overline{az_1^2+bz_1+c} = 0$$

$$\Rightarrow \bar{a}\bar{z}_1^2+\bar{b}\bar{z}_1+\bar{c} = 0$$

$$\Rightarrow \bar{a}z_1^2-\bar{b}z_1+\bar{c} = 0 \quad (\text{as } \bar{z}_1 = -z_1) \quad (2)$$

Now Eqs. (1) and (2) must have one common root.

$$\therefore (c\bar{a}-a\bar{c})^2 = (b\bar{c}+c\bar{b})(-a\bar{b}-\bar{a}b)$$

Let z_1 and z_2 be two purely imaginary roots. Then,

$\bar{z}_1 = -z_1,\ \bar{z}_2 = -z_2$

Now, $az^2+bz+c = 0 \quad (3)$

or $\overline{az^2+bz+c} = \bar{0}$

or $\bar{a}\bar{z}^2+\bar{b}\bar{z}+\bar{c} = 0$

or $\bar{a}z^2-\bar{b}z+\bar{c} = 0 \quad (4)$

Equations (3) and (4) must be identical as their roots are same.

$$\therefore \frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$$

$\Rightarrow a\bar{c} = \bar{a}c$, $a\bar{b}+\bar{a}b = 0$ and $b\bar{c}+\bar{b}c = 0$

Hence, $a\bar{c}$ is purely real and $a\bar{b}$ and $b\bar{c}$ are purely imaginary.

Let z_1 (purely real) be a root of the given equation. Then,

$z_1 = \bar{z}_1$

and $az_1^2+bz_1+c = 0 \quad (5)$

or $\overline{az_1^2+bz_1+c} = \bar{0}$

or $\bar{a}\bar{z}_1^2+\bar{b}\bar{z}_1+\bar{c} = 0$

or $\bar{a}z_1^2+\bar{b}z_1+\bar{c} = 0 \quad (6)$

Now (5) and (6) must have one common root. Hence,

$$(c\bar{a}-a\bar{c})^2 = (b\bar{c}-c\bar{b})(a\bar{b}-\bar{a}b)$$

17. (4) We know that

$$|z + i\omega| \le |z| + |i\omega| = |z| + |i|\,|\omega| \le 2$$

Given $|z + i\omega| = 2 \Rightarrow |z| = |\omega| = 1$.

18. (4) Suppose $z = x + iy$, $\omega = \alpha + i\beta$

$$|z + i\omega| = 2$$

$$\Rightarrow \quad |z|^2 + |\omega|^2 + i\omega\bar{z} - i\bar{\omega}z = 4$$

$$\Rightarrow \quad i\omega\bar{z} - i\bar{\omega}z = 2 \qquad (1)$$

Also $|z - i\bar{\omega}| = 2$

$$\Rightarrow \quad |z|^2 + |\omega|^2 + i\omega z - i\bar{\omega}\bar{z} = 4$$

$$\Rightarrow \quad i\omega z - i\bar{\omega}\bar{z} = 2 \qquad (2)$$

Adding (1) and (2), we get

$$\Rightarrow \quad i(\omega - \bar{\omega})(z + \bar{z}) = 4$$

$$\Rightarrow \quad i(2i\beta)(2x) = 4$$

$$\Rightarrow \quad \beta x = -1 \qquad (3)$$

Subtracting (1) from (2), we get

$$i\,(\omega + \bar{\omega})\,(z - \bar{z}) = 0 \Rightarrow \alpha y = 0 \qquad (4)$$

$\Rightarrow$ either $\alpha = 0$ or $y = 0$

If $y = 0$, then from $x^2 + y^2 = 1$,

$x = \pm 1$ and $z = 1$ or -1

If $\alpha = 0$ then from $\alpha^2 + \beta^2 = 1$,

$\beta = \pm 1 \Rightarrow \omega = \pm i$

So $\mathrm{Im}(z) = \mathrm{Re}(\omega) = 0$.

19. (3) As obtained earlier $\omega = \pm i$.

20. (3) 21. (4), 22. (2)

Given equation of line is $a\bar{z} + \bar{a}z + b = 0 \;\forall\; b \in R$.

Let the PQ be the segment intercepted between axes.

For intercept on real axis Z_R,

$$z = \bar{z}$$

$$\Rightarrow \quad Z_R(a + \bar{a}) + b = 0$$

$$\Rightarrow \quad Z_R = \frac{-b}{a + \bar{a}}$$

For intercept on imaginary Z_I,

$$z + \bar{z} = 0$$

$$\Rightarrow \quad Z_I(\bar{a} - a) + b = 0$$

$$\Rightarrow \quad Z_I = -\frac{b}{\bar{a} - a}$$

For mid-point,

$$z = \frac{Z_R + Z_I}{2}$$

$$\Rightarrow \quad z = \frac{-b}{2}\left[\frac{1}{\bar{a} + a} + \frac{1}{\bar{a} - a}\right]$$

$$\Rightarrow \quad z = \frac{\bar{a}b}{(a + \bar{a})(a - \bar{a})}$$

$$\Rightarrow \quad z = \frac{\bar{a}b}{a^2 - (\bar{a})^2}$$

$$\Rightarrow \quad \frac{z\left[a^2 - (\bar{a})^2\right]}{\bar{a}} = b \;(= \text{real})$$

$$\Rightarrow \quad z\frac{a^2 - (\bar{a})^2}{\bar{a}} = \bar{z}\frac{(\bar{a})^2 - (a)^2}{a}$$

$$\Rightarrow \quad az + \overline{az} = 0$$

23. (3), 24. (2), 25. (4)

$$az + b\bar{z} + c = 0 \qquad (1)$$

or $\quad \bar{a}\bar{z} + \bar{b}z + \bar{c} = 0 \qquad (2)$

Eliminating $\bar{z}$ from (1) and (2), we get

$$z = \frac{c\bar{a} - b\bar{c}}{|b|^2 - |a|^2}$$

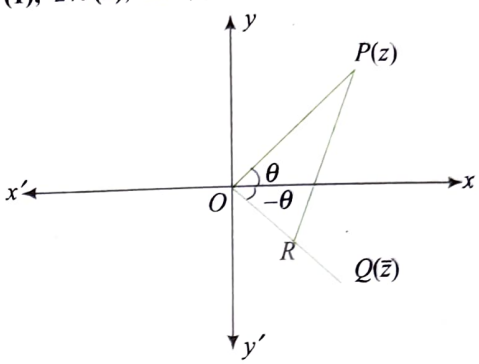
If $|a| \ne |b|$, then z represents one point on the Argand plane. If $|a| = |b|$ and $\bar{a}c \ne b\bar{c}$, then no such z exists. Adding (1) and (2),

$$(\bar{a} + b)\bar{z} + (a + \bar{b})z + (c + \bar{c}) = 0$$

This is of the form $A\bar{z} + \bar{A}z + B = 0$, where $B = c + \bar{c}$ is real.

Hence, locus of z is a straight line.

26. (1), 27. (2), 28. (2)



We have $\left|\,|z| - \left|\frac{4}{z}\right|\right| \le \left|z - \frac{4}{z}\right| = 2$

$$\Rightarrow \quad -2 \le |z| - \frac{4}{|z|} \le 2$$

$$\Rightarrow \quad |z|^2 + 2|z| - 4 \ge 0 \text{ and } |z|^2 - 2|z| - 4 \le 0$$

$$\Rightarrow \quad (|z| + 1)^2 - 5 \ge 0 \text{ and } (|z| - 1)^2 \le 5$$

$$\Rightarrow \quad (|z| + 1 + \sqrt{5})(|z| + 1 - \sqrt{5}) \ge 0$$

$$\Rightarrow \quad |z| \ge \sqrt{5} - 1$$

and $\quad (|z| - 1 + \sqrt{5}) \times (|z| - 1 - \sqrt{5}) \le 0$

$$\Rightarrow \quad \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

$$\Rightarrow \quad \sqrt{5} - 1 \le |z| \le \sqrt{5} + 1$$

Hence, the least modulus is $\sqrt{5} - 1$ and the greatest modulus is $\sqrt{5} + 1$. Also,

$$|z| = \sqrt{5} + 1$$

$$\Rightarrow \quad \frac{4}{|z|} = \sqrt{5} - 1$$

Now, $\quad \frac{4}{z} = \frac{4\bar{z}}{|z|^2}$

Hence, $4/z$ lies in the direction of $\bar{z}$.

$$\left|z - \frac{4}{z}\right| = PR = 2 \text{ (given)}$$

We have

$$OP = \sqrt{5} + 1 \text{ and } OR = \sqrt{5} - 1$$

$$\Rightarrow \quad \cos 2\theta = \frac{OP^2 + OR^2 - PR^2}{2\,OP \cdot OR}$$

$$= \frac{(\sqrt{5} + 1)^2 + (\sqrt{5} - 1)^2 - 4}{2(5 - 1)} = 1$$

$$\Rightarrow \quad 2\theta = 0, 2\pi$$

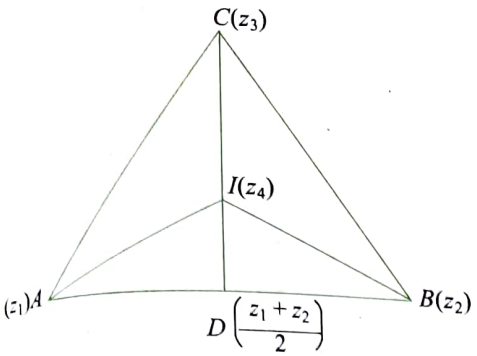
$$\Rightarrow \quad \theta = 0, \pi$$

$\Rightarrow$ z is purely real

$$\Rightarrow \quad z = \pm(\sqrt{5} + 1)$$

Similarly for $|z| = \sqrt{5} - 1$, we have $z = \pm(\sqrt{5} - 1)$.

29. (1)



$$\frac{AB\times AC}{(IA)^2}=\frac{AB}{IA}\times\frac{AC}{IA}$$

$$\angle IAB=\frac{\theta}{2}\,,\ \angle IAC=\frac{\theta}{2}$$

$$\frac{z_2-z_1}{z_4-z_1}=\frac{|z_2-z_1|}{|z_4-z_1|}\,e^{-\frac{i\theta}{2}}$$

and $$\frac{z_3-z_1}{z_4-z_1}=\frac{|z_3-z_1|}{|z_4-z_1|}e^{\frac{i\theta}{2}}$$

Multiplying,

$$\frac{z_2-z_1}{z_4-z_1}\,\frac{z_3-z_1}{z_4-z_1}=\frac{|z_2-z_1|}{|z_4-z_1|}\,\frac{|z_3-z_1|}{|z_4-z_1|}$$

$$\Rightarrow\quad\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=\frac{AB\times AC}{IA^2}\qquad(1)$$

30. (2) From (1),

$$\frac{(z_2-z_1)(z_3-z_1)}{(z_4-z_1)^2}=2\left(\frac{AD}{IA}\right)^2\left(\frac{AC}{AD}\right)\qquad(\because AB=2AD)$$

$$\Rightarrow\quad(z_2-z_1)(z_3-z_1)=(z_4-z_1)^2\,2\cos^2\frac{\theta}{2}\sec\theta$$

$$=(z_4-z_1)^2(\cos\theta+1)\sec\theta$$

31. (3) Keeping in mind that $\tan\theta=CD/AD$ and $\tan\theta/2=ID/BD$, we have,

$$\frac{z_3-\frac{z_1+z_2}{2}}{z_1-\frac{z_1+z_2}{2}}=\frac{\left|z_3-\frac{z_1+z_2}{2}\right|}{\left|z_1-\frac{z_1+z_2}{2}\right|}e^{-i\frac{\pi}{2}}$$

$$\Rightarrow\quad\frac{2z_3-z_1-z_2}{z_1-z_2}=\frac{CD}{AD}e^{-i\frac{\pi}{2}}\qquad(1)$$

and $$\frac{z_4-\frac{z_1+z_2}{2}}{z_2-\frac{z_1+z_2}{2}}=\frac{\left|z_4-\frac{z_1+z_2}{2}\right|}{\left|z_2-\frac{z_1+z_2}{2}\right|}e^{i\frac{\pi}{2}}$$

$$\Rightarrow\quad\frac{2z_4-z_1-z_2}{z_2-z_1}=\frac{ID}{BD}e^{i\frac{\pi}{2}}\qquad(2)$$

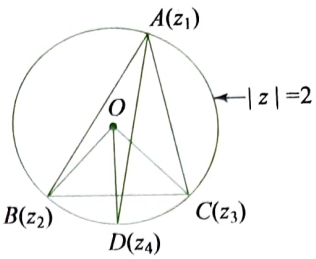
Multiplying (1) and (2), we have

$$\frac{2z_3-z_1-z_2}{z_1-z_2}\,\frac{2z_4-z_1-z_2}{z_2-z_1}=\frac{CD}{AD}\,\frac{ID}{BD}=\tan\theta\tan\frac{\theta}{2}$$

$$\Rightarrow\quad(z_2-z_1)^2\tan\theta\tan\frac{\theta}{2}=-(z_1+z_2-2z_3)(z_1+z_2-2z_4)$$

32. (4) $\angle BOD=2\angle BAD=A$

$\angle COD=2\angle CAD=A$



$$\frac{z_4}{z_2}=e^{iA}\,,\ \frac{z_3}{z_4}=e^{iA}\qquad\text{(From rotation about the point } O\text{)}$$

$$\Rightarrow\quad z_4^2=z_2z_3$$

33. (3) Clearly, OD bisects $\angle BOC$ of isosceles triangle BOC.

Thus angle between segments OD and BC is $\pi/2$.

$$\therefore\quad\arg\left(\frac{z_4}{z_2-z_3}\right)=\frac{\pi}{2}$$

34. (3)

Matrix Match Type

1. **a → q; b → s; c → p; d → r**

Since graph is concave upward, $a>0$.

Graph intersects y-axis at negative y-axis. So, $c<0$.

Also, vertex lies to the right of y-axis.

So, $\quad-\frac{b}{2a}>0\,.\quad\therefore\ b<0$

Complex number $a+ib$ lies in fourth quadrant.

$$\therefore\quad\arg(a+ib)=\tan^{-1}\frac{b}{a}$$

Complex number $ac-ibc$ lies in third quadrant.

$$\therefore\quad\arg(ac-ibc)=-\pi-\tan^{-1}\frac{b}{a}$$

Complex number $-ac+ibc$ lies in first quadrant.

$$\therefore\quad\arg(-ac+ibc)=-\tan^{-1}\frac{b}{a}$$

Complex number $abc-ib^2c$ lies in second quadrant.

$$\therefore\quad\arg(abc-ib^2c)=\pi+\tan^{-1}\frac{b}{a}$$

2. **a → s; b → p; c → q; d → r**

a. $$\frac{z_3-z_1}{z_2-z_1}=\frac{1+i\sqrt{3}}{2}$$

$$\Rightarrow\quad\frac{z_3-z_1}{z_2-z_1}=e^{i\frac{\pi}{3}}$$

$$\Rightarrow\quad\left|\frac{z_3-z_1}{z_2-z_1}\right|=1\ \text{ and }\ \arg\left(\frac{z_3-z_1}{z_2-z_1}\right)=\frac{\pi}{3}$$

So, triangle is equilateral.

b. $$\text{Re}\left(\frac{z_3-z_1}{z_3-z_2}\right)=0$$

$$\Rightarrow\quad\frac{z_3-z_1}{z_3-z_2}\ \text{is purely imaginary}$$

$\Rightarrow \quad \arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \pm\frac{\pi}{2}$

So, triangle is right angled.

c. Let $\frac{z_3 - z_1}{z_3 - z_2} = r(\cos\theta + i\sin\theta)$

Since $\cos\theta < 0$, $\arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) > \frac{\pi}{2}$

So, triangle is obtuse angled.

d. $\frac{z_3 - z_1}{z_3 - z_2} = i$

$\Rightarrow \quad |z_3 - z_1| = |z_3 - z_2|$

and $\arg\left(\frac{z_3 - z_1}{z_3 - z_2}\right) = \frac{\pi}{2}$

So, triangle is isosceles right angled.

3. a → q; b → p; c → s; d → r

a. Minimum value is equal to distance between (0, 0) and $(\cos\alpha, \sin\alpha)$ which is equal to 1.

b. $|z-1| + |z+1| = 2$ represents the line segment joining points '1' and '–1'. $|z+2| + |z-2| = 8$ represents the ellipse with foci '–2' and '2'. Clearly, line segment does not intersect the ellipse.

c. $(z_1 - 3z_2)(\bar{z}_1 - 3\bar{z}_2) = (3 - z_1\bar{z}_2)(3 - \bar{z}_1 z_2)$

$\Rightarrow \quad |z_1|^2 - |z_1|^2|z_2|^2 + 9|z_2|^2 - 9 = 0$

$\Rightarrow \quad (1 - |z_2|^2)(|z_1|^2 - 9) = 0$

$\Rightarrow \quad |z_1| = 3$

d. $z = \frac{(2\sqrt{3}+2i)^8}{(1-i)^6} + \frac{(1+i)^6}{(2\sqrt{3}-2i)^8}$

$= \frac{z_1^8}{\bar{z}_2^6} + \frac{z_2^6}{\bar{z}_1^8} = \frac{|z_1|^{16} + |z_2|^{12}}{(\bar{z}_2)^6(\bar{z}_1)^8}$

$\Rightarrow \quad \arg(z) = -8\arg(\bar{z}_1) - 6\arg(\bar{z}_2)$

$= 8\arg(z_1) + 6\arg(z_2)$

$= 8.\frac{\pi}{6} + 6.\frac{\pi}{4} = \frac{4\pi}{3} + \frac{3\pi}{2} = \frac{17\pi}{6}$

$\therefore \quad |4\sin\theta| = 4\left|\sin\frac{17\pi}{6}\right| = 2$

4. a → r; b → p; c → s; d → q.

We have equation $||z-5i| + m|z-12i|| = n$

a. $m = 1$ and $n = 13$.

$\therefore \quad |z-5i| + |z-12i| = 13$

or $\quad |z-5i| + |z-12i| = |5-12i|$

So, z lies on the line segment joining complex numbers '5' and '$12i$'.

b. $m = 1, n = 15$

$\therefore \quad |z-5i| + |z-12i| = 15 > 13$

So, z lies on the ellipse whose foci are '5' and '$12i$'.

c. $m = -1, n = 13$

$\therefore \quad ||z-5i| - |z-12i|| = 13$

or $\quad ||z-5i| - |z-12i|| = |5-12i|$

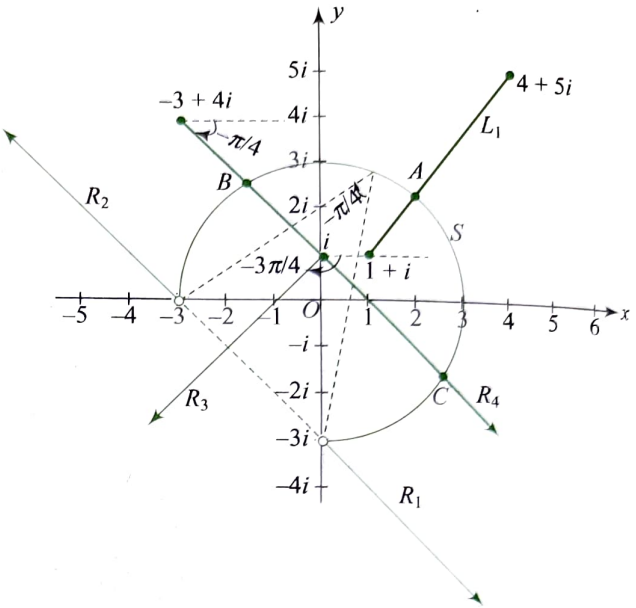
So, z lies on either of the two rays emanating from '5' and '$12i$' and moving in opposite directions.

d. $m = -1, n = 12$

$\therefore \quad ||z-5i| - |z-12i|| = 12 < 13$

So, z lies on the hyperbola whose foci are '5' and '$12i$'.

5. (1) $\arg\frac{z+3}{z+3i} = -\frac{\pi}{4}$ is the major arc excluding complex numbers '–3' and '$-3i$'.



a. $||z+3| - |z+3i|| = 3\sqrt{2} = |3-3i|$

This implies that z lies either on ray R_1 emanating from '$-3i$' or ray R_2 emanating from '–3'.

So, there is no point of intersection.

b. $|z-(1+i)| + |z-(4+5i)| = 5 = |(4+5i)-(1+i)|$

This implies that z lies on the line segment L_1 joining complex numbers '$1+i$' and '$4+5i$'.

Clearly, there is one point of intersection.

c. $\arg(z-i) = -\frac{3\pi}{4}$

So, z lies on the ray R_3 emanating from 'i' and making an angle $-3\pi/4$ with horizontal. Clearly, there is no point of intersection.

d. $\arg(z-(-3+4i)) = -\frac{\pi}{4}$

So, z lies on the ray R_4 emanating from $-3+4i$ and making an angle $-\pi/4$ with horizontal. Clearly, there are two points of intersection.

6. (2) We have $z^{27} - 1 = 0$.

The roots of the equation are $\cos\frac{2k\pi}{27} + i\sin\frac{2k\pi}{27}$; $k = 0, 1, 2, 3, \ldots, 26$.

This equation has one real root '1'. The other 26 roots are in the form of 13 conjugate pairs.

Of these, 13 roots lie above real-axis and the other 13 are lie below real axis.

So, there are 13 roots having negative imaginary part.

The other equation is $z^{36} - 1 = 0$.

This equation has two real roots, ±1 and two purely imaginary roots, ±i.

Of remaining 32 roots, 16 lie to the right of imaginary axis.

So, there are 17 roots having positive real part.

The roots of the equation are $\cos\frac{2k\pi}{36}+i\sin\frac{2k\pi}{36}$; $k = 0, 1, 2, 3, \ldots, 35$.

Number of common roots = HCF(27, 36) = 9

$\therefore \quad n(A \cap B) = 9$

$\therefore \quad n(A \cup B) = 27 + 36 - 9 = 54$

Numerical Value Type

1. (3) $x = \frac{x^3}{x^2} = \frac{2+11i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{50+25i}{25} = 2+i$

2. (21) We have

$x^3 - y^3 = 98i$

or $(x-y)^3 + 3xy(x-y) = 98i$

or $-343i + 3(a+ib)(7i) = 98i$

or $-343 + 3(a+bi)7 = 98$

or $a + ib = 21$

$\Rightarrow \quad a = 21$ and $b = 0$

$\Rightarrow \quad a + b = 21$

3. (1) We have $x = \omega - \omega^2 - 2$ or $x + 2 = \omega - \omega^2$

Squaring, $x^2 + 4x + 4 = \omega^2 + \omega^4 - 2\omega^3 = \omega^2 + \omega - 2 = -3$

$\Rightarrow \quad x^2 + 4x + 7 = 0.$

Dividing $x^4 + 3x^3 + 2x^2 - 11x - 6$ by $x^2 + 4x + 7$, we get

$x^4 + 3x^3 + 2x^2 - 11x - 6 = (x^2 + 4x + 7)(x^2 - x - 1) + 1$

$= (0)(x^2 - x - 1) + 1 = 0 + 1 = 1$

4. (15) $z^2 = 81 - b^2 + 18bi$

$z^3 = 729 + 243bi - 27b^2 - b^3 i$

$\Rightarrow \quad 243b - b^3 = 18b$

$\Rightarrow \quad b = 15$

5. (2) $\bar{z} + z = 0$

$\Rightarrow \quad \bar{z} = -z \qquad (1)$

Now $|z|^2 - 4zi = z^2$

$\Rightarrow \quad -z^2 - 4zi = z^2 \qquad$ [from (1)]

$\Rightarrow \quad 2z = -4i$

$\Rightarrow \quad z = -2i$

$\Rightarrow \quad |z| = 2$

6. (3) $(1 + ri)^3 = s(1 + i)$

$\Rightarrow \quad 1 + 3ri + 3r^2 i^2 + r^3 i^3 = s(1 + i)$

$\Rightarrow \quad 1 - 3r^2 + i(3r - r^3) = s + si$

$\Rightarrow \quad 1 - 3r^2 = s = 3r - r^3$

Hence, $1 - 3r^2 = 3r - r^3$

$\Rightarrow \quad r^3 - 3r^2 - 3r + 1 = 0$

Thus, sum of three roots is 3.

7. (4) We have $|z|^2 + \frac{16}{|z|^3} = z^2 - 4z = \bar{z}^2 - 4\bar{z}$

$\Rightarrow \quad (z - \bar{z})(z + \bar{z} - 4) = 0$

$\Rightarrow \quad z = \bar{z} = x \ (x \neq 2)$

So, $x^2 = 4x + x^2 + \frac{16}{|x|^3}$

$\Rightarrow \quad x = \frac{-4}{|x|^3}$

$\Rightarrow \quad x = -\sqrt{2}$

$\therefore \quad z = -\sqrt{2}$

$\therefore \quad |z|^4 = 4$

8. (17) Let $z = a + bi$.

$\Rightarrow \quad |z|^2 = a^2 + b^2.$

Now $z + |z| = 2 + 8i$

$\Rightarrow \quad a + bi + \sqrt{a^2 + b^2} = 2 + 8i$

$\Rightarrow \quad a + \sqrt{a^2 + b^2} = 2, \ b = 8$

$\Rightarrow \quad a + \sqrt{a^2 + 64} = 2$

$\Rightarrow \quad a^2 + 64 = (2 - a)^2 = a^2 - 4a + 4,$

$\Rightarrow \quad 4a = -60, \ a = -15.$

Thus, $a^2 + b^2 = 225 + 64 = 289$

$\therefore \quad |z| = \sqrt{a^2 + b^2} = \sqrt{289} = 17$

9. (1) Let $z = a + ib$

Given $|z| = 2$

$\Rightarrow \quad a^2 + b^2 = 4 \quad \Rightarrow \quad a, b \in [-2, 2]$

Now $w = \frac{(a+1)+ib}{(a-1)+ib}$;

$\Rightarrow \quad |w| = \sqrt{\frac{(a+1)^2 + b^2}{(a-1)^2 + b^2}} = \sqrt{\frac{a^2 + b^2 + 2a + 1}{a^2 + b^2 - 2a + 1}} = \sqrt{\frac{5 + 2a}{5 - 2a}}$

$|w|_{max} = \sqrt{\frac{5+4}{1}} = 3$ (when $a = 2$)

$|w|_{min} = \sqrt{\frac{5-4}{9}} = \frac{1}{3}$ (when $a = -2$)

Hence, required product is 1.

10. (1) $z^4 + z^3 + z^2 + z^2 + z + 1 = 0$

or $(z^2(z^2 + z + 1) + (z^2 + z + 1) = 0$

or $(z^2 + z + 1)(z^2 + 1) = 0$

$\therefore \quad z = i, -i, \omega, \omega^2$. For each, $|z| = 1$

11. (5) Roots are $2\omega, (2 + 3\omega), (2 + 3\omega^2), (2 - \omega - \omega^2)$

$2 + 3\omega$ and $2 + 3\omega^2$ are conjugate to each other.

2ω is complex root, then other root must be $2\omega^2$ (as complex roots occur in conjugate pair)

$2 - \omega - \omega^2 = 2 - (-1) = 3$ which is real.

Hence, least degree of the polynomial is 5.

12. (6) We have $|a\omega + b| = 1$

$\Rightarrow \quad |a\omega + b|^2 = 1$

$\Rightarrow \quad (a\omega + b)(a\bar{\omega} + b) = 1$

$\Rightarrow \quad a^2 + ab\ (\omega + \bar{\omega}) + b^2 = 1$

$\Rightarrow \quad a^2 - ab + b^2 = 1$

$\Rightarrow \quad (a - b)^2 + ab = 1 \qquad (1)$

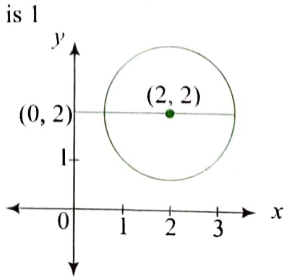
when $(a - b)^2 = 0$ and $ab = 1$, then (1, 1); (–1, –1)

when $(a - b)^2 = 1$ and $ab = 0$, then (0, 1); (1, 0); (0, –1); (–1, 0)

Hence, there are 6 ordered pairs.

13. (3) $|z - 2 - 2i| \leq 1$

denotes the region inside a circle with center (2, 2) and radius is 1



$|2iz + 4| = |2i\,(z - 2i)|$
$= |2i|\,|z - 2i\,|$
$= 2|z - 2i\,|$
$|z - 2i|$ = distance of z from $P(0, 2)$
Hence, maximum value is 3.

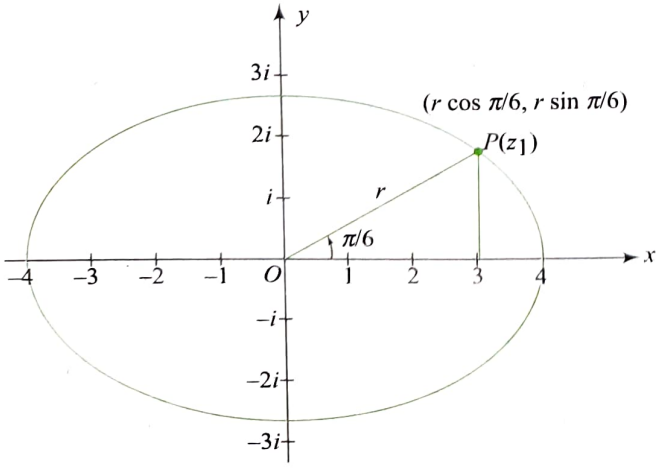
14. (20) $|3z + 9 - 7i| = |(3z + 6 - 3i) + (3 - 4i)| \le |3z + 6 - 3i| + |3 - 4i|$
$= 3|z + 2 - i| + 5$
$= 20$

15. (125) $Z_1 = (8 \sin\theta + 7\cos\theta) + i\,(\sin\theta + 4\cos\theta)$
$Z_2 = (\sin\theta + 4\cos\theta) + i\,(8\sin\theta + 4\cos\theta)$
Hence, $Z_1 = x + iy$ and $Z_2 = y + ix$
where $x = (8\sin\theta + 7\cos\theta)$ and $y = (\sin\theta + 4\cos\theta)$
$Z_1 \cdot Z_2 = (xy - xy) + i(x^2 + y^2) = i(x^2 + y^2) = a + ib$
$\Rightarrow \quad a = 0;\ b = x^2 + y^2$
Now, $x^2 + y^2 = (8\sin\theta + 7\cos\theta)^2 + (\sin\theta + 4\cos\theta)^2$
$= 65\sin^2\theta + 65\cos^2\theta + 120\sin\theta\cdot\cos\theta$
$= 65 + 60\sin 2\theta$
$\Rightarrow \quad Z_1 \cdot Z_2\big|_{\max} = 125$

16. (18) $A \equiv (1 + 2i)x^3 - 2\,(3 + i)\,x^2 + (5 - 4i)x + 2a^2 = 0$
Let the real root of equation be α. Then
$(1 + 2i)\,\alpha^3 - 2(3 + i)\,\alpha^2 + (5 - 4i)\,\alpha + 2a^2 = 0$
Equating imaginary part zero, we get
$2\alpha^3 - 2\alpha^2 - 4\alpha = 0$
$\Rightarrow$ or $\alpha\,(\alpha^2 - \alpha - 2) = 0$
$\Rightarrow \quad \alpha = 0$ or $\alpha = -1, 2$
Now equating real part zero, we have
$\alpha^3 - 6\alpha^2 + 5\alpha + 2a^2 = 0$
$\alpha = 0 \quad \Rightarrow \quad a = 0$
$\alpha = -1 \quad \Rightarrow \quad a = \pm\sqrt{6}$
$\alpha = 2 \quad \Rightarrow \quad a = \pm\sqrt{3}$
$\Rightarrow \quad \sum a^2 = (0)^2 + (+\sqrt{6})^2 + (-\sqrt{6})^2 + (+\sqrt{3})^2 + (-\sqrt{3})^2$
$= 18$

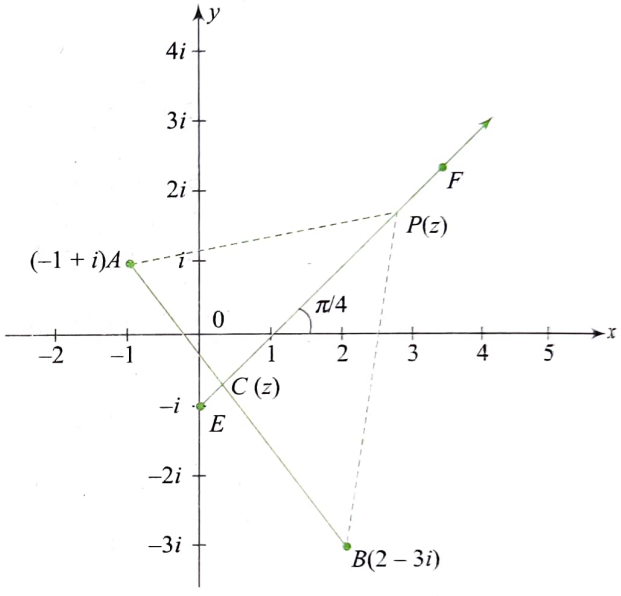
17. (30) Let $z = x + iy$
$\therefore \quad E = z\bar{z} + (z - 3)(\bar{z} - 3) + (z - 6i)(\bar{z} + 6i)$
$= 3z\bar{z} - 3(z + \bar{z}) + 9 + 6(z - \bar{z})i + 36$
$= 3(x^2 + y^2) - 6x - 12y + 45$
$= 3[x^2 + y^2 - 2x - 4y + 15]$
$= 3[(x - 1)^2 + (y - 2)^2 + 10]$
$\therefore \quad E_{\min} = 30$ when $x = 1$ and $y = 2$

18. (448) $|z - 3| + |z + 3| = 8$



Given locus is ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with length of major axis
$2a = 8$ and $e = \dfrac{6}{8} = \dfrac{3}{4}$.
$\therefore \quad b^2 = a^2(1 - e^2) = 7$
Therefore, ellipse is $\dfrac{x^2}{16} + \dfrac{y^2}{7} = 1$.
Let $|z_1| = r$. Then point $P(z_1)$
$\Rightarrow \quad \left(r\cos\dfrac{\pi}{6}, r\sin\dfrac{\pi}{6}\right)$.
$\therefore \quad \dfrac{3r^2}{64} + \dfrac{r^2}{28} = 1 \Rightarrow r^2 = \dfrac{448}{37}$

19. (5) Locus given by equation $\arg(z + i) = \dfrac{\pi}{4}$ is a ray EF as shown in the following figure.



Complex number z moves on ray EF.
Now, for any position of complex number $P(z)$ on the ray EF.
$|z - (-1 + i)| + |z - (2 - 3i)| = PA + PB$
Clearly, $PA + PB$ is minimum when point P is at point C, where points A, C and B are collinear.
So, $(PA + PB)_{\min.} = AB = 5$

20. (33) $(4 + 5\omega + 6\omega^2)^{n^2+2} + (6 + 5\omega^2 + 4\omega)^{n^2+2} + (5 + 6\omega + 4\omega^2)^{n^2+2} = 0$
$\Rightarrow \quad \left(4 + 5\omega + 6\omega^2\right)^{n^2+2}\left[1 + \omega^{n^2+2} + \omega^{2n^2+4}\right] = 0$
$\Rightarrow \quad n = 3\lambda,\ n \ne 3\lambda + 1, 3\lambda + 2$
$\therefore \quad n = 3, 6, 9, 12, \ldots, 99$
So, number of values of n is 33.

21. (15) $\displaystyle\sum_{k=1}^{22} \frac{1}{1 + z^{8k} + z^{16k}} = \sum_{k=1}^{22} \frac{z^{8k} - 1}{(z^{8k} - 1)(z^{16k} + z^{8k} + 1)}$
$\displaystyle= \sum_{k=1}^{22} \frac{z^{8k} - 1}{z^{24k} - 1}$
$\displaystyle= \sum_{k=1}^{22} \frac{z^{8k} - 1}{z^{k} - 1} \quad (\because z^{24k} = z^{23k}z^k = 1\cdot z^k = z^k)$

$$= \sum_{k=1}^{22} (1 + z^k + z^{2k} + \ldots + z^{7k})$$

$$= 22 + (0 - 1) \times 7$$

$$= 15$$

(0.5) We know that

$$|\bar{z}_2 - z_1|^2 = |z_2|^2 + |z_1|^2 - 2\,\mathrm{Re}(z_1 z_2)$$

$$= (|z_2| - |z_1|)^2 + 2(|z| - \mathrm{Re}(z))$$

$$\therefore \quad |z| - \mathrm{Re}(z) = \frac{|\bar{z}_2 - z_1|^2}{2} - \frac{(|z_2| - |z_1|)^2}{2}$$

$$\leq \frac{1}{2} - \frac{(|z_2| - |z_1|)^2}{2} \leq \frac{1}{2}$$

$$\therefore \quad (|z| - \mathrm{Re}(z))_{\max.} = \frac{1}{2}$$

(1) $z_1 + z_2 + z_3 = z_1 z_2 + z_2 z_3 + z_3 z_1 = z_1 z_2 z_3 = 1$

i.e., Sum of roots = Sum of roots taken two at a time = Product of roots = 1

So, equation will be

$$z^3 - z^2 + z - 1 = 0$$

$$\Rightarrow \quad (z - 1)(z^2 + 1) = 0$$

$$\Rightarrow \quad z = 1, -i, i$$

Area of the triangle formed by the point $A(z_1)$, $B(z_2)$ and $C(z_3) = \frac{1}{2} \times 2 \times 1 = 1$

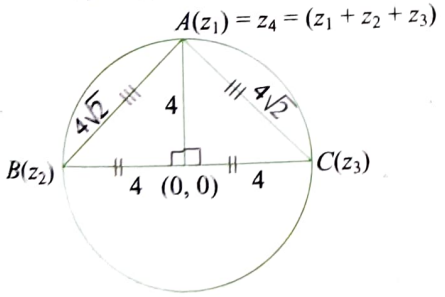
(40) $$\left|z_1 + \alpha^t z_2\right|^2 = |z_1|^2 + |z_2|^2 + \alpha^t z_2 \bar{z}_1 + (\bar{\alpha})^t \bar{z}_2 z_1$$

Now, $\sum_{t=0}^{4} \alpha^t = 0$ and $\sum_{t=0}^{4} (\bar{\alpha})^t = 0$

$$\therefore \quad \sum_{t=0}^{4} \left|z_1 + \alpha^t z_2\right|^2 = 5(|z_1|^2 + |z_2|^2)$$

$$= 5(4 + 4) = 40$$

(32) Since, $|z_1| = |z_2| = |z_3| = 4$, origin is circumcentre of triangle formed by $A(z_1)$, $B(z_2)$ and $C(z_3)$.



So, orthocentre of triangle is $z_1 + z_2 + z_3 = z_4$ (say).

Also, given that $|z_1 + z_2 + z_3| = 4$.

So, z_4 lies on the circumcircle of triangle ABC.

Given that $|z_1 - z_2| = |z_1 - z_3|$

$$\Rightarrow \quad AB = AC$$

$$\therefore \quad \angle BAC = 90^\circ \text{ (orthocentre is at vertex } A)$$

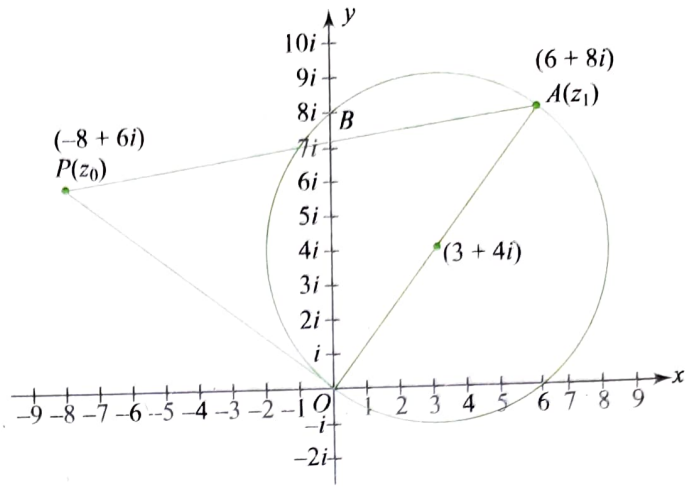
Thus, $|z_1 + z_2| \cdot |z_1 + z_3| = |z_4 - z_3|\,|z_4 - z_2|$

$$= (AC)(AB) = (4\sqrt{2})(4\sqrt{2}) = 32$$

26. (100)

We have $|z - (3 + 4i)| = 5$

So, z lies on the circle having center at $3 + 4i$ and radius 5.



This circle passes through the origin.

z_1 lies on circle such that $|z_1|$ is maximum.

Therefore, z_1 is at other end of the diameter through origin.

$$\therefore \quad z_1 = 6 + 8i.$$

OA is rotated by 90° in anticlockwise direction.

So, $z_0 = i(6 + 8i) = -8 + 6i$

Since A, B and P are collinear,

$$|z_0 - z_1| \cdot |z_0 - z_2| = PA \cdot PB = OP^2 = 10^2 = 100$$

27. (12) $|z| = |z_2| = |z_3| = 2$

Now, $|z_1 + z_2 + z_3|^2 \geq 0$

$$\Rightarrow \quad (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) \geq 0$$

$$\Rightarrow \quad \sum |z_1|^2 + \sum (z_1 \bar{z}_2 + \bar{z}_1 z_2) \geq 0$$

$$\Rightarrow \quad 2\sum |z_1|^2 + \sum (z_1 \bar{z}_2 + \bar{z}_1 z_2) \geq \sum |z_1|^2$$

$$\Rightarrow \quad |z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 \geq 12$$

28. (2) $|z_1 + 1| + |z_2 + 1| + |z_1 z_2 + 1|$

$$\geq |z_1 + 1| + |z_2 + 1 - z_1 z_2 - 1|$$

$$\geq |z_1 + 1| + |z_2|\,|z_1 - 1|$$

$$= |z_1 + 1| + |z_1 - 1|$$

$$\geq 2$$

29. (2) $|z_1| = 2$

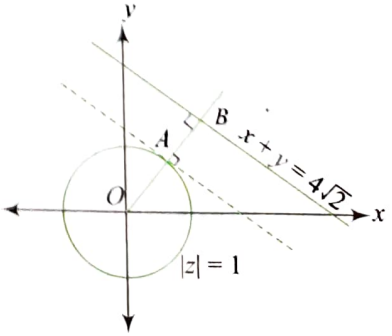
This implies that z_1 lies on the circle having center at origin and radius 2.

$$(1 - i) z_2 + (1 + i) \bar{z}_2 = 8\sqrt{2}$$

$$\therefore \quad (1 - i)(x_2 + iy_2) + (1 + i)(x_2 - iy_2) = 8\sqrt{2}$$

$$\Rightarrow \quad x_2 + y_2 = 4\sqrt{2}$$

So, z_2 lies on the straight line $x + y = 4\sqrt{2}$.



$|z_1 - z_2|_{\min.}$ = shortest distance between circle and straight line
$= AB$
$= OB - OA$
$= 4 - 2 = 2$

30. (1) $1 + 2|z|^2 = |z^2 + 1|^2 + 2|z + 1|^2$
$\Rightarrow \quad 1 + 2z\bar{z} = (z^2 + 1)(\bar{z}^2 + 1) + 2(z + 1)(\bar{z} + 1)$
$\Rightarrow \quad (z\bar{z})^2 + 2(z + \bar{z}) + z^2 + \bar{z}^2 + 2 = 0$
$\Rightarrow \quad (z\bar{z} - 1)^2 + (z + \bar{z} + 1)^2 = 0$
$\Rightarrow \quad z\bar{z} = 1 \quad \Rightarrow \bar{z} = \frac{1}{z}$
and $\quad z + \bar{z} + 1 = 0$
or $\quad z + \frac{1}{z} + 1 = 0$
$\Rightarrow \quad z^2 + z = -1$
$\Rightarrow \quad |z^2 + z| = 1$

Archives

JEE Main

Single Correct Answer Type

1. (2) $|Z| = \left|\left(Z - \frac{4}{\bar{Z}}\right) + \frac{4}{\bar{Z}}\right| \le \left|Z - \frac{4}{\bar{Z}}\right| + \frac{4}{|\bar{Z}|} \le 2 + \frac{4}{|Z|}$
$\Rightarrow \quad |Z|^2 - 2|Z| - 4 \le 0$
$\Rightarrow \quad [|Z| - (\sqrt{5} + 1)]\,[|Z| - (1 - \sqrt{5})] \le 0$
$\Rightarrow \quad 0 \le |Z| \le \sqrt{5} + 1$

2. (3) Let $z = x + iy$. So
$|z - 1| = |z + 1| \Rightarrow \text{Re}(z) = 0 \Rightarrow x = 0$
$|z - 1| = |z + i| \Rightarrow x = y$
$|z + 1| = |z - i| \Rightarrow y = -x$
Only (0, 0) will satisfy all conditions.
Therefore, the number of complex numbers (z) is 1.

3. (1) Let the roots of the given equation be $1 + ip$ and $1 - ip$, where $p \in R$
$\Rightarrow \quad \beta =$ product of roots
$= (1 + ip)(1 - ip) = 1 + p^2 > 1 \forall\, p \in R$
$\Rightarrow \quad \beta \in (1, \infty)$

4. (3) $(1 + \omega)^7 = A + B\omega$
$\therefore \quad (-\omega^2)^7 = A + B\omega$
$\therefore \quad -\omega^2 = A + B\omega$
$\therefore \quad 1 + \omega = A + B\omega$
$\therefore \quad A = 1, B = 1$
$\therefore \quad (A, B) = (1, 1)$

5. (1) $\frac{z^2}{z - 1}$ is purely real
$\therefore \quad \frac{z^2}{z - 1} = \frac{\bar{z}^2}{\bar{z} - 1}$
$\Rightarrow \quad z\bar{z}z - z^2 = z\bar{z}\bar{z} - \bar{z}^2$
$\Rightarrow \quad |\bar{z}|^2 (z - \bar{z}) - (z - \bar{z})(z + \bar{z}) = 0$
$\Rightarrow \quad (z - \bar{z})(|z|^2 - (z + \bar{z})) = 0$
Either $z = \bar{z}$
$\Rightarrow \quad z$ lies on real axis
or $\quad |z|^2 = z + z$
$\Rightarrow \quad z\bar{z} - z - \bar{z} = 0$
$\Rightarrow \quad x^2 + y^2 - 2x = 0$
Which represents a circle passing through origin.

6. (3) Since $|z| = 1, z \cdot \bar{z} = 1$
$\therefore \quad \arg\left(\frac{1 + z}{1 + \bar{z}}\right) = \arg\left(\frac{1 + z}{1 + \frac{1}{z}}\right)$
$= \arg\, z\left(\frac{1 + z}{z + 1}\right)$
$= \arg z = \theta$

7. (2) $|z| \ge 2$
$\therefore \quad \left|z + \frac{1}{2}\right| \ge \left||z| - \left|\frac{1}{2}\right|\right| \ge \left|2 - \frac{1}{2}\right| \ge \frac{3}{2}$
Hence, minimum distance between z and $\left(-\frac{1}{2}, 0\right)$ is $\frac{3}{2}$.

8. (3) $|z_1| = 1; |z_2| \ne 1$
Given $\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$
$\therefore \quad |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_1|^2$
$\Rightarrow \quad |z_1|^2 + 4|z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2 = 4 + |z_1|^2 |z_2|^2 - 2\bar{z}_1 z_2 - 2z_1\bar{z}_2$
$\Rightarrow \quad (|z_2|^2 - 1)(|z_1|^2 - 4) = 0$
$\because \quad |z_1| \ne 1$
$\therefore \quad |z_2| = 2$

9. (3) $\frac{2 + 3i\sin\theta}{1 - 2i\sin\theta} = \frac{2 + 3i\sin\theta}{1 - 2i\sin\theta} \times \frac{1 + 2i\sin\theta}{1 + 2i\sin\theta}$
$= \frac{(2 - 6\sin^2\theta) + i(7\sin\theta)}{1 + 4\sin^2\theta}$
For purely imaginary number
$\sin^2\theta = \frac{1}{3}$ or $\sin\theta = \frac{1}{\sqrt{3}}$
$\therefore \quad \theta = \sin^{-1}\frac{1}{\sqrt{3}}$

10. (2) Here, ω is complex cube root of unity.
Applying $R_1 \to R_1 + R_2 + R_3$, the given matrix reduces to
$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3(-1 - \omega - \omega) = -3z$$
$\Rightarrow \quad k = -z$

11. (4) $x^2 - x + 1 = 0$
$\Rightarrow \quad x = \frac{1 \pm \sqrt{-3}}{2}$
$= \frac{1 \pm i\sqrt{3}}{2}$
$= -\left[\frac{-1 \mp i\sqrt{3}}{2}\right]$
$\therefore \quad x = -\omega, -\omega^2$, where ω is imaginary cube root of unity.
Let $\quad \alpha = -\omega$ and $\beta = -\omega^2$
$\therefore \quad \alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$
$= -[\omega^{101} + \omega^{214}]$
$= -[\omega^{99}\omega^2 + \omega^{213}\omega]$
$= -[\omega^2 + \omega]$
$= -(-1) = 1 \qquad (\text{as } 1 + \omega + \omega^2 = 0)$
If we consider $\alpha = -\omega^2$ and $\beta = -\omega$, we get the same results.

Single Correct Answer Type

1. (1) $z\bar{z}(\bar{z}^2 + z^2) = 350$

Puting $z = x + iy$, we have

$(x^2 + y^2)(x^2 - y^2) = 175$

$(x^2 + y^2)(x^2 - y^2) = 5 \times 5 \times 7$

$x^2 + y^2 = 25$

and $x^2 - y^2 = 7$

(as other combinations give non-integral values of x and y)

$\therefore$ $x = \pm 4, y = \pm 3$ $(x, y \in I)$

Hence, area is $8 \times 6 = 48$ sq. units.

2. (4) Given equation is $z^2 + z + 1 - a = 0$

Clearly this equation do not have real roots if

$D < 0$

$\Rightarrow$ $1 - 4(1 - a) < 0$

$\Rightarrow$ $4a < 3$

$a < \frac{3}{4}$

3. (3) Given circles are

$(x - x_0)^2 + (y - y_0)^2 = r^2$

and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$

or $|z - z_0| = r$ (1)

and $|z - z_0| = 2r$ (2)

where $z_0 = x_0 + iy_0$

Now α and $\frac{1}{\bar{\alpha}}$ lies on circle (1) and (2), respectively. Then

$|\alpha - z_0| = r$ and $\left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$

$\Rightarrow$ $|\alpha - z_0| = r$ and $|1 - \bar{\alpha} z_0| = 2r|\bar{\alpha}|$

$\Rightarrow$ $|\alpha - z_0|^2 = r^2$ and $|1 - \bar{\alpha} z_0|^2 = 4r^2 |\alpha|^2$

Subtracting, we get

$|1 - \bar{\alpha} z_0|^2 - |\alpha - z_0|^2 = 4r^2 |\alpha|^2 - r^2$

$\Rightarrow$ $1 + |\alpha z_0|^2 - \bar{\alpha} z_0 - \alpha \bar{z}_0 - (|\alpha|^2 + |z_0|^2 - \bar{\alpha} z_0 - \alpha \bar{z}_0)$

$= 4r^2 |\alpha|^2 - r^2$

$\Rightarrow$ $1 + |\alpha|^2 |z_0|^2 - |\alpha|^2 - |z_0|^2 = 4r^2 |\alpha|^2 - r^2$

$\Rightarrow$ $(1 - |\alpha|^2)(1 - |z_0|^2) = 4r^2 |\alpha|^2 - r^2$

Given $2|z_0|^2 = r^2 + 2$

$\Rightarrow$ $(1 - |\alpha|^2)\left(1 - \frac{r^2 + 2}{2}\right) = 4r^2 |\alpha|^2 - r^2$

$\Rightarrow$ $(1 - |\alpha|^2)\left(\frac{-r^2}{2}\right) = 4r^2 |\alpha|^2 - r^2$

$\Rightarrow$ $|\alpha|^2 - 1 = 8|\alpha|^2 - 2$

$\Rightarrow$ $|\alpha|^2 = \frac{1}{7}$ $\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

Multiple Correct Answers Type

1. (1), (3), (4)

Given $z = (1 - t) z_1 + t z_2$

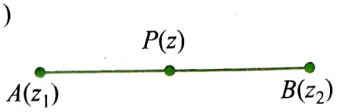
$\Rightarrow$ $z = \frac{(1-t)z_1 + t z_2}{(1-t) + t}$

$\Rightarrow$ z divides the line segment joining z_1 and z_2 in ration $(1 - t) : t$ internally as $0 < t < 1$

$\Rightarrow$ z, z_1, and z_2 are collinear.

$\Rightarrow$ $\arg(z - z_1) = \arg(z_2 - z_1)$

$\Rightarrow$ $\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$



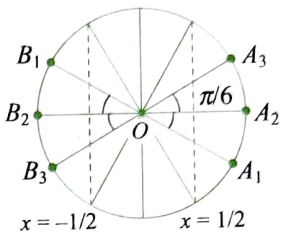
$\Rightarrow$ $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

$AP + PB = AB$

$\Rightarrow$ $|z - z_1| + |z - z_2| = |z_1 - z_2|$

2. (3), (4)

$w = \frac{\sqrt{3} + i}{2} = e^{\frac{i\pi}{6}}$, so $w^n = e^{i\left(\frac{n\pi}{6}\right)}$, $n = 0, 1, 2, 3, \ldots, 12$



Now, for z_1, $\cos\frac{n\pi}{6} > \frac{1}{2}$ and for z_2, $\cos\frac{n\pi}{6} < -\frac{1}{2}$

Possible position of z_1 are A_1, A_2, A_3 whereas of z_2 are B_1, B_2, B_3 (as shown in the figure)

So possible value of $\angle z_1 O z_2$ according to the given options is $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

3. (1), (3), (4)

$z = \frac{1}{a + ibt}$

$\Rightarrow$ $x + iy = \frac{a - ibt}{a^2 + b^2 t^2}$

$\Rightarrow$ $x = \frac{a}{a^2 + b^2 t^2}$, $y = \frac{-bt}{a^2 + b^2 t^2}$

Eliminating t, we get

$x^2 + y^2 = \frac{x}{a}$

$\Rightarrow$ $\left(x - \frac{1}{2a}\right)^2 + y^2 = \left(\frac{1}{2a}\right)^2$

$\therefore$ Option (1) is correct.

(3), (4) can be verified by putting $b = 0$ and $a = 0$ respectively.

4. (1), (4)

We have $\text{Im}\left(\frac{az + b}{z + 1}\right) = y$ and $z = x + iy$

$\therefore$ $\text{Im}\left(\frac{a(x + iy) + b}{x + iy + 1}\right) = y$

$\Rightarrow$ $\text{Im}\left(\frac{(ax + b + iay)(x + 1 - iy)}{(x + 1)^2 + y^2}\right) = y$

$\Rightarrow$ $-y(ax + b) + ay(x + 1) = y((x + 1)^2 + y^2)$

$\Rightarrow$ $(a - b)y = y((x + 1)^2 + y^2)$

$\because$ $y \neq 0$ and $a - b = 1$

$\therefore$ $(x + 1)^2 + y^2 = 1$

$\Rightarrow$ $x = -1 \pm \sqrt{1 - y^2}$

5. (1), (2), (4)

(1) Complex number $-1-i$ lies in third quadrant.

$$\therefore \quad \arg(-1-i) = -\frac{3\pi}{4}$$

(2) We have $f: R \to (-\pi, \pi], f(t) = \arg(-1+it)$ for all $t \in R$

Complex number $-1+it$ lies in second quadrant if $t > 0$ and in third quadrant if $t < 0$.

$$\therefore \quad f(t) = \arg(-1+it)$$

$$= \begin{cases} \pi - \tan^{-1}\left|\frac{t}{-1}\right|, & t \geq 0 \\ -\left(\pi - \tan^{-1}\left|\frac{t}{-1}\right|\right), & t < 0 \end{cases}$$

$$= \begin{cases} \pi - \tan^{-1} t, & t \geq 0 \\ -\pi + \tan^{-1} t, & t < 0 \end{cases}$$

Clearly, $f(t)$ is discontinuous at $t = 0$.

(3) $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$

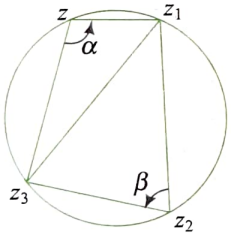
$= \arg z_1 - \arg z_2 + 2n\pi - \arg z_1 + \arg z_2 = 2n\pi,\ n \in I$

(4) $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$

$$\Rightarrow \quad \arg\underset{\alpha}{\frac{z-z_1}{z-z_3}} + \arg\underset{\beta}{\frac{z_2-z_3}{z_2-z_1}} = \pi$$

Where α and β are angles subtended by line segment joining z_1 and z_3 at z and z_2 respectively.

So, we get the following figure:



Thus, locus of z is a circle.

6. (1), (3), (4)

Given $sz + t\bar{z} + r = 0,\ \bar{z} = x - iy$ ...(i)

Taking conjugate, we get

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$(t+\bar{s})\bar{z} + (\bar{t}+s)z + (r+\bar{r}) = 0$$

Clearly, this is the equation of a straight line.

Now, eliminating $\bar{z}$ from (i) and (ii), we get

$$s\bar{s}z + r\bar{s} - t\bar{t}z - \bar{r}t = 0$$

$$\Rightarrow \quad (|s|^2 - |t|^2)z = \bar{r}t - r\bar{s}$$

If $|s| \neq |t|$, then L has unique solution.

If $|s| = |t|$ and $\bar{r}t = r\bar{s}$, then L has infinite solutions.

If $|s| = |t|$ and $\bar{r}t \neq r\bar{s}$, then L has no solution.

Thus, L has either unique, infinite or no solution.

Also, line can intersect circle in maximum two points.

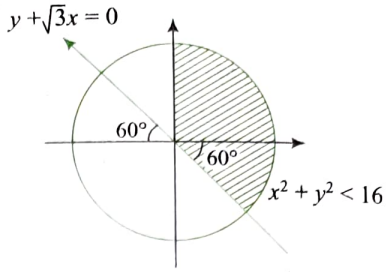
So, number of elements in $L \cap \{z : |z-1+i| = 5\}$ is at most 2.

Linked Comprehension Type

1. (2) $S_1 : |z| < 4$, z lies inside the circle or radius 4

$S_2 : \sqrt{3}x + y > 0$, z lies above the line $\sqrt{3}x + y = 0$

S_3: $\mathrm{Re}(z) > 0$, z lies to the right of imaginary axis.



Area of region $S_1 \cap S_2 \cap S_3$ = Shaded area

$$= \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6} = 4^2\pi\left\{\frac{1}{4} + \frac{1}{6}\right\} = \frac{20\pi}{3}$$

2. (3) Distance of $(1, -3)$ from $y + \sqrt{3}x = 0$ is

$$\left|\frac{\sqrt{3} \times 1 - 3}{2}\right| = \frac{3-\sqrt{3}}{2}$$

$$\Rightarrow \quad \min_{z \in S} |1 - 3i - z| = \frac{3-\sqrt{3}}{2}$$

Matrix Match Type

1. a → q

$$\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|, z \neq 0$$

$\frac{z}{|z|}$ is unimodular complex number and lies on perpendicular bisector of i and $-i$

$$\Rightarrow \quad \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1|z|$$

$\Rightarrow$ z is real number $\Rightarrow$ $\mathrm{Im}(z) = 0$.

b → p

$|z+4| + |z-4| = 10$

z lies on an ellipse whose focus are $(4,0)$ and $(-4,0)$ and length of major axis is 10

$\Rightarrow$ $2ae = 8$ and $2a = 10 \Rightarrow e = 4/5$

$|\mathrm{Re}(z)| \leq 5$.

c → p, t

$|\omega| = 2 \Rightarrow \omega = 2(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad z = x + iy = 2(\cos\theta + i\sin\theta) - \frac{1}{2}(\cos\theta - i\sin\theta)$$

$$= \frac{3}{2}\cos\theta + i\frac{5}{2}\sin\theta \Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$$

$$\Rightarrow \quad e^2 = 1 - \frac{9/4}{25/4} = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow e = \frac{4}{5}$$

d → q, t

$|\omega| = 1 \Rightarrow x + iy = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$x + iy = 2\cos\theta$

$|\mathrm{Re}(z)| \leq 1$, $\mathrm{Im}(z) = 0$.

2. (1) a. z_k is 10^{th} root of unity

So, $\bar{z}_k$ will also be 10^{th} root of unity.

Take z_j as $\bar{z}_k$.

b. $z_1 \neq 0$ take $z = \frac{z_k}{z_1}$, we can always find z.

c. $z^{10} - 1 = (z-1)(z-z_1) \ldots (z-z_9)$

$\Rightarrow (z-z_1)(z-z_2) \ldots\ldots (z-z_9)$

$= 1 + z + z^2 + \ldots + z^9 \quad \forall\, z \in$ complex number

Put $z = 1$

$\Rightarrow (1-z_1)(1-z_2) \ldots\ldots (1-z_9) = 10$

$\Rightarrow \frac{|1-z_1||1-z_2| \ldots\ldots |1-z_9|}{10} = 1$

d. $1 + z_1 + z_2 + \ldots + z_9 = 0$

$\Rightarrow \text{Re}(1) + \text{Re}(z_1) + \ldots + \text{Re}(z_9) = 0$

$\Rightarrow \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = -1.$

3. **c → p, q, s,t.**

Let $a = 3 - 3\omega + 2\omega^2$

$\Rightarrow a\omega = 3\omega - 3\omega^2 + 2$

$\Rightarrow a\omega^2 = 3\omega^2 - 3 + 2\omega$

Now given $a^{4n+3}(1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$

Thus, n should not be a multiple of 3.

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (1) $\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

Applying $(C_1 \to C_1 + C_2 + C_3)$

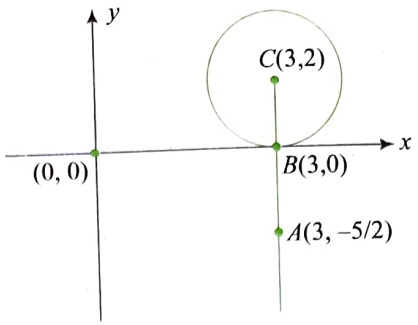
$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow z^3 = 0$

$z = 0$ is only solution.

2. (5) $|z - 3 - 2i| \leq 2$

$\Rightarrow$ z lies on or inside the circle radius 2 and center (3, 2)



$|2z - 6 + 5i|_{\min.}$

$= 2|z - 3 + (5/2)i|_{\min.}$

= 2(minimum distance of any point on the circle to the point (3, – 5/2)

$= 2(5/2) = 5$

3. (4) $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = e^{\frac{k\pi}{7}i}$

$$\therefore \frac{\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{\frac{(k+1)\pi}{7}i} - e^{\frac{k\pi}{7}i}\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-1)\pi}{7}i} - e^{\frac{(4k-2)\pi}{7}i}\right|}$$

$$= \frac{\sum_{k=1}^{12}\left|e^{\frac{k\pi}{7}i}\right|\left|e^{\frac{\pi i}{7}} - 1\right|}{\sum_{k=1}^{3}\left|e^{\frac{(4k-2)\pi}{7}i}\right|\left|e^{\frac{\pi i}{7}} - 1\right|}$$

$$= \frac{\sum_{k=1}^{12}1}{\sum_{k=1}^{3}1} \qquad \left(\because \left|e^{\frac{k\pi}{7}i}\right| = \left|e^{\frac{(4k-2)\pi}{7}i}\right| = 1\right)$$

$$= \frac{12}{3} = 4$$

Chapter 4

1. We have

$P(n)$: $1 \times 3 + 3 \times 5 + 5 \times 7 + \ldots + (2n-1) \times (2n+1)$

$$= \frac{n(4n^2 + 6n - 1)}{3}$$

For $n = 1$,

L.H.S. $= 1 \times 3 = 3$;

R.H.S. $= \frac{1 \cdot (4 + 6 - 1)}{3} = 3$

Thus, $P(1)$ is holds.

Let $P(n)$ be true for some $n = k$.

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \ldots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3} \quad \ldots (1)$$

Adding $(2k+1)(2k+3)$ on both sides, we get

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \ldots + (2k-1)(2k+1) + (2k+1)(2k+3)$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+1)(2k+3)$$

$$= \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 8k + 3)}{3}$$

$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

$$= \frac{(k+1)(4k^2 + 14k + 9)}{3}$$

$$= \frac{(k+1)\{4(k+1)^2 + 6(k+1) - 1\}}{3}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers.

2. We have

$$P(n) = \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots + \frac{1}{n(n+1)(n+2)}$$

$$= \frac{n(n+3)}{4(n+1)(n+2)}$$

For $n = 1$,

L.H.S. $= \frac{1}{1 \cdot 2 \cdot 3} = \frac{1}{6}$ and R.H.S. $= \frac{1 \cdot 4}{4 \cdot 2 \cdot 3} = \frac{1}{6}$

Thus, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

Then $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad \ldots (1)$

Adding $\frac{1}{(k+1)(k+2)(k+3)}$ on both sides, we get

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \ldots + \frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4} + \frac{1}{k+3}\right\}$$

$$= \frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2 + 4}{4(k+3)}\right\}$$

$$= \frac{1}{(k+1)(k+2)}\left\{\frac{k^3 + 6k^2 + 9k + 4}{4(k+3)}\right\}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers.

3. Let $P(n)$: $2^{3n} - 1$ is divisible by 7.

For $n = 1$, $2^3 - 1 = 7$, which is divisible by 7.

Thus, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

Then $2^{3k} - 1 = 7m$, $m \in N$ ...(1)

Now, $2^{3(k+1)} - 1 = 8 \times 2^{3k} - 1$

$= 8(7m + 1) - 1$ [Using (1)]

$= 56m + 7$

$= 7(8m + 1)$, which is divisible by 7.

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for all natural numbers.

4. Let $P(n)$: $x^n - y^n$ is divisible by $x - y$, where x and y are any integers such that $x \neq y$, $n \in N$.

For $n = 1$,

$P(1) = x^1 - y^1 = x - y$, which is divisible by $(x - y)$.

So, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

Then $x^k - y^k$ is divisible by $(x - y)$.

or $x^k - y^k = m(x - y)$, $m \in N$ ...(1)

Now, $x^{k+1} - y^{k+1}$

$= x^k \times x - x^k \times y + x^k \times y - y^k \times y$

$= x^k(x - y) + y(x^k - y^k)$

$= x^k(x - y) + ym(x - y)$ [Using (i)]

$= (x - y)[x^k + ym]$, which is divisible by $(x - y)$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for any natural number n.

et $P(n)$: $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}}$, for all natural numbers, ≥ 2.

or $n = 2$, $\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$, which is true.

o, $P(2)$ is true.

et $P(n)$ be true for some $n = k$.

Then $\sqrt{k} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}$ ...(1)

Adding $\frac{1}{\sqrt{k+1}}$ on both sides, we get

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \quad ...(2)$$

Now, $\sqrt{k+1} < \sqrt{k} + \frac{1}{\sqrt{k+1}}$ ...(3)

or $k+1 < \sqrt{k}\sqrt{k+1} + 1$

or $k < \sqrt{k^2 + k}$, which is true.

So, from (2) and (3),

$$\sqrt{k+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for any natural number n.

6. Let $P(n)$: $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$ is a natural number, for all $n \in N$.

$P(1)$: $\frac{1^5}{5} + \frac{1^3}{3} + \frac{7(1)}{15} = \frac{3+5+7}{15} = \frac{15}{15} = 1$, which is a natural number.

Hence, $P(1)$ is true.

Let $P(n)$ be true for some $n = k$.

Then $\frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15}$ is natural number.

Now, $\frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{7(k+1)}{15}$

$$= \frac{k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1}{5} + \frac{k^3 + 1 + 3k^2 + 3k}{3} + \frac{7k+7}{15}$$

$$= \frac{k^5}{5} + \frac{k^3}{3} + \frac{7k}{15} + k^4 + 2k^3 + 3k^2 + 2k + 1$$

$$= P(k) + k^4 + 2k^3 + 2k^2 + 2k + 1$$

= Natural number

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for any natural number n.

7. Let $P(n) = 7^{4^n} - 1$ be divisible by 2^{2n+3}.

$$P(1) = 7^4 - 1 = (7^2 - 1)(7^2 + 1)$$
$$= 48 \times 50 = 32 \times 75$$
$$= 2^5 \times 75,$$

which is divisible by $2^{2\times 1+3}$

Let us assume that the result is true for $n = k$.

i.e., $7^{4^k} - 1$ is divisible by 2^{2k+3} but not by 2^{2k+4}.

$\Rightarrow$ $7^{4^k} - 1 = 2^{2k+3} m$; where m is some odd natural number

Now, $7^{4^{k+1}} - 1 = (7^{4^k})^4 - 1$

$$= ((2^{2k+3} m + 1)^2 + 1)((2^{2k+3} m + 1)^2 - 1)$$
$$= ((2^{2k+3} m + 1)^2 + 1)(2^{2k+3} m + 2)(2^{2k+3} m)$$
$$= (2^{4k+6} m^2 + 2^{2k+4} m + 2)(2^{2k+3} m + 2)(2^{2k+3} m)$$
$$= 2^{2k+5}(2^{4k+5} m^2 + 2^{2k+3} m + 1)(2^{2k+2} m + 1)(m);$$

which is divisible by 2^{2k+5}.

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for any natural number n.

8. We have to prove that $n^5 - n$ is divisible by 10.

For $n = 1$, $1^5 - 1 = 0$ is divisible by 10.

Also, $n = 2$, $2^5 - 2 = 30$ is divisible by 10.

Thus, $P(1)$ and $P(2)$ are true.

Let $P(n)$ be true for some $n = k$.

$k^5 - k = 10m$ ...(1)

Now, $(k+1)^5 - (k+1)$

$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$
$$= (k^5 - k) + 5k(k^3 + 1) + 10k^3 + 10k^2$$
$$= 10m + 10k^3 + 10k^2 + 5k(k^3 + 1) \quad \text{[Using (1)]}$$

Clearly, $k(k^3 + 1)$ is even for $k \in N$.

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction, $P(n)$ is true for any natural number n.

9. We have a sequence $b_0, b_1, b_2, \ldots$ is defined by letting $b_0 = 5$ and $b_k = 4 + b_{k-1}$, for all natural numbers k.

Let $P(n) : b_n = 5 + 4n$, for all natural numbers n

For $n = 1$,

$b_1 = 5 + 4 \times 1 = 9$

Also $b_0 = 5$

$\therefore$ $b_1 = 4 + b_0 = 4 + 5 = 9$

Thus, $P(1)$ is true.

Now, assume that $P(n)$ is true for $n = k$.

$\therefore$ $P(k) : b_k = 5 + 4k$ ...(1)

Now, to prove $P(k+1)$ is true, we have to show that

$P(k+1) : b_{k+1} = 5 + 4(k+1)$

$b_{k+1} = 4 + b_{k+1-1}$ (as $b_k = 4 + b_{k-1}$)

$= 4 + b_k$

$= 4 + 5 + 4k = 5 + 4(k+1)$

Hence, $P(k+1)$ is true whenever $P(k)$ is true.

So, by the principle of mathematical induction $P(n)$ is true for any natural number n.

Chapter 5

Concept Application Exercises

Exercise 5.1

1. $a_1 = a_2 = 2,\ a_n = a_{n-1} - 1,\ n > 2$

$\Rightarrow \quad a_3 = a_2 - 1 = 2 - 1 = 1$

$a_4 = a_3 - 1 = 1 - 1 = 0$

$a_5 = a_4 - 1 = 0 - 1 = -1$

Hence, the first five terms of the sequence are 2, 2, 1, 0, and -1.

The corresponding series is $2 + 2 + 1 + 0 + (-1) + \cdots$

2. We have,

$$a_{n+1} = \frac{1}{1 - a_n}$$

$$\therefore \quad a_2 = \frac{1}{1 - a_1}$$

and $$a_3 = \frac{1}{1 - a_2} = \frac{1}{1 - \frac{1}{1 - a_1}} = \frac{1 - a_1}{-a_1}$$

Since $a_3 = a_1$, we have $\frac{1 - a_1}{-a_1} = a_1$

$\Rightarrow \quad a_1^2 - a_1 + 1 = 0$

$\Rightarrow \quad a_1 = -\omega$ or $-\omega^2$, where ω is cube root of unity.

Now, $a_5 = \frac{1}{1 - a_4} = \frac{1}{1 - \frac{1}{1 - a_3}}$

$$= \frac{1 - a_3}{-a_3}$$

$$= \frac{1 - a_1}{-a_1} = a_1 = a_3 \text{ and so on}$$

$\therefore \quad a_1 = a_3 = a_5 \ldots a_{2001}$

Thus, $(a_{2001})^{2001} = (-\omega)^{2001}$

or $\quad (-\omega^2)^{2001} = -1$

or $\quad (-1)^{2001}(\omega^3)^{1334} = -1 \qquad (\because\ \omega^3 = 1)$

3. $a_1 = 1$

$$3a_{n+1} - 3a_n = 1$$

$$\therefore \quad a_{n+1} = \frac{3a_n + 1}{3} = a_n + \frac{1}{3},$$

$$\therefore \quad a_2 = a_1 + \frac{1}{3} = 1 + \frac{1}{3}$$

$$\therefore \quad a_3 = a_2 + \frac{1}{3} = a_1 + \frac{1}{3} + \frac{1}{3} = 1 + \frac{2}{3}$$

$$\therefore \quad a_4 = a_3 + \frac{1}{3} = 1 + \frac{2}{3} + \frac{1}{3} = 1 + \frac{3}{3}$$

$$\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots \quad \ldots\ldots$$

$$\ldots \quad \ldots\ldots \quad \ldots\ldots\ldots \quad \ldots \quad \ldots\ldots$$

$$\therefore \quad a_{2002} = 1 + \frac{2001}{3} = 1 + 667 = 668$$

Exercise 5.2

1. $a + (p - 1)d = q$ and $a + (q - 1)d = p$

$\Rightarrow \quad (p - q)d = q - p$

$\Rightarrow \quad d = -1$

$T_r = a + (r - 1)d = a + (p - 1)d + (r - p)d$

$= q - (r - p) = p + q - r$

2. We have

$$\frac{1}{1-x} - \frac{1}{1+\sqrt{x}} = \frac{1}{(1-\sqrt{x})(1+\sqrt{x})} - \frac{1}{1+\sqrt{x}} = \frac{\sqrt{x}}{(1-x)}$$

and $$\frac{1}{1-\sqrt{x}} - \frac{1}{1-x} = \frac{1+\sqrt{x}-1}{1-x} = \frac{\sqrt{x}}{1-x}$$

Therefore, the given numbers are in A.P. with common difference $\sqrt{x}/(1-x)$.

3. Let the A.P. be $11, 11 + d, 11 + 2d, 11 + 3d, \ldots, 11 + (n-2)d, 11 + (n-1)d$.

Sum of first four terms $= 11 + (11 + d) + (11 + 2d) + (11 + 3d)$

$= 44 + 6d$

Sum of last four terms $= [11 + (n-4)d] + [11 + (n-3)d] +$

$[11 + (n-2)d] + [11 + (n-1)d]$

$= 44 + (4n - 10)d$

According to the given condition,

$44 + 6d = 56 \Rightarrow d = 2$

and $\quad 44 + (4n - 10)d = 112$

or $\quad 44 + 2(4n - 10) = 112$

or $\quad n = 11$

Thus, the number of terms of the A.P. is 11.

4. Let $a - 3d$, $a - d$, $a + d$ and $a + 3d$ be four consecutive terms of an A.P. with common difference $2d$.

Hence $P = (2d)^4 + (a - 3d)(a - d)(a + d)(a + 3d)$

$= 16d^4 + (a^2 - 9d^2)(a^2 - d^2)$

$= (a^2 - 5d^2)^2$

which is integer.

5. Let four member be $a - 3d$, $a - d$, $a + b$, $a + 3d$.

Here, $4a = 28$ or $a = 7$

Also, $\frac{(a-3d)(a+d)}{(a-d)(a+3d)} = \frac{8}{15}$

or $\quad 15[a^2 - 3d^2 - 2ad] = 8[a^2 - 3d^2 + 2ad]$

or $\quad 7(a^2 - 3d^2) = 46ad$

or $\quad 7(49 - 3d^2) = 46 \times 7 \times d$

or $\quad 49 - 3d^2 = 46d$

or $\quad 3d^2 + 46d - 49 = 0$

or $\quad (d - 1)(3d + 49) = 0$

or $\quad d = 1$

Therefore, the required numbers are 4, 6, 8, 10.

6. $\frac{1}{c-a} - \frac{1}{b-c} = \frac{1}{a-b} - \frac{1}{c-a}$

$$\frac{(a+b-2c)}{b-c} = \frac{(c+b-2a)}{a-b}$$

or $\quad (a - b)(a + b - 2c) = (b - c)(b + c - 2a) \qquad (1)$

Above is true by given condition as shown below by (2). We are given

$$(c-a)^2-(b-c)^2=(a-b)^2-(c-a)^2$$

or $(c-a+b-c)(c-a-b+c)$

$$=(a-b+c-a)(a-b-c+a)$$

or $(b-a)(2c-a-b)=(c-b)(2a-b-c)$

or $(b-c)(b+c-2a)=(a-b)(a+b-2c)$ (2)

7. Series 17, 21, 25, ..., 417 has common difference 4. Series 16, 21, 26, ..., 466 has common difference 5. Hence, the series with common terms has common difference equal to the L.C.M. of 4 and 5 i.e., 20. The first common term is 21. Hence, the series is 21, 41, 61, ..., 401 which has 20 terms.

8. $a+3k=a^2+(a+k)^2+(a+2k)^2$ (1)

(where k is the common difference of A.P.)

$\Rightarrow$ $5k^2+3(2a-1)k+3a^2-a=0$ (2)

$\Rightarrow$ $9(2a-1)^2-20(3a^2-a)\geq 0$ ($\because k$ is real)

$\Rightarrow$ $24a^2+16a-9\leq 0$

$\Rightarrow$ $-\frac{1}{3}-\frac{\sqrt{70}}{12}<a<-\frac{1}{3}+\frac{\sqrt{70}}{12}$

$\Rightarrow$ $a=-1, 0$ ($\because a\in I$)

From (2), when $a=0$,

$5k^2-3k=0$

$\Rightarrow$ $k=0, 3/5$ [Not possible]

When $a=-1$,

$5k^2-9k+4=0$

$\Rightarrow$ $k=1, \frac{4}{5}$ $\Rightarrow k=1$ (since k is an integer)

$\therefore$ $a=-1, b=0, c=1, d=2$

$\Rightarrow$ $a+b+c+d=2$

9. According to the given condition, we have

$$\frac{a+b}{2}=\frac{a^n+b^n}{a^{n-1}+b^{n-1}}$$

or $(a+b)(a^{n-1}+b^{n-1})=2(a^n+b^n)$

or $a^n+ab^{n-1}+ba^{n-1}+b^n=2a^n+2b^n$

or $ab^{n-1}+a^{n-1}b=a^n+b^n$

or $ab^{n-1}-b^n=a^n-a^{n-1}b$

or $b^{n-1}(a-b)=a^{n-1}(a-b)$

or $b^{n-1}=a^{n-1}$

or $\left(\frac{a}{b}\right)^{n-1}=1=\left(\frac{a}{b}\right)^0$

$\Rightarrow$ $n-1=0$

or $n=1$

10. $a, x_1, x_2, \ldots, x_n, b$ are in A.P. Hence,

$b=T_{n+2}=a+(n+1)d$

or $\frac{b-a}{n+1}=d$

$\Rightarrow$ $x_r=T_{r+1}=a+rd$

$$=a+\frac{r(b-a)}{n+1}$$

$$=\frac{a(n-r+1)+rb}{n+1}$$ (1)

Put $a=x$ and $b=2y$ and then again put $a=2x$ and $b=y$ and equate the results as the two means are equal. Then

$$\frac{x(n-r+1)+2yr}{n+1}=\frac{2x(n-r+1)+yr}{n+1}$$

$\Rightarrow$ $x(n-r+1)=yr$

$\Rightarrow$ $\frac{x}{y}=\frac{r}{n-r+1}$

Exercise 5.3

1. $S_n=nP+\frac{n(n-1)}{2}Q$

$$=\frac{n}{2}[2P+(n-1)Q]$$

Comparing with

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$d=Q$

2. We have

$(x+1)+(x+4)+\cdots+(x+28)=155$

L.H.S. has 10 terms in A.P. with common difference 3.

$\Rightarrow$ $\frac{10}{2}[(x+1)+(x+28)]=155$

$\Rightarrow$ $x=1$

3. Let a be the first term and d be the common difference of A.P.

According to the question,

$S_{10}=4S_5$

$\Rightarrow$ $5[2a+9d]=4\times\frac{5}{2}[2a+4d]$

$\Rightarrow$ $10a+45d=20a+40d$

$\Rightarrow$ $10a-5d=0$

$\Rightarrow$ $\frac{a}{d}=\frac{1}{2}$

4. Let a be the first term and d be the common difference of the given A.P. Then, sum of n terms is

$$S_1=\frac{n}{2}[2a+(n-1)d]$$ (1)

Sum of $2n$ terms,

$$S_2=\text{Sum of } 2n \text{ terms}=\frac{2n}{2}[2a+(2n-1)d]$$ (2)

Sumof $3n$ terms,

$$S_3=\frac{3n}{2}[2a+(3n-1)d]$$ (3)

Now, $S_2-S_1=\frac{2n}{2}[2a+(2n-1)d]-\frac{n}{2}[2a+(n-1)d]$

$$=\frac{n}{2}[2\{2a+(2n-1)d\}-\{2a+(n-1)d\}]$$

$$=\frac{n}{2}[2a+(3n-1)d]$$

$\therefore$ $3(S_2-S_1)=\frac{3n}{2}[2a+(3n-1)d]=S_3$ [Using (3)]

5. Given, $S_{2n}=3S_n$

$\Rightarrow$ $\frac{2n}{2}[2a+(2n-1)d]=3\frac{n}{2}[2a+(n-1)d]$

$\Rightarrow$ $4a+(4n-2)d=6a+(3n-3)d$

or $2a=(n+1)d$ (1)

Now, $\frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a+(3n-1)d]}{\frac{n}{2}[2a+(n-1)d]}$

$= \frac{3[(n+1)d+(3n-1)d]}{[(n+1)d+(n-1)d]}$

$= \frac{3[4nd]}{[2nd]} = 6$

6. Let a be the first term and d the common difference of the given A.P. Then, the sums of m and n terms are given by

$$S_m = \frac{m}{2}[2a+(m-1)d]$$

and $S_n = \frac{n}{2}[2a+(n-1)d]$

Given, $\frac{S_m}{S_n} = \frac{m^2}{n^2}$

$\Rightarrow \quad \frac{\frac{m}{2}[2a+(m-1)d]}{\frac{n}{2}[2a+(n-1)d]} = \frac{m^2}{n^2}$

or $\frac{2a+(m-1)d}{2a+(n-1)d} = \frac{m}{n}$

Replacing m by $2m-1$ and n by $2n-1$, we have

$$\frac{2a+(2m-1-1)d}{2a+(2n-1-1)d} = \frac{2m-1}{2n-1}$$

or $\frac{a+(m-1)d}{a+(n-1)d} = \frac{2m-1}{2n-1}$

$\Rightarrow \quad \frac{T_m}{T_n} = \frac{2m-1}{2n-1}$

7. Clearly, nth term of the given series is negative or positive accordingly as n is even or odd, respectively.

(a) n is even:

$1^2-2^2+3^2-4^2+5^2-6^2+\cdots+(n-1)^2-n^2$

$= (1^2-2^2)+(3^2-4^2)+(5^2-6^2)+\cdots+((n-1)^2-n^2)$

$= (1-2)(1+2)+(3-4)(3+4)+(5-6)(5+6)$

$\quad +\cdots+((n-1)-(n))(n-1+n)$

$= -(1+2+3+4+\cdots+(n-1)+n)$

$= -\frac{n(n+1)}{2}$

(b) n is odd:

$(1^2-2^2)+(3^2-4^2)+\cdots+\{(n-2)^2-(n-1)^2\}+n^2$

$= (1-2)(1+2)+(3-4)(3+4)+\cdots$

$\quad +[(n-2)-(n-1)][(n-2)+(n-1)]+n^2$

$= -(1+2+3+4+\cdots+(n-2)+(n-1))+n^2$

$= -\frac{(n-1)(n-1+1)}{2}+n^2$

$= \frac{n(n+1)}{2}$

8. Let there be n sides in the polygon. Then by geometry, the sum of all n interior angles of polygon is $(n-2)\times 180°$. Also the angles are in A.P. with the smallest angle $120°$ and common difference $5°$.

Therefore, sum of all interior angles of polygon is

$$\frac{n}{2}[2\times 120+(n-1)\times 5]$$

Thus, we must have

$$\frac{n}{2}[2\times 120+(n-1)5] = (n-2)\times 180$$

or $\frac{n}{2}[5n+235] = (n-2)\times 180$

or $5n^2+235n = 360n-720$

or $5n^2-125n+720 = 0$

or $n^2-25n+144 = 0$

or $(n-16)(n-9) = 0$

or $n = 16, 9$

But if $n = 16$, then 16^{th} angle $= 120+15\times 5 = 195 > 180°$ which is not possible. Hence, $n = 9$.

9. Suppose the work is completed in n days. Since 4 workers dropped every day except the first day, the total amount of work done in n days is the sum of n terms of an A.P. with first term 150 and common difference -4.

$\therefore$ Total amount of work done

$$= \frac{n}{2}[2\times 150+(n-1)\times(-4)] = n(152-2n)$$

Had the workers not dropped, the work would have finished in $(n-8)$ days with 150 workers working each day. Therefore, the total amount of work done in n days is $150(n-8)$.

$\therefore \quad n(152-2n) = 150(n-8)$

$\Rightarrow \quad n^2-n-600 = 0$

$\Rightarrow \quad (n-25)(n+24) = 0$

$\Rightarrow \quad n = 25$

Thus, the work is completed in 25 days.

Exercise 5.4

1. Given

$$r = \frac{T_2}{T_1} = \frac{x^n}{x^{-4}} = x^{n+4}$$

and $T_8 = ar^7 = x^{-4}\times(x^{n+4})^7 = x^{52}$ (given)

$\Rightarrow \quad 7n+24 = 52$

or $n = 4$

2. Let A be the first term and R the common ratio of the given G.P.

Then, $a = p$th term

$\Rightarrow \quad a = AR^{p-1}$

$\Rightarrow \quad \log a = \log A+(p-1)\log R \qquad (1)$

Similarly,

$$\log b = \log A+(q-1)\log R \qquad (2)$$

and $\log c = \log A+(r-1)\log R \qquad (3)$

Now, $(q-r)\log a+(r-p)\log b+(p-q)\log c$

$= (q-r)\{\log A+(p-1)\log R\}+(r-p)$

$\{\log A+(q-1)\log R\}+(p-q)\{\log A+(r-1)\log R\}$

[Using (1), (2), and (3)]

$= \log A\{(q-r)+(r-p)+(p-q)\}+\log R\{(p-1)\times(q-r)$

$+(q-1)(r-p)+(r-1)(p-q)\}$

$= \log A \times 0 + \log R\{p(q-r) + q(r-p) + r(p-q) - (q-r) - (r-p) - (p-q)\}$

$= \log A \times 0 + \log R \times 0 = 0$

3. Let A be the first term and R the common ratio of a G.P. Then, $a_p = AR^{p-1}$, $a_q = AR^{q-1}$, and $a_r = AR^{r-1}$. We have to prove that a_p, a_q, and a_r are in G.P. given that p, q, r and in A.P.

$(a_q)^2 = (AR^{q-1})^2$

$= A^2 R^{2q-2}$

$= A^2 R^{p+r-2}$ [$\because$ p, q, r are in A.P. $\therefore$ $2q = p + r$]

$= (AR^{p-1})(AR^{r-1}) = a_p a_r$

Hence, a_p, a_q, and a_r are in G.P.

4. It is given that a, b, c, and d are in G.P.

$\therefore$ $b = ar$, $c = ar^2$, $d = ar^3$.

Now $(a^n + b^n) = (a^n + (ar)^n) = a^n(1 + r^n)$

$(b^n + c^n) = ((ar)^n + (ar^2)^n) = a^n r^n (1 + r^n)$

$(c^n + d^n) = ((ar^2)^n + (ar^3)^n) = a^n r^{2n}(1 + r^n)$

Clearly, $\dfrac{b^n + c^n}{a^n + b^n} = \dfrac{c^n + d^n}{b^n + c^n} = r^n$

Thus, $(a^n + b^n)$, $(b^n + c^n)$, $(c^n + d^n)$ are in G.P.

5. Given that

$$T_m = AR^{m-1} = \frac{1}{n^2} \text{ and } T_n = AR^{n-1} = \frac{1}{m^2}$$

$$\Rightarrow \quad A^2 R^{m+n-2} = \frac{1}{m^2 n^2}$$

$$\Rightarrow \quad AR^{\frac{m+n}{2}-1} = \frac{1}{mn}$$

$$\Rightarrow \quad T_{\frac{m+n}{2}} = \frac{1}{mn}$$

6. Let r be the common ratio of the G.P., a, b, c, d. Then

$b = ar$, $c = ar^2$, and $d = ar^3$.

L.H.S. $= (ab + bc + cd)^2$

$= (aar + arar^2 + ar^2 ar^3)^2$

$= a^4 r^2 (1 + r^2 + r^4)^2$

R.H.S. $= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

$= (a^2 + a^2 r^2 + a^2 r^4)(a^2 r^2 + a^2 r^4 + a^2 r^6)$

$= a^2(1 + r^2 + r^4) a^2 r^2 (1 + r^2 + r^4)$

$= a^4 r^2 (1 + r^2 + r^4)^2$

$\therefore$ L.H.S. = R.H.S.

7. Let the three number in G.P. be a, ar, ar^2

Sum of numbers is $a(1 + r + r^2) = 56$ (1)

Subtracting 1, 7, 21 from these terms, we get terms

$a - 1$, $ar - 7$, $ar^2 - 21$

Now these numbers are in A.P. Thus,

$2(ar - 7) = a - 1 + ar^2 - 21$ (2)

or $a(r^2 - 2r + 1) = 8$

From (1) and (2), we get

$7(r^2 - 2r + 1) = 1 + r + r^2$

or $6r^2 - 15r + 6 = 0$

or $(6r - 3)(r - 2) = 0$

or $r = 2, \dfrac{1}{2}$

When $r = 2$, $a = 8$

When $r = \dfrac{1}{2}$, $a = 32$

Therefore, when $r = 2$, the three numbers in G.P. are 8, 16, and 32.

When $r = \dfrac{1}{2}$, the three numbers in G.P. are 32, 16, and 8.

Thus, in either case, the three required numbers are 8, 16, and 32.

8. Given that x, y, z are the pth, qth and rth terms of an A.P., respectively.

Let A be the first term and D be the common difference.

$\therefore$ $x = A + (p-1)D$;

$y = A + (q-1)D$;

and $z = A + (r-1)D$

Using these, we get the following results:

$x - y = (p - q)D$;

$y - z = (q - r)D$;

and $z - x = (r - p)D$

Also x, y, z are the pth, qth and rth terms of a GP.

Let a be the first term and R be the common ratio.

$\therefore$ $x = aR^{p-1}$;

$y = aR^{q-1}$;

and $z = aR^{r-1}$

Now, $x^{y-z} y^{z-x} z^{x-y} = (aR^{p-1})^{y-z} (aR^{q-1})^{z-x} (aR^{r-1})^{x-y}$

$= a^{y-z+z-x+x-y} R^{(p-1)(y-z)+(q-1)(z-x)+(r-1)(x-y)}$

$= a^0 R^{(r-1)(q-r)D + (q-1)(r-p)D + (r-1)(p-q)D}$

$= a^0 R^0 = 1$

9. Let three terms in G.P. be a/r, a, ar. Given that the product is 125.

$\therefore$ $a = 5$

Also, $a^2\left(r \times 1 + 1 \times \dfrac{1}{r} + r \times \dfrac{1}{r}\right) = \dfrac{175}{2}$

or $25(r^2 + r + 1) = \dfrac{175}{2} r$

or $2(r^2 + r + 1) = 7r$

or $2r^2 - 5r + 2 = 0$

or $(r - 2)(2r - 1) = 0$

or $r = 2, \dfrac{1}{2}$.

Hence, the numbers are 10, 5, 5/2 or 5/2, 5, 10.

10. Product of n G.M.'s is $(\sqrt{ab})^n = \left(\sqrt{4\dfrac{1}{4}}\right)^3 = 1$

11. Let the two numbers be a and b. Then,

A.M. = 34

$$\Rightarrow \quad \frac{a+b}{2} = 34 \quad \text{or } a + b = 68$$

G.M. = 16

$\Rightarrow$ $\sqrt{ab} = 16$ or $ab = 256$ (1)

$\therefore$ $(a - b)^2 = (a + b)^2 - 4ab$

or $(a - b)^2 = (68)^2 - 4 \times 256 = 3600$

or $a - b = 60$ (2)

On solving (1) and (2), we get $a = 64$ and $b = 4$. Hence, the required numbers are 64 and 4.

12. Let A be the A.M. and G be the G.M. of a and b. Then the numbers a and b are roots of the equation $x^2 - 2Ax + G^2 = 0$, i.e.,

$x^2 - 4Gx + G^2 = 0$ [$\because$ $A = 2G$]

$$\Rightarrow \quad x = \frac{4G \pm \sqrt{16G^2 - 4G^2}}{2}$$

$$= \frac{4G \pm 2\sqrt{3}G}{2} = (2 \pm \sqrt{3})G$$

Hence, $a:b = (2+\sqrt{3}):(2-\sqrt{3})$

13. $a_1 = 2\sqrt{3}$ and $b_1 = \frac{52}{9}\sqrt{3}$

Also, $3a_{99}b_{99} = 104$

$\Rightarrow \quad a_1 r_1^{98} b_1 r_2^{98} = 104/3$

$\Rightarrow \quad r_1^{98} r_2^{98} = 1$ (Putting the values of a_1 and b_1)

$\Rightarrow \quad r_1 r_2 = 1$

$\Rightarrow \quad a_i b_i = \frac{104}{3} \ \forall i$

$\therefore \quad a_1b_1 + a_2b_2 + \cdots + a_nb_n = 102 \times \frac{104}{3}$

$= 34 \times 104$

$= 3536$

Exercise 5.5

1. $S_{2n} = 5(t_1 + t_3 + t_5 + \cdots + n)$

$$\Rightarrow \quad \frac{a\left(1-r^{2n}\right)}{1-r} = 5 \times \frac{a\left\{1-\left(r^2\right)^n\right\}}{1-r^2}$$

$$\Rightarrow \quad 1 = \frac{5}{1+r}$$

or $\quad 1 + r = 5$

or $\quad r = 4$

2. $S_n = 3 - \frac{3^{n+1}}{4^{2n}}$

Putting $n = 1, 2$, we get

$$T_1 = S_1 = 3 - \frac{9}{16} = \frac{39}{16}$$

$$S_2 = 3 - \frac{27}{256} = T_1 + T_2$$

$\Rightarrow \quad T_2 = S_2 - T_1$

$$= \left(3 - \frac{27}{256}\right) - \left(3 - \frac{9}{16}\right) = \frac{117}{256}$$

$$\Rightarrow \quad r = \frac{T_2}{T_1} = \frac{\frac{117}{256}}{\frac{39}{16}}$$

$$= \frac{117 \times 16}{256 \times 39} = \frac{3}{16}$$

3. $S_1 = \underset{n \text{ digits}}{6666\ldots}$

$= 6 + 6 \times 10^1 + 6 \times 10^2 + \cdots + 6 \times 10^{n-1}$

$$= 6 \times \frac{\left(10^n - 1\right)}{10-1} = \frac{2}{3}\left(10^n - 1\right)$$

Similarly,

$$S_2 = \frac{8}{9}\left(10^n - 1\right)$$

$$\Rightarrow \quad S_1^2 + S_2 = \frac{4}{9}(10^n - 1)^2 + \frac{8}{9}\left(10^n - 1\right)$$

$$= \frac{4}{9}\left(10^n - 1\right)\ [10^n - 1 + 2]$$

$$= \frac{4}{9}\ [10^{2n} - 1]$$

Also, $\underset{2n \text{ digits}}{4444\cdots4} = 4 + 4 \times 10 + 4 \times 10^2 + \cdots + 4 \times 10^{2n-1}$

$$= 4\frac{\left(10^{2n} - 1\right)}{10-1}$$

Hence, proved.

4. $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots n$ terms

$$= \frac{1}{x-y}[(x+y)(x-y) + (x^2 + xy + y^2)(x-y) + \cdots n \text{ terms}]$$

$$= \frac{1}{x-y}[(x^2 - y^2) + (x^3 - y^3) + \cdots n \text{ terms}]$$

$$= \frac{1}{x-y}[(x^2 + x^3 + \cdots n \text{ terms}) - (y^2 + y^3 + \cdots n \text{ terms})]$$

$$= \frac{1}{x-y}\left[\frac{x^2(x^n-1)}{x-1} - \frac{y^2(y^n-1)}{y-1}\right]$$

5. $\frac{4}{3} + \frac{10}{9} + \frac{28}{27} + \cdots$ up to n terms

$$= \left(1 + \frac{1}{3}\right) + \left(1 + \frac{1}{9}\right) + \left(1 + \frac{1}{27}\right) + \cdots$$

$$= n + \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \text{ up to } n \text{ terms}\right)$$

$$= n + \frac{\frac{1}{3}\left(1 - \frac{1}{3^n}\right)}{1 - \frac{1}{3}}$$

$$= n + \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

$$= n + \frac{3^n - 1}{2 \times 3^n}$$

$$= \frac{2 \cdot n\, 3^n + 3^n - 1}{2 \times 3^n}$$

$$= \frac{(2n+1) \times 3^n - 1}{2(3^n)}$$

6. $p(x) = \left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right) = \frac{1+x^n}{1+x}$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n$, i.e., $1 + (-1)^n = 0$. Hence, n is odd.

7. $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^3 + \cdots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$

$$= \frac{\frac{3}{4}\left(1 - \left(-\frac{3}{4}\right)^n\right)}{1 - \left(-\frac{3}{4}\right)} = \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$$

Now, $b_n = 1 - a_n$ and $b_n > a_n$ for $n \geq n_0$.

$\therefore \quad 1 - a_n > a_n$ or $2a_n < 1$

or $\quad \frac{6}{7}\left[1-\left(-\frac{3}{4}\right)^n\right] < 1$

or $\quad -\left(-\frac{3}{4}\right)^n < \frac{1}{6}$

or $\quad (-3)^{n+1} < 2^{2n-1}$

For n to be even, inequality always holds. For n to be odd, it holds for $n \geq 7$. Therefore, the least natural number for which it holds is 6.

8. $s = \sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots \text{ to } \infty = \frac{1}{1-r}$

$\Rightarrow \quad r = 1 - \frac{1}{s} = \frac{s-1}{s}$

$\Rightarrow \quad \sum_{n=0}^{\infty} r^{2n} = \frac{1}{1-r^2} = \frac{1}{1-\frac{(s-1)^2}{s^2}} = \frac{s^2}{2s-1}$

9. We have,

$6^{1/2} \times 6^{1/4} \times 6^{1/8} \cdots \infty$

$= 6^{[1/2 + 1/4 + 1/8 + \cdots \text{ to } \infty]}$

$= 6^{[(1/2)/(1-1/2)]} = 6^1 = 6$

$\left[\because \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \text{ to } \infty = \frac{1/2}{1-1/2} = 1\right]$

10. $S_n = 1 + \frac{1-\left(\frac{1}{2}\right)^3}{1-\frac{1}{2}} + \frac{1-\left(\frac{1}{2}\right)^5}{1-\frac{1}{2}} + \cdots \; n \text{ terms}$

$= 1 + 2\left(1-\left(\frac{1}{2}\right)^3\right) + 2\left(1-\left(\frac{1}{2}\right)^5\right) + \cdots n \text{ times}$

$= 1 + (2 + 2 + \ldots (n-1) \text{ times}) - \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \ldots (n-1) \text{ times}\right]$

$= 1 + 2(n-1) - \frac{(1/2)^2}{1-\left(\frac{1}{2}\right)^2}\left(1-\left(\frac{1}{2}\right)^{2n-2}\right)$

$= 2n - 1 - \frac{1}{3} + \frac{4}{3.2^{2n}}$

$= 2n - \frac{4}{3}\left(1 - \frac{1}{2^{2n}}\right)$

Exercise 5.6

1. Let the H.P. be $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}, \ldots$

Then, $a_8 = \frac{1}{2}$ and $a_{14} = \frac{1}{3}$

$\Rightarrow \quad \frac{1}{a+7d} = \frac{1}{2}$

and $\quad \frac{1}{a+13d} = \frac{1}{3} \qquad \left[\because a_n = \frac{1}{a+(n-1)d}\right]$

$\Rightarrow \quad a + 7d = 2$ and $a + 13d = 3$

$\Rightarrow \quad a = \frac{5}{6}, d = \frac{1}{6}$

Now, $a_{20} = \frac{1}{a+19d} = \frac{1}{\frac{5}{6}+\frac{19}{6}} = \frac{1}{4}$

and $\quad a_n = \frac{1}{a+(n-1)d} = \frac{1}{\frac{5}{6}+(n-1)\times\frac{1}{6}} = \frac{6}{n+4}$

2. Let the H.P. be

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \cdots$

Then $\quad \frac{1}{a} = \frac{2}{5}$ and $\frac{1}{a+d} = \frac{12}{13}$

$\Rightarrow \quad a = \frac{5}{2}$ and $d = -\frac{17}{12}$

Now, nth term of the H.P. is

$\frac{1}{a+(n-1)d} = \frac{12}{47-17n}$

So, the nth term is largest when $47 - 17n$ has the least positive value. Clearly, $12/(47 - 17n)$ is least for $n = 2$. Hence, 2nd term is the largest term.

3. a, b, c in G.P.

$\Rightarrow \quad b = ar, c = ar^2$

$\Rightarrow \quad \frac{1}{a-b}, \frac{1}{c-a}, \frac{1}{b-c}$ are in A.P.

$\Rightarrow \quad \frac{2}{a(r^2-1)} = \frac{1}{a(1-r)} + \frac{1}{ar(1-r)} = \frac{1+r}{ar(1-r)}$

$\Rightarrow \quad -2r = (r+1)^2$

$\Rightarrow \quad r^2 + 4r + 1 = 0 \qquad (1)$

Also, $c + 4b + a = a(r^2 + 4r + 1) = 0$ [Using (1)]

4. By the given conditions, we have

$2y = x + z \qquad (1)$

$b^2y^2 = ax \times cz \qquad (2)$

$b = 2ac/(a + c) \qquad (3)$

It is clear from the question that we have to eliminate b and y. Substituting for y and b from (1) and (3) in (2), we get

$\frac{4a^2c^2}{(a+c)^2} \times \frac{(x+z)^2}{4} = axcz$

or $\quad \frac{(x+z)^2}{xz} = \frac{(a+c)^2}{ac}$

or $\quad \frac{x^2+z^2+2xz}{xz} = \frac{a^2+c^2+2ac}{ac}$

or $\quad \frac{x}{z} + \frac{z}{x} + 2 = \frac{a}{c} + \frac{c}{a} + 2$

or $\quad \frac{x}{z} + \frac{z}{x} = \frac{a}{c} + \frac{c}{a}$

5. Let $1/a = A$, $1/b = A + D$, $1/c = A + 2D$, $1/d = A + 3D$. Then,

$a^{-2} - d^{-2} = A^2 - (A + 3D)^2 = -3D(2A + 3D)$

and $\quad b^{-2} - c^{-2} = -D(2A + 3D)$

$\Rightarrow \quad \frac{a^{-2}-d^{-2}}{b^{-2}-c^{-2}} = 3$

6. Here, $x = \frac{1}{1-a}, y = \frac{1}{1-b}, z = \frac{1}{1-c}$

Since a, b, c are in A.P., so

$1-a, 1-b, 1-c$ are in A.P.

$\Rightarrow \quad \frac{1}{1-a}, \frac{1}{1-b}, \frac{1}{1-c}$ are in H.P.

$\Rightarrow \quad x, y, z$ are in H. P.

7. $\left.\begin{array}{l} x, 1, z \text{ are in A.P.} \Rightarrow 2 = x + z \\ x, 2, z \text{ are in G.P.} \Rightarrow 4 = xz \end{array}\right]$ (1)

$\therefore \quad 4 = \frac{2xz}{x+z}$

Therefore, $x, 4, z$ are in H.P.

8. Here,

$a + b = a_1 + a_{2n} = a_2 + a_{2n-1} = \cdots = a_n + a_{n+1}$

$ab = g_1 \times g_{2n} = g_2 \times g_{2n-1} = \cdots = g_n \times g_{n+1}$

$\Rightarrow \quad$ and $h = \frac{2ab}{a+b}$

$\therefore \quad$ Sum of given series $= \frac{n(a+b)}{ab} = \frac{2n}{h}$

9. Let α, β be roots of $ax^2 + bx + c = 0$. Then

$$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2 + \alpha^2}{\alpha^2\beta^2}$$

$$\Rightarrow \quad \alpha + \beta = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2}$$

$$\Rightarrow \quad \frac{-b}{a} = \frac{(-b/a)^2 - 2c/a}{(c/a)^2}$$

or $\quad \frac{-b}{a} = \frac{b^2 - 2ac}{c^2}$

or $\quad -bc^2 = ab^2 - 2a^2c$

or $\quad 2a^2c = ab^2 + bc^2$

or $\quad \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$

$\Rightarrow \quad \frac{c}{a}, \frac{a}{b}, \frac{b}{c}$ are in A.P.

$\Rightarrow \quad \frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in H.P.

10. Let a, b $(a > b)$ be the two numbers. Given,

$$\frac{a+b}{2} = 2 \text{ or } a + b = 4 \qquad (1)$$

and $\quad \sqrt{(a+1)b} = 2$

or $\quad (a+1)b = 4$

or $\quad b^2 - 5b + 4 = 0 \qquad$ [Using (1)]

or $\quad b = 1, 4$

But $b \neq 4$, therefore, $b = 1$, and hence $a = 3$.

Hence, harmonic mean,

$$H = \frac{2\times3\times1}{3+1} = \frac{3}{2}$$

11. $H = 21/5, 3A + G^2 = 36$

or $\quad 3A + AH = 36$

or $\quad A = 5$

or $\quad a + b = 10$

Also, from $H = 2ab/(a+b) = 21/5$, $ab = 21$, we have

$a^2 + b^2 = (a+b)^2 - 2ab = 100 - 42 = 58$

Exercise 5.7

1. α is nth root of unity,

Therefore, α is root of the equation $z^n = 1$.

Let $\quad S = 1 + 2\alpha + 3\alpha^2 + \ldots + n\alpha^{n-1}$

$\Rightarrow \quad \alpha S = \alpha + 2\alpha^2 + 3\alpha^3 + \ldots + (n-1)\alpha^{n-1} + n\alpha^n$

On subtracting, we get

$$S(1-\alpha) = 1 + [\alpha + \alpha^2 + \alpha^{n-1}] - n\alpha^n$$

$$\Rightarrow \quad S(1-\alpha) = 1 + \frac{\alpha(1-\alpha^{n-1})}{1-\alpha} - n\alpha^n$$

$$\Rightarrow \quad S = \frac{1}{1-\alpha} + \frac{\alpha - \alpha^n}{(1-\alpha)^2} - \frac{n\alpha^n}{1-\alpha}$$

$$= \frac{1}{1-\alpha} + \frac{\alpha - 1}{(1-\alpha)^2} - \frac{n}{1-\alpha} \qquad [\because \alpha^n = 1]$$

$$= -\frac{n}{1-\alpha}$$

2. Clearly, the given series is an arithmetico-geometric series whose corresponding A.P. and G.P. are, respectively, 1, 4, 7, 10, ... and $1, 1/5, 1/5^2, 1/5^3, \ldots$.

The nth term of A.P. is $[1 + (n-1) \times 3] = 3n - 2$.

The nth term of G.P. is $[1 \times (1/5)^{n-1}] = (1/5)^{n-1}$.

So, the nth term of the given series is

$(3n-2) \times (1/5^{n-1}) = (3n-2)/5^{n-1}$.

Let $\quad S_n = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{3n-5}{5^{n-2}} + \frac{3n-2}{5^{n-1}} \qquad (1)$

$$\frac{1}{5}S_n = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots + \frac{(3n-5)}{5^{n-1}} + \frac{3n-2}{5^n} \qquad (2)$$

Subtracting (2) from (1), we get

$$S_n - \frac{1}{5}S_n = 1 + \left[\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^2} + \cdots + \frac{3}{5^{n-1}}\right] - \frac{(3n-2)}{5^n}$$

or $\quad \frac{4}{5}S_n = 1 + \frac{3}{5}\frac{\left(1 - \left(\frac{1}{5}\right)^{n-1}\right)}{\left(1 - \frac{1}{5}\right)} - \frac{(3n-2)}{5^n}$

$$= 1 + \frac{3}{5}\frac{\left[1 - \frac{1}{5^{n-1}}\right]}{\left(\frac{4}{5}\right)} - \frac{(3n-2)}{5^n}$$

$$= 1 + \frac{3}{4}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{5^n}$$

or $\quad S_n = \frac{5}{4} + \frac{15}{16}\left(1 - \frac{1}{5^{n-1}}\right) - \frac{(3n-2)}{4(5^{n-1})}$

3. $S = \frac{3}{2} - \frac{5}{6} + \frac{7}{18} - \frac{9}{54} + \cdots \infty$

$$\Rightarrow \quad S = 3\left(\frac{1}{2}\right) + 5\left(-\frac{1}{6}\right) + 7\left(\frac{1}{18}\right) + 9\left(-\frac{1}{54}\right) + \cdots \qquad (1)$$

$$\Rightarrow \quad -\frac{1}{3}S = 3\left(-\frac{1}{6}\right) + 5\left(\frac{1}{18}\right) + 9\left(-\frac{1}{54}\right) + \cdots \qquad (2)$$

Subtracting (2) from (1), we have

$$\frac{4}{3}S = 3\left(\frac{1}{2}\right) + \left[2\left(-\frac{1}{6}\right) + 2\left(\frac{1}{18}\right) + 2\left(-\frac{1}{54}\right) + \cdots \infty\right]$$

$$= \frac{3}{2} + \frac{2\left(-\frac{1}{6}\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{3}{2} - \frac{\frac{1}{3}}{\frac{4}{3}}$$

$$= \frac{3}{2} - \frac{1}{4}$$

$$= \frac{5}{4}$$

$$\Rightarrow \quad S = \frac{15}{16}$$

4. $S = \frac{1^2}{2} + \frac{3^2}{2^2} + \frac{5^2}{2^3} + \frac{7^2}{2^4} + \cdots \infty \qquad (1)$

$$\Rightarrow \quad \frac{1}{2}S = \frac{1^2}{2^2} + \frac{3^2}{2^3} + \frac{5^2}{2^4} + \cdots \infty \qquad (2)$$

Subtracting (2) from (1),

$$\frac{1}{2}S = \frac{1^2}{2} + \frac{8}{2^2} + \frac{16}{2^3} + \frac{24}{2^4} + \frac{32}{2^5} + \cdots \infty \qquad (3)$$

Let $S_1 = \frac{8}{2^2} + \frac{16}{2^3} + \frac{24}{2^4} + \frac{32}{2^5} + \cdots \infty \qquad (4)$

$$\Rightarrow \quad \frac{1}{2}S_1 = \frac{8}{2^3} + \frac{16}{2^4} + \frac{24}{2^5} + \cdots \infty \qquad (5)$$

Subtracting (5) from (4),

$$\frac{1}{2}S_1 = \frac{8}{2^2} + \frac{8}{2^3} + \frac{8}{2^4} + \frac{8}{2^5} + \cdots \infty$$

$$= \frac{\frac{8}{2^2}}{1-\frac{1}{2}}$$

$$= 4$$

or $S_1 = 8$

Hence, from (3),

$$\frac{1}{2}S = \frac{1}{2} + 8$$

$$\Rightarrow \quad S = 17$$

Exercise 5.8

1. $T_n = n(n+1)(n+2)$

Let S_n denote the sum to n terms of the given series. Then,

$$S_n = \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k(k+1)(k+2)$$

$$= \sum_{k=1}^{n} (k^3 + 3k^2 + 2k)$$

$$= \left(\sum_{k=1}^{n} k^3\right) + 3\left(\sum_{k=1}^{n} k^2\right) + 2\left(\sum_{k=1}^{n} k\right)$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}\{n^2 + n + 4n + 2 + 4\}$$

$$= \frac{n(n+1)}{4}(n^2 + 5n + 6)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

2. $1^2 + 3^2 + 5^2 + \cdots = \Sigma(2n-1)^2$

$$= \Sigma(4n^2 - 4n + 1)$$

$$= 4\Sigma n^2 - 4\Sigma n + n$$

$$= 4\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= \frac{n}{3}(4n^2 - 1)$$

$$= \frac{n(2n-1)(2n+1)}{3}$$

3. $S = (1^3 + 2^3 + \cdots + 50^3) - (1^3 + 2^3 + \cdots + 30^3)$

$$= \left(\frac{50\times 51}{2}\right)^2 - \left(\frac{30\times 31}{2}\right)^2 \qquad \left[\text{Using } \Sigma n^3 = \left(\frac{n(n+1)}{2}\right)^2\right]$$

$$= \frac{1}{4}(50\times 51 - 30\times 31)(50\times 51 + 30\times 31)$$

$$= 1409400$$

4. $T_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n}{6}(2n^2 + 3n + 1)$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$S_n = \sum T_n$$

$$= \frac{1}{6}\left[2\sum n^3 + 3\sum n^2 + \sum n\right]$$

$$= \frac{1}{6}\left\{2\left[\frac{n(n+1)}{2}\right]^2 + 3\times\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right\}$$

$$= \frac{1}{6}\left[\frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{6}\times\frac{1}{2}n(n+1)\left[n(n+1) + (2n+1) + 1\right]$$

$$= \frac{n(n+1)}{12}\left[n^2 + 3n + 2\right]$$

For $n = 22$

$$S_{22} = \frac{22 \times 23}{12}\left[22^2 + 66 + 2\right]$$

$$= \frac{22 \times 23}{12}[552]$$

$$= 22 \times 23 \times 46$$

$$= 23276$$

5. When n is even, last term of the series is n^2. So, we have

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + (n-1)^2 + 2 \times n^2$$

$$= \frac{n(n+1)^2}{2} \qquad \ldots(1)$$

When n is odd, the series is

$$1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2 + \ldots + 2(n-1)^2 + n^2$$

$\therefore$ Sum of series $= [1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + \ldots + 2 \times (n-1)^2] + n^2$

$$= \frac{(n-1)\,n^2}{2} + n^2 \qquad \text{[In (1) replacing } n \text{ by } (n-1)\text{]}$$

$$= \frac{n^2(n+1)}{2}$$

6. $11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \ldots + 20^2 - 10^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + 20^2) - 2(1^2 + 2^2 + 3^2 + \ldots + 10^2)$$

$$= \frac{20 \times 21 \times 41}{6} - 2 \times \frac{10 \times 11 \times 21}{6}$$

$$= \frac{20 \times 21}{6}(41 - 21)$$

$$= 1400$$

7. Clearly, here the difference between the successive terms are $7 - 3, 14 - 7, 24 - 14, \ldots$

i.e., 4, 7, 10, ... which are in A.P.

$\therefore \quad T_n = an^2 + bn + c$

Thus, we have

$3 = a + b + c,$

$7 = 4a + 2b + c$

and $14 = 9a + 3b + c$

Solving we get,

$$a = \frac{3}{2},\ b = -\frac{1}{2},\ c = 2.$$

Hence, $T_n = \frac{1}{2}\left(3n^2 - n + 4\right)$

$$\therefore \quad S_n = \frac{1}{2}\left[3\Sigma n^2 - \Sigma n + 4n\right]$$

$$= \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$

$$= \frac{n}{2}(n^2 + n + 4)$$

$$\Rightarrow \quad S_{20} = 4240$$

8. $$\sum_{j=1}^{n}\sum_{i=1}^{n} i \times 3^j = \left(\sum_{j=1}^{n} 3^j\right)\left(\sum_{i=1}^{n} i\right)$$

$$= (3 + 3^2 + 3^3 + \ldots + 3^n) \times (1 + 2 + 3 + \ldots + n)$$

$$= \frac{3(3^n - 1)}{3 - 1} \times \frac{n(n+1)}{2}$$

$$= \frac{3n(3^n - 1)(n+1)}{4}$$

9. We have $S_n = 2n^2 + 3n$

$\therefore \quad a_n = S_n - S_{n-1}$

$$= 2n^2 + 3n - 2(n-1)^2 - 3(n-1)$$

$$= 4n + 1$$

Therefore, the sequence is 5, 9, 13, ...

$$\therefore \quad \sum_{1 \le i < j \le 10}\sum a_i a_j = \frac{\sum_{i=1}^{10}\sum_{j=1}^{10}(4i+1)\cdot(4j+1) - \sum_{i=1}^{10}(4i+1)^2}{2}$$

$$= \frac{\left(\sum_{i=1}^{10}(4i+1)\right)^2 - \sum_{i=1}^{10}(4i+1)^2}{2}$$

$$= 23145$$

10. $$S = \sum_{1 \le i < j}\sum i \times \left(\frac{1}{2}\right)^j$$

$$= 1 \times \left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots\right]$$

$$+ 2 \times \left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \ldots\right]$$

$$+ 3 \times \left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \ldots\right]$$

.....

$$= 1 \times \frac{\left(\frac{1}{2}\right)^2}{1 - \left(\frac{1}{2}\right)} + 2 \times \frac{\left(\frac{1}{2}\right)^3}{1 - \left(\frac{1}{2}\right)} + 3 \times \frac{\left(\frac{1}{2}\right)^4}{1 - \left(\frac{1}{2}\right)} + \ldots$$

$$\therefore \quad S = 1 \times \left(\frac{1}{2}\right) + 2 \times \left(\frac{1}{2}\right)^2 + 3 \times \left(\frac{1}{2}\right)^3 + \ldots$$

$$\therefore \quad \left(\frac{1}{2}\right)S = 1 \times \left(\frac{1}{2}\right)^2 + 2 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^4 + \ldots$$

Subtracting, we get

$$\left(\frac{1}{2}\right)S = \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

$\therefore \quad S = 2$

Exercise 5.9

1. $$T_r = \frac{1}{(2r-1)(2r+1)(2r+3)}$$

$$= \frac{1}{4} \cdot \frac{(2r+3) - (2r-1)}{(2r-1)(2r+1)(2r+3)}$$

$$= \frac{1}{4}\left[\frac{1}{(2r-1)(2r+1)} - \frac{1}{(2r+1)(2r+3)}\right]$$

$$= \frac{1}{4}\left[V(r-1) - V(r)\right], \qquad \text{where } V(r) = \frac{1}{(2r+1)(2r+3)}$$

$$\therefore \quad \sum_{r=1}^{n} T_r = \sum_{r=1}^{n} \frac{1}{4}[V(r-1) - V(r)]$$

$$= \frac{1}{4}[V(0) - V(n)]$$

$$= \frac{1}{4}\left[\frac{1}{3} - \frac{1}{(2n+1)(2n+3)}\right]$$

$\therefore$ Sum of infinite terms $= \frac{1}{4}\left[\frac{1}{3} - 0\right] = \frac{1}{12}$

2. $T_n = \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$

$$= \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8}$$

$$= \frac{n(n+1)(n+2)}{2}$$

$$\therefore \quad \frac{1}{T_r} = \frac{2}{r(r+1)(r+2)} = \frac{r+2-r}{r(r+1)(r+2)}$$

$$= \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= V(r) - V(r+1)$$

$$\therefore \quad \sum_{r=1}^{n} \frac{1}{T_r} = \sum_{r=1}^{n} (V(r) - V(r+1))$$

$$= V(1) - V(n+1)$$

$$= \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+3)}{2(n+1)(n+2)}$$

3. $T_n = \frac{3n^2+1}{(n^2-1)^3} = \frac{1}{2}\frac{6n^2+2}{(n^2-1)^3}$

$$= \frac{1}{2}\left(\frac{(n+1)^3 - (n-1)^3}{(n+1)^3\,(n-1)^3}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(n-1)^3} - \frac{1}{(n+1)^3}\right)$$

$$\therefore \quad S = \frac{1}{2}\left[\left(\frac{1}{1^3} - \frac{1}{3^3}\right) + \left(\frac{1}{2^3} - \frac{1}{4^3}\right) + \left(\frac{1}{3^3} - \frac{1}{5^3}\right) + \left(\frac{1}{4^3} - \frac{1}{6^3}\right) + \ldots\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{8}\right) = \frac{9}{16}$$

4. $T_r = \frac{r}{r^4 + \frac{1}{4}} = \frac{r}{r^4 + r^2 + \frac{1}{4} - r^2} = \frac{r}{\left(r^2 + \frac{1}{2}\right)^2 - r^2}$

$$= \frac{1}{2}\left[\frac{2r}{\left(r^2 + r + \frac{1}{2}\right)\left(r^2 - r + \frac{1}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{\left(r^2 + r + \frac{1}{2}\right) - \left(r^2 - r + \frac{1}{2}\right)}{\left(r^2 + r + \frac{1}{2}\right)\left(r^2 - r + \frac{1}{2}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{r^2 - r + \frac{1}{2}} - \frac{1}{r^2 + r + \frac{1}{2}}\right]$$

$$= \left[\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right]$$

$$\therefore \quad \sum_{r=1}^{20} T_r = \sum_{r=1}^{20}\left[\frac{1}{2r^2 - 2r + 1} - \frac{1}{2r^2 + 2r + 1}\right]$$

$$= \frac{1}{2(1)^2 - 2(1) + 1} - \frac{1}{2(20)^2 + 2(20) + 1}$$

$$= 1 - \frac{1}{841} = \frac{840}{841}$$

5. $T_n = \frac{n+2}{n! + (n+1)! + (n+2)!}$

$$= \frac{n+2}{n![1 + (n+1) + (n+1)(n+2)]}$$

$$= \frac{n+2}{n!(n^2 + 4n + 4)} = \frac{1}{n!(n+2)}$$

$$= \frac{n+1}{(n+2)!} = \frac{(n+2)-1}{(n+2)!} = \frac{1}{(n+1)!} - \frac{1}{(n+2)!}$$

$$\therefore \quad S = \sum_{n=1}^{998} T_n = \sum_{n=1}^{998}\left(\frac{1}{(n+1)!} - \frac{1}{(n+2)!}\right) = \frac{1}{2} - \frac{1}{1000!}$$

6. $T_r = \frac{\sqrt{r}}{1 + \sqrt{r} + \sqrt{r+1}} = \frac{\sqrt{r}\left\{1 + \sqrt{r} - \sqrt{r+1}\right\}}{1 + r + 2\sqrt{r} - (r+1)}$

$$= \frac{1}{2}\left\{1 + \sqrt{r} - \sqrt{r+1}\right\}$$

$$\therefore \quad S_n = \frac{1}{2}(n + 1 - \sqrt{n+1}) = 10$$

Let $\sqrt{n+1} = x$

$\therefore \quad x^2 - x = 20$

$\Rightarrow \quad x^2 - x - 20 = 0$

$\Rightarrow \quad x = \sqrt{n+1} = 5$

$\therefore \quad n = 24$

7. $T_r = \frac{r \times 2^r}{(r+2)!} = \frac{(r+2-2)\cdot 2^r}{(r+2)!}$

$$= \frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}$$

$\therefore$ Required sum $= \sum_{r=1}^{20}\left[\frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}\right]$

$$= \frac{2}{2!} - \frac{2^{21}}{22!} = 1 - \frac{2^{21}}{22!}$$

8. $T_r = \frac{r-2}{(r+2)(r+3)(r+4)} = \frac{r(r+4) - (r+1)(r+2)}{(r+2)(r+3)(r+4)}$

$$= \frac{r}{(r+2)(r+3)} - \frac{(r+1)}{(r+3)(r+4)}$$

$$\therefore \quad S_n = \sum_{r=1}^{n} T_r = \frac{1}{12} - \frac{(n+1)}{(n+3)(n+4)}$$

$$\therefore \quad S_\infty = \frac{1}{12}$$

9. $T_r = r(1-x)(1-2x)(1-3x)\ldots(1-(r-1)x)$

$$= (1-x)(1-2x)(1-3x)\ldots(1-(r-1)x)\frac{xr-1+1}{x}$$

$$= \frac{1}{x}\{(1-x)(1-2x)\ldots(1-(r-1)x) - (1-x)(1-2x)\ldots (1-(r-1)x)(1-rx)\}$$

$$= \frac{1}{x}[V(r-1) - V(r)], \text{ where}$$

$$V(r) = (1-x)(1-2x)\ldots(1-rx)$$

$$\therefore \quad \text{Required sum} = \sum_{r=1}^{n} T_r = \frac{1}{x}\sum_{r=1}^{n} V(r-1) - V(r)$$

$$= \frac{1}{x}(V(0) - V(n))$$

$$= \frac{1}{x}[1-(1-x)(1-2x)\ldots(1-nx)]$$

Exercises

Single Correct Answer Type

1. (4) $2b = a + c$

$\Rightarrow \quad 8b^3 = (a+c)^3 = a^3 + c^3 + 3ac(a+c)$

or $\quad 8b^3 = a^3 + c^3 + 3ac(2b)$

or $\quad a^3 + c^3 - 8b^3 = -6abc$

2. (2) Since a, b, c are in A.P., therefore, $b - a = d$ and $c - b = d$, where d is the common difference of the A.P.

$\therefore \quad a = b - d$ and $c = b + d$

Now,

$abc = 4$

$\Rightarrow \quad (b-d)\,b\,(b+d) = 4$

$\Rightarrow \quad b(b^2 - d^2) = 4$

But, $\quad b(b^2 - d^2) < b \times b^2$

$\Rightarrow \quad b(b^2 - d^2) < b^3$

$\Rightarrow \quad 4 < b^3$

$\Rightarrow \quad b^3 > 4$

$\Rightarrow \quad b > 2^{2/3}$

Hence, the minimum value of b is $2^{2/3}$.

3. (4) The given numbers are in A.P. Therefore,

$$2\log_4(2^{1-x}+1) = \log_2(5\times 2^x + 1) + 1$$

or $\quad 2\log_{2^2}\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x+1) + \log_2 2$

or $\quad \frac{2}{2}\log_2\left(\frac{2}{2^x}+1\right) = \log_2(5\times 2^x+1)2$

or $\quad \log_2\left(\frac{2}{2^x}+1\right) = \log_2(10\times 2^x + 2)$

or $\quad \frac{2}{2^x} + 1 = 10\times 2^x + 2$

$\Rightarrow \quad \frac{2}{y} + 1 = 10y + 2$, where $2^x = y$

or $\quad 10y^2 + y - 2 = 0$

or $\quad (5y-2)(2y+1) = 0$

$\Rightarrow \quad y = 2/5$ or $y = -1/2$

$\Rightarrow \quad 2^x = 2/5$ or $2^x = -1/2$

$\Rightarrow \quad x = \log_2(2/5)$ [$\because 2^x$ cannot be negative]

$\Rightarrow \quad x = \log_2 2 - \log_2 5$

$\Rightarrow \quad x = 1 - \log_2 5$

4. (4) 100^{th} term of 1, 11, 21, 31, ... is $1 + (100-1)10 = 991$.

100^{th} term of 31, 36, 41, 46, ... is $31 + (100-1)5 = 526$.

Let the largest common term be 526. Then,

$526 = 31 + (n-1)10$

or $\quad n = 50.5$

But n is an integer; hence $n = 50$. Hence, the largest common term is $31 + (50-1)10 = 521$.

5. (2) According to the question,

$$\frac{6}{2}(t_1 + t_6) = 5 \times \frac{6}{2}(t_7 + t_{12})$$

$\Rightarrow \quad t_1 + (t_1 + 5d) = 5(t_1 + 6d + t_1 + 11d)$

$\Rightarrow \quad 2t_1 + 5d = 10t_1 + 85d$

$\Rightarrow \quad 8t_1 + 80d = 0$

$\Rightarrow \quad t_1 + 10d = 0$

$\Rightarrow \quad t_{11} = 0$

6. (1) Let $\angle C = 90°$ being greatest and $B = 90° - A$.

The sides are $a - d$, a, and $a + d$

We have $(a+d)^2 = (a-d)^2 + a^2$ (using Pythagoras theorem)

$\therefore \quad 4ad - a^2 = 0$ or $a = 4d$

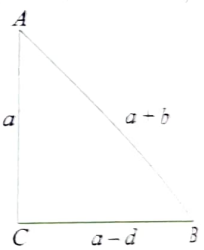
Hence, the sides are $3d$, $4d$, $5d$

Clearly,

$$\sin A = \frac{BC}{AB} = \frac{a-d}{a+d} = \frac{3d}{5d} = \frac{3}{5}$$

$$\sin B = \frac{AC}{AB} = \frac{a}{a+d} = \frac{4d}{5d}$$

$$= \frac{4}{5}$$



7. (2) Since a, q and c are in A.P., so

$2q = a + c$

$$\Rightarrow \quad \frac{1}{p} + \frac{1}{r} = \frac{2}{b}$$

$\Rightarrow \quad \frac{1}{p}, \frac{1}{b}, \frac{1}{r}$ are in A.P.

8. (2) Given,

$$F(n+1) = \frac{2F(n)+1}{2}$$

$\Rightarrow \quad F(n+1) - F(n) = 1/2$

Hence, the given series is an A.P. with common difference $1/2$ and first term being 2. $F(101)$ is 101st term of A.P. given by $2 + (101-1)(1/2) = 52$.

9. (3) Given that

$a_3 + a_5 + a_8 = 11$

$\Rightarrow \quad a + 2d + a + 4d + a + 7d = 11$ (1)

or $\quad 3a + 13d = 11$

Given,

$a_4 + a_2 = -2$

$\Rightarrow \quad a + 3d + a + d = -2$ (2)

or $\quad a = -1 - 2d$

Putting value of a from (2) in (1), we get

$3(-1-2d)+13d=11$

$\Rightarrow \quad 7d=14$

$\Rightarrow \quad d=2$ and $a=-5$

$\Rightarrow \quad a_1+a_6+a_7=7$

11. (1) If p, q, r are in A.P., then in an A.P. or G.P. or an H.P. $a_1, a_2, a_3, \ldots,$ the terms a_p, a_q, a_r are in A.P., G.P. or H.P., respectively.

12. (2) For the equation $x^2-px+1=0$,

the product of roots, $\alpha\beta^2=1$

and for the equation $x^2-qx+8=0$,

the product of roots $\alpha^2\beta=8$

Hence, $(\alpha\beta^2)(\alpha^2\beta)=8$

$\Rightarrow \quad \alpha^3\beta^3=8 \quad \Rightarrow \alpha\beta=2$

$\therefore$ From $\alpha\beta^2=1$, we have $\beta=\frac{1}{2}$ and from $\alpha^2 . \beta=8$, we have $\alpha=4$

Hence, from sum of roots $=-\frac{b}{a}$, we have

$$p=\alpha+\beta^2=4+\frac{1}{4}=\frac{17}{4}$$

and $\quad q=\alpha^2+\beta=16+\frac{1}{2}=\frac{33}{2}$

$\frac{r}{8}$ is arithmetic mean of p and q

$$\therefore \quad \frac{r}{8}=\frac{p+q}{2}$$

$$\Rightarrow \quad r=4(p+q)=4\left(\frac{17}{4}+\frac{33}{2}\right)=17+66=83$$

12. (4) Let a be the first term and d be the common difference of the given A.P. Then

$$S_m=S_n$$

$$\Rightarrow \quad \frac{m}{2}[2a+(m-1)d]=\frac{n}{2}[2a+(n-1)d]$$

or $\quad 2a(m-n)+\{m(m-1)-n(n-1)\}\,d=0$

or $\quad 2a(m-n)+\{(m^2-n^2)-(m-n)\}d=0$

or $\quad (m-n)\,[2a+(m+n-1)d]=0$

or $\quad 2a+(m+n-1)d=0 \qquad [\because m-n\neq 0] \quad (1)$

Now,

$$S_{m+n}=\frac{m+n}{2}[2a+(m+n-1)d]=\frac{m+n}{2}\times 0=0$$

[Using (1)]

13. (4) $S_{n+3}-3S_{n+2}+3S_{n+1}-S_n$

$=(S_{n+3}-S_{n+2})-2(S_{n+2}-S_{n-1})+(S_{n+1}-S_n)$

$=T_{n+3}-2T_{n+2}+T_{n+1}$

$=(T_{n+3}-T_{n+2})-(T_{n+2}-T_{n+1})$

$=d-d=0.$

14. (3) Given, $S_p=0$. Therefore,

$$\frac{p}{2}[2a+(p-1)d]=0 \text{ or } d=\frac{-2a}{p-1} \qquad (1)$$

Sum of next q terms is sum of an A.P. whose first term will be

$T_{p+1}=a+pd.$

$$\therefore \quad S=\frac{q}{2}[2(a+pd)+(q-1)d]$$

$$=\frac{q}{2}[2a+(p-1)d+(p+q)d]$$

$$=\frac{q}{2}\left[0-(p+q)\frac{2a}{p-1}\right]$$

$$=-a\,\frac{(p+q)q}{p-1} \qquad \text{[Using (1)]}$$

15. (2) $S_{3n}=\frac{3n}{2}[2a+(3n-1)d]$

$$S_{n-1}=\frac{n-1}{2}[2a+(n-2)d]$$

$$\Rightarrow \quad S_{3n}-S_{n-1}=\frac{1}{2}[2a(3n-n+1)]$$

$$+\frac{d}{2}[3n(3n-1)-(n-1)(n-2)]$$

$$=\frac{1}{2}[2a(2n+1)+d(8n^2-2)]$$

$$=a(2n+1)+d(4n^2-1)$$

$$=(2n+1)\,[a+(2n-1)d]$$

$$S_{2n}-S_{2n-1}=T_{2n}=a+(2n-1)\,d$$

$$\Rightarrow \quad \frac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=(2n+1)$$

Given,

$$\frac{S_{3n}-S_{n-1}}{S_{2n}-S_{2n-1}}=31$$

$$\Rightarrow \quad n=15$$

16. (1) Let the series have $2n$ terms and the series is a, $a+d$, $a+2d$, …, $a+(2n-1)d$.

According to the given conditions, we have

$$[a+(a+2d)+(a+4d)+\cdots+(a+(2n-2)d)]=24$$

$$\Rightarrow \quad \frac{n}{2}[2a+(n-1)2d]=24$$

or $\quad n[a+(n-1)d]=24 \qquad (1)$

Also,

$$[(a+d)+(a+3d)+\cdots+(a+(2n-1)d)]=30$$

$$\Rightarrow \quad \frac{n}{2}[2(a+d)+(n-1)2d]=30$$

or $\quad n[(a+d)+(n-1)d]=30 \qquad (2)$

Also, the last term exceeds the first by 21/2. Therefore,

$$a+(2n-1)d-a=21/2$$

or $\quad (2n-1)d=21/2 \qquad (3)$

Now, subtracting (1) from (2),

$$nd=6 \qquad (4)$$

Dividing (3) by (4), we get

$$\frac{2n-1}{n}=\frac{21}{12}$$

$$\Rightarrow \quad n=4$$

17. (4) Given, $a_1, a_2, a_3, \ldots$ are terms of A.P.

$$\therefore \quad \frac{a_1+a_2+\cdots+a_p}{a_1+a_2+\cdots+a_q}=\frac{p^2}{q^2}$$

$$\Rightarrow \quad \frac{\frac{p}{2}[2a_1+(p-1)d]}{\frac{q}{2}[2a_1+(q-1)d]}=\frac{p^2}{q^2}$$

or $\quad \frac{2a_1+(p-1)d}{2a_1+(q-1)d}=\frac{p}{q}$

or $[2a_1 + (p-1)d]\, q = [2a_1 + (q-1)d]\, p$

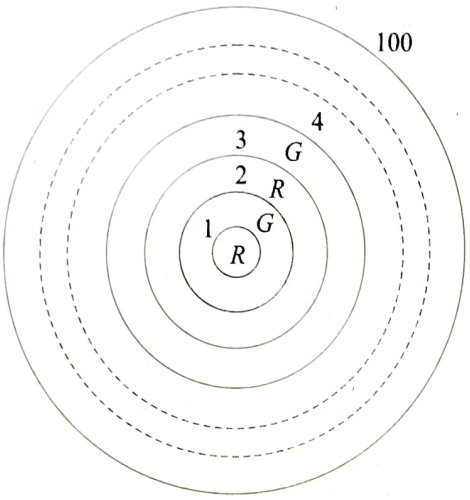
or $2a_1 (q-p) = d[(q-1)\, p - (p-1)q]$

or $2a_1 (q-p) = d(q-p)$

or $2a_1 = d$

$\therefore \quad \dfrac{a_6}{a_{21}} = \dfrac{a_1 + 5d}{a_1 + 20d} = \dfrac{a_1 + 10a_1}{a_1 + 40a_1} = \dfrac{11}{41}$

18. (2)



Area of green regions

$= \pi\left[(r_2^2 - r_1^2) + (r_4^2 - r_3^2) + \cdots + (r_{100}^2 - r_{99}^2)\right]$

$= \pi\,[r_1 + r_2 + r_3 + r_4 + \cdots + r_{100}]$ $(\because\ r_2 - r_1 = r_4 - r_3 = \cdots = r_{100} - r_{99} = 1)$

$= \pi\,[1 + 2 + 3 + \cdots + 100]$

$= 5050\pi$ sq. cm

19. (1) The general term can be given by

$$t_{r+1} = \frac{a_{2n+1-r} - a_{r+1}}{a_{2n+1-r} + a_{r+1}},\ r = 0, 1, 2, \ldots, n-1$$

$$= \frac{a_1 + (2n-r)\,d - \{a_1 + rd\}}{a_1 + (2n-r)\,d + \{a_1 + rd\}}$$

$$= \frac{(n-r)d}{a_1 + nd}$$

Therefore, the required sum is

$$S_n = \sum_{r=0}^{n-1} t_{r+1}$$

$$= \sum_{r=0}^{n-1} \frac{(n-r)\,d}{a_1 + nd}$$

$$= \left[\frac{n + (n-1) + (n-2) + \cdots + 1}{a_1 + nd}\right] d$$

$$= \frac{n(n+1)\,d}{2a_{n+1}}$$

$$= \frac{n(n+1)}{2}\,\frac{a_2 - a_1}{a_{n+1}} \qquad [\because\ d = a_2 - a_1]$$

20. (4) As $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are in A.P., hence

$d = a_2 - a_1 = a_3 - a_2 = \cdots = a_n - a_{n-1}$

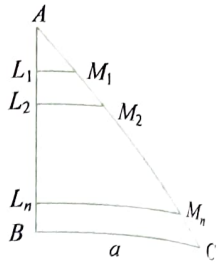
$\sin d[\sec a_1 \sec a_2 + \sec a_2 \sec a_3 + \cdots + \sec a_{n-1} \sec a_n]$

$$= \frac{\sin(a_2 - a_1)}{\cos a_1 \cos a_2} + \frac{\sin(a_3 - a_2)}{\cos a_2 \cos a_3} + \cdots + \frac{\sin(a_n - a_{n-1})}{\cos a_{n-1} \cos a_n}$$

$= (\tan a_2 - \tan a_1) + (\tan a_3 - \tan a_2) + \cdots + (\tan a_n - \tan a_{n-1})$

$= \tan a_n - \tan a_1$

21. (3) $\dfrac{AL_1}{AB} = \dfrac{L_1M_1}{BC}$



$$\Rightarrow \quad \frac{1}{n+1} = \frac{L_1M_1}{a}$$

$$\Rightarrow \quad L_1M_1 = \frac{a}{n+1}$$

$$\frac{AL_2}{AB} = \frac{L_2M_2}{BC}$$

$$\Rightarrow \quad \frac{2}{n+1} = \frac{L_2M_2}{a}$$

$$\Rightarrow \quad L_2M_2 = \frac{2a}{n+1}, \quad \text{etc.}$$

Hence, the required sum is

$$\frac{a}{n+1} + \frac{2a}{n+1} + \frac{3a}{n+1} + \cdots + \frac{na}{n+1}$$

$$= \frac{a}{n+1}\,\frac{n(n+1)}{2}$$

$$= \frac{an}{2}$$

22. (1) Let r be the common ratio of the G.P., a, b, c, d. Then,

$b = ar$, $c = ar^2$ and $d = ar^3$

$\therefore \quad (b-c)^2 + (c-a)^2 + (d-b)^2$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= a^2r^2\,(1-r)^2 + a^2(r^2-1)^2 + a^2r^2(r^2-1)^2$

$= a^2(r^6 - 2r^3 + 1)$

$= a^2(1-r^3)^2$

$= (a - ar^3)^2$

$= (a-d)^2$

23. (1) $\dfrac{t_4}{t_6} = \dfrac{1}{4} \Rightarrow \dfrac{ar^3}{ar^5} = \dfrac{1}{4} \Rightarrow r^2 = 4 \Rightarrow r = \pm 2$

Also, $t_2 + t_5 = 216$

$\Rightarrow \quad ar + ar^4 = 216$

$\Rightarrow \quad a + 8a = 108$

$\Rightarrow \quad a = 12$ (when $r = 2$)

24. (2) x, y, and z are in G.P. Hence,

$y = xr$, $z = xr^2$

Also, x, $2y$, and $3z$ are in A.P. Hence,

$4y = x + 3z$

$\Rightarrow \quad 4xr = x + 3xr^2$

$\Rightarrow \quad 3r^2 - 4r + 1 = 0$

$\Rightarrow \quad (3r-1)(r-1) = 0$

$\Rightarrow \quad r = 1/3 \quad (r \neq 1$ is not possible as x, y, z are distinct)

25. (1) We have

$2b = a + c$

$(c-b)^2 = (b-a)a$

$\Rightarrow \quad (b-a)^2 = (b-a)a \quad [2b = a + c \Rightarrow b - a = c - b]$

$\Rightarrow \quad b = 2a$

$\Rightarrow \quad c = 3a$ [Using $2b = a + c$]

$\Rightarrow \quad a:b:c = 1:2:3$

26. (2) Let the sides of the triangle be a/r, a and ar, with $a > 0$ and $r > 1$. Let α be the smallest angle, so that the largest angle is 2α. Then α is opposite to the side a/r, and 2α is positive to the side ar. Applying sine rule, we get

$$\frac{a/r}{\sin\alpha} = \frac{ar}{\sin 2\alpha}$$

or $\dfrac{\sin 2\alpha}{\sin\alpha} = r^2$

or $2\cos\alpha = r^2$

$\Rightarrow$ $r^2 < 2$

or $r < \sqrt{2}$

Hence, $1 < r < \sqrt{2}$.

27. (3) x, y, and z are in G.P. Hence,

$$y^2 = xz \qquad (1)$$

We have,

$$a^x = b^y = c^z = \lambda \text{ (say)}$$

$\Rightarrow$ $x \log a = y \log b = z \log c = \log \lambda$

$\Rightarrow$ $x = \dfrac{\log\lambda}{\log a}, y = \dfrac{\log\lambda}{\log b}, z = \dfrac{\log\lambda}{\log c}$

Putting the values of x, y and z in (1), we get

$$\left(\frac{\log\lambda}{\log b}\right)^2 = \frac{\log\lambda}{\log a}\frac{\log\lambda}{\log c}$$

$\Rightarrow$ $(\log b)^2 = \log a \log c$

$\Rightarrow$ $\log_b a = \log_c b$

28. (3) For G.P., $t_n = 2^{n-1}$; for A.P., $T_m = 1 + (m-1)3 = 3m - 2$.

They are common if $2^{n-1} = 3m - 2$. For G.P. 100^{th} term is 2^{99}. For A.P. 100^{th} term is $1 + (100 - 1)3 = 298$. Now we must choose value of m such that $3m - 2$ is of the form 2^{n-1}. Hence, $3m - 2 = 1, 2, 4, 8, 16, 32, 64, 128, 256$ for which $m = 1, 4/3, 2, 10/3, 6, 34/3, 22, 130/3, 86$. Hence, possible values of m are $1, 2, 6, 22, 86$. Hence, there are five common terms.

29. (2) $a^2 + b^2$, $ab + bc$, $b^2 + c^2$ are in G.P.

$\Rightarrow$ $(ab + bc)^2 = (a^2 + b^2)(b^2 + c^2)$

or $a^2b^2 + b^2c^2 + 2ab^2c = a^2b^2 + a^2c^2 + b^2c^2 + b^4$

or $b^4 + a^2c^2 - 2ab^2c = 0$

or $(b^2 - ac)^2 = 0$

or $b^2 = ac$

Hence, a, b, and c are in G.P.

30. (4) $a = 5$, $ar^2 = a + 3d$, $ar^4 = a + 15d$

$\therefore$ $5r^2 = 5 + 3d$, $5r^4 = 5 + 15d$

$\Rightarrow$ $r^4 = 1 + 3d$

$\Rightarrow$ $25r^4 = 25 + 75d$

$\Rightarrow$ $(5 + 3d)^2 = 25 + 75d$

$\Rightarrow$ $25 + 30d + 9d^2 = 25 + 75d$

$\Rightarrow$ $9d^2 - 45d = 0$

$\Rightarrow$ $d = 5, 0$

$\Rightarrow$ $T_4 = a + 3d = 5 + 15 = 20$

31. (4) pth, qth, rth terms of A.P. are

$$a + (p-1)d = x \qquad (1)$$

$$a + (q-1)d = xR \qquad (2)$$

$$a + (r-1)d = xR^2 \qquad (3)$$

where R is common ratio of G.P.

Subtracting (2) from (3) and (1) from (2) and then dividing the former by the later, we have

$$\frac{q-r}{p-q} = \frac{xR^2 - xR}{xR - x} = R$$

32. (2) Given that

$$a + (p-1)d = A$$

$$a + (q-1)d = AR$$

$$a + (r-1)d = AR^2$$

$$a + (s-1)d = AR^3$$

where R is common ratio of G.P. Now,

$$p - q = \frac{A - AR}{d}, q - r = R\left(\frac{A - AR}{d}\right),$$

$$r - s = R^2\left(\frac{A - AR}{d}\right)$$

Clearly, $p - q$, $q - r$, $r - s$ are in G.P.

33. (2) Given, $b^2 = ac$ and $x = \dfrac{a+b}{2}$, $y = \dfrac{b+c}{2}$. Therefore,

$$\frac{a}{x} + \frac{c}{y} = \frac{2a}{a+b} + \frac{2c}{b+c}$$

$$= \frac{2a(b+c) + 2c(a+b)}{(a+b)(b+c)}$$

$$= 2\frac{2ac + ab + bc}{ab + ac + b^2 + bc}$$

$$= 2\frac{2ac + ab + bc}{2ac + ab + bc} \qquad (\because b^2 = ac)$$

$$= 2$$

34. (3) $2b = a + c$

a, p, b, q, c are in A.P. Hence,

$$p = \frac{a+b}{2} \text{ and } q = \frac{b+c}{2}$$

Again, a, p', b, q', and c are in G.P. Hence,

$$p' = \sqrt{ab} \text{ and } q' = \sqrt{bc}$$

$\Rightarrow$ $p^2 - q^2 = \dfrac{(a-c)(a+c+2b)}{4}$

$$= (a - c)b$$

$$= ab - bc$$

$$= p'^2 - q'^2$$

35. (2) Degree of x on L.H.S. is

$$1 + 2 + 4 + \cdots + 128 = 1 + 2 + 2^2 + \cdots + 2^7$$

$$= \frac{2^8 - 1}{2 - 1} = 255$$

36. (2) $(1 - p)(1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5) = 1 - p^6$

or $1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5 = \dfrac{1 - p^6}{1 - p}$

or $1 + 3x + 9x^2 + 27x^3 + 81x^4 + 243x^5$

$$= 1 + p + p^2 + p^3 + p^4 + p^5$$

Comparing, we get $p = 3x$ or $p/x = 3$.

37. (2) Given $\dfrac{ar(r^{10} - 1)}{r - 1} = 18 \qquad (1)$

Also $\dfrac{\dfrac{1}{ar}\left(1 - \dfrac{1}{r^{10}}\right)}{1 - \dfrac{1}{r}} = 6$

or $\dfrac{1}{ar^{11}} \cdot \dfrac{(r^{10} - 1)r}{r - 1} = 6$

or $\dfrac{1}{a^2r^{11}} \cdot \dfrac{ar(r^{10} - 1)}{r - 1} = 6 \qquad (2)$

From (1) and (2),

$$\frac{1}{a^2r^{11}} \times 18 = 6$$

or $a^2r^{11} = 3$

Now $P = a^{10}r^{55} = (a^2r^{11})^5 = 3^5 = 243$

38. (1) x, y, z are in G.P.

$\Leftrightarrow \quad y^2 = xz$

$\Leftrightarrow \quad x$ is a factor of y (not possible)

Taking $x = 3, y = 5, z = 7$, we have x, y, z are in A.P. Thus x, y, z may be in A.P. but not in G.P.

39. (2)

$$a = 1 + 10 + 10^2 + \cdots + 10^{54}$$

$$= \frac{10^{55}-1}{10-1} = \frac{10^{55}-1}{10^5-1} \times \frac{10^5-1}{10-1} = bc$$

40. (1) Let a be the first term and r be the common ratio of the given G.P. Then,

$$\alpha = \sum_{n=1}^{100} a_{2n}$$

$$\Rightarrow \quad \alpha = a_2 + a_4 + \cdots + a_{200}$$
$$= ar + ar^3 + \cdots + ar^{199}$$
$$= ar(1 + r^2 + r^4 + \cdots + r^{198})$$

$$\beta = \sum_{n=1}^{100} a_{2n-1}$$

$$\Rightarrow \quad \beta = a_1 + a_3 + \cdots + a_{199}$$
$$= a + ar^2 + \cdots + ar^{198}$$
$$= a(1 + r^2 + \cdots + r^{198})$$

Clearly, $\alpha/\beta = r$.

41. (3) The series is

$$1 + 2 + 2\times 3 + 2^2\times 3 + 2^2\times 3^2 + 2^3\times 3^2 + \cdots \text{ to 20 terms}$$
$$= (1 + 2\times 3 + 2^2\times 3^2 + \cdots \text{ to 10 terms})$$
$$+ (2 + 2^2\times 3 + 2^3\times 3^2 + \cdots \text{ to 10 terms})$$
$$= \frac{1(2^{10}3^{10}-1)}{6-1} + \frac{2(2^{10}3^{10}-1)}{6-1} = \left(\frac{3}{5}\right)(6^{10}-1)$$

42. (1) Let $a = 1$, then $S_1 = 2008$

If $\quad a \neq 1$ then $S = \dfrac{a^{2008}-1}{a-1}$

But $\quad a^{2008} = 2a - 1$; therefore,

$$S_2 = \frac{2(a-1)}{a-1} = 2$$

$\therefore \quad S = S_1 + S_2 = 2010$

43. (3) $S_n = \dfrac{a(r^n-1)}{r-1}$

$$\Rightarrow \quad U_n = \sum_{n=1}^{n} \frac{a(r^n-1)}{r-1} = \frac{a}{r-1}\sum_{n=1}^{n}(r^n-1)$$
$$= \frac{a}{r-1}\{r + r^2 + \cdots + r^n - n\}$$
$$= \frac{a}{r-1}\left\{\frac{r(r^n-1)}{r-1} - n\right\}$$

$$\Rightarrow \quad (r-1)\,U_n = \frac{ar(r^n-1)}{r-1} - an$$

$$\Rightarrow \quad (r-1)\,U_n = rS_n - an$$

$$\Rightarrow \quad rS_n + (1-r)\,U_n = an$$

44. (4) We know that

$$-1 \le \cos x \le 1$$

$$\Rightarrow \quad |\cos x| \le 1$$

But, $x \in S \Rightarrow x \in (0, \pi) \Rightarrow |\cos x| < 1$

Now,

$$8^{1+|\cos x|+\cos^2 x+|\cos^3 x|+\cdots \text{ to } \infty} = 4^3$$

$$\Rightarrow \quad 8^{1/(1-|\cos x|)} = 8^2$$

or $\quad \dfrac{1}{1-|\cos x|} = 2$

or $\quad |\cos x| = \dfrac{1}{2}$

or $\quad \cos x = \pm\dfrac{1}{2}$

or $\quad x = \pi/3, 2\pi/3$

$$\Rightarrow \quad S = \{\pi/3, 2\pi/3\}$$

45. (3) We have

$$1 + (1 + a)b + (1 + a + a^2)b^2 + (1 + a + a^2 + a^3)b^3 + \cdots \infty$$

$$= \sum_{n=1}^{\infty}(1 + a + a^2 + \cdots + a^{n-1})\,b^{n-1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1-a^n}{1-a}\right)b^{n-1}$$

$$= \sum_{n=1}^{\infty}\frac{b^{n-1}}{1-a} - \sum_{n=1}^{\infty}\frac{a^n b^{n-1}}{1-a}$$

$$= \frac{1}{1-a}\sum_{n=1}^{\infty}b^{n-1} - \frac{a}{1-a}\sum_{n=1}^{\infty}(ab)^{n-1}$$

$$= \frac{1}{1-a}[1 + b + b^2 + \cdots \infty] - \frac{a}{1-a}[1 + ab + (ab)^2 + \cdots \infty]$$

$$= \frac{1}{1-a} \times \frac{1}{1-b} - \frac{a}{(1-a)(1-ab)}$$

$$= \frac{1}{(1-ab)(1-b)}$$

46. (1) We have,

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{1/4}{1-1/2} = \frac{1}{2}$$

Hence,

$$0.2^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)} = 0.2^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= \left(\frac{1}{5}\right)^{\log_{\sqrt{5}}\frac{1}{2}}$$

$$= (5^{-1})^{2\log_5\frac{1}{2}}$$

$$= (5)^{-2\log_5\frac{1}{2}}$$

$$= (5)^{\log_5 4} = 4$$

47. (3)

$$x = 9^{\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots} = 9^{\frac{\frac{1}{3}}{1-\frac{1}{3}}} = 9^{\frac{1}{2}} = 3$$

$$y = 4^{\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\cdots} = 4^{\frac{\frac{1}{3}}{1+\frac{1}{3}}} = 4^{\frac{1}{4}} = \sqrt{2}$$

$$z=\sum_{r=1}^{\infty}(1+i)^{-r}=\frac{1}{1+i}+\frac{1}{(1+i)^2}+\frac{1}{(1+i)^3}+\ldots$$

$$=\frac{\frac{1}{1+i}}{1-\frac{1}{1+i}}=\frac{1}{i}=-i$$

Let $\alpha=x+yz=3-i\sqrt{2}$

$\therefore$ $\arg\alpha=-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

48. (3) $x=1-x+x^2-x^3+x^4-x^5+\ldots\infty$

$\therefore$ $x=\frac{1}{1+x}$

$\Rightarrow$ $x^2+x-1=0$

$\Rightarrow$ $x=\frac{-1\pm\sqrt{5}}{2}$

But for infinite G.P., $x=\frac{-1-\sqrt{5}}{2}$ is rejected.

$\therefore$ $x=\frac{-1+\sqrt{5}}{2}=2\sin 18^\circ$

49. (4) We have

$$S=\frac{1}{1-\frac{1}{2}}=2$$

$$S_n=\frac{(1-1/2^n)}{(1-1/2)}=2\left(1-\frac{1}{2^n}\right)=2-\frac{1}{2^{n-1}}$$

$\therefore$ $S-S_n<\frac{1}{1000}$

$\Rightarrow$ $\frac{1}{2^{n-1}}<\frac{1}{1000}$

$\Rightarrow$ $2^{n-1}>1000$

$\Rightarrow$ $n-1\geq 10$

$\Rightarrow$ $n\geq 11$

Hence, the least value of n is 11.

50. (4) Let the series be $21, 21r, 21r^2, \ldots$

Sum, $S=\frac{21}{1-r}$ is a positive integer

also $21r$ is a positive integer

$$S=\frac{(21)(21)}{21-21r}$$

Since, $21r\in N$, $21-21r$ must be an integer. Also, $21r<21$

Hence $21-21r$ may be equal to 1, 3, 7 or 9

i.e., must be a divisor of (21)(21)

Hence, $21-21r=1, 3, 7, 9$

Therefore, $21r=20, 18, 14$ or 12

51. (2) Let a be the first term and r the common ratio of the G.P. Then, the sum is given by

$$\frac{a}{1-r}=57 \qquad (1)$$

Sum of the cubes is 9747. Hence,

$$a^3+a^3r^3+a^3r^6+\cdots=9747$$

$$\Rightarrow \frac{a^3}{1-r^3}=9747 \qquad (2)$$

Dividing the cube of (1) by (2), we get

$$\frac{a^3}{(1-r)^3}\frac{(1-r^3)}{a^3}=\frac{(57)^3}{9747}$$

or $\frac{1-r^3}{(1-r)^3}=19$

or $\frac{1+r+r^2}{(1-r)^2}=19$

or $18r^2-39r+18=0$

or $(3r-2)(6r-9)=0$

or $r=2/3$[$\because r\neq 3/2$, because $0<|r|<1$ for an infinite G.P.]

51. (1) $S_p=\frac{1}{1-r^p}$, $s_p=\frac{1}{1+r^p}$, $S_{2p}=\frac{1}{1-r^{2p}}$

Clearly,

$$S_p+s_p=\frac{2}{1-r^{2p}}=2S_{2p}$$

53. (2) $S=1+2r+3r^2+4r^3+\cdots$

$\therefore$ $rS=r+2r^2+3r^3+4r^4+\cdots$

$\Rightarrow$ $(1-r)S=1+r+r^2+r^3+\cdots$

$$=\frac{1}{1-r}$$

$\Rightarrow$ $S=\frac{1}{(1-r)^2}$

Given, $S=\frac{9}{4}$

$\Rightarrow$ $\frac{1}{(1-r)^2}=\frac{9}{4}$

$\Rightarrow$ $1-r=\pm\frac{2}{3}$

$\Rightarrow$ $r=1/3$ or $5/3$

Hence, $r=1/3$ as $0<|r|<1$.

54. (4) Let

$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\ldots\infty$$

Then

$$\frac{1}{5}S=\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\ldots\infty$$

or $S\left(1-\frac{1}{5}\right)=1+3\left[\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\infty\right]$

or $\frac{4}{5}S=1+3\times\frac{1/5}{1-(1/5)}=1+\frac{3}{4}=\frac{7}{4}$

or $S=\frac{35}{16}$

55. (4) $S=\frac{2}{10}+\frac{4}{10^3}+\frac{6}{10^5}+\frac{8}{10^7}+\ldots$ to ∞ (1)

$$=\frac{\frac{2}{10}}{1-\frac{1}{10^2}}+\frac{2\times\left(\frac{1}{10^2}\right)}{\left(1-\frac{1}{10^2}\right)^2}$$

$$=\frac{20}{99}+\frac{200}{9801}$$

$$=\frac{2180}{9801}$$

56. (4) We have

$$2^{n+10} = 2 \times 2^2 + 3 \times 2^3 + 4 \times 2^4 + \cdots + n \times 2^n$$

or $\quad 2(2^{n+10}) = 2 \times 2^3 + 3 \times 2^4 + \cdots + (n-1) \times 2^n + n \times 2^{n+1}$

Subtracting, we get

$$-2^{n+10} = 2 \times 2^2 + 2^3 + 2^4 + \cdots + 2^n - n \times 2^{n+1}$$

$$= 8 + \frac{8(2^{n-2} - 1)}{2-1} - n \cdot 2^{n+1}$$

$$= 8 + 2^{n+1} - 8 - n \times 2^{n+1} = 2^{n+1} - (n)\, 2^{n+1}$$

$$\Rightarrow \quad 2^{10} = 2n - 2 \Rightarrow n = 513$$

57. (3) $S = \sum_{r=1}^{n} (ar + b)\omega^{r-1}$

$$= (a+b) + (2a+b)\,\omega + (3a+b)\,\omega^2 + \ldots + (na+b)\,\omega^{n-1}$$

$$= b(1 + \omega + \omega^2 + \ldots + \omega^{n-1}) + a(1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1})$$

$$= 0 + a(1 + 2\omega + 2\omega^2 + \cdots + n\omega^{n-1})$$

Now, $S_1 = 1 + 2\omega + 3\omega^2 + \ldots + n\omega^{n-1}$

$\therefore \quad \omega S_1 = \omega + 2\omega^2 + \ldots + (n-1)\,\omega^{n-1} + n\omega^n$

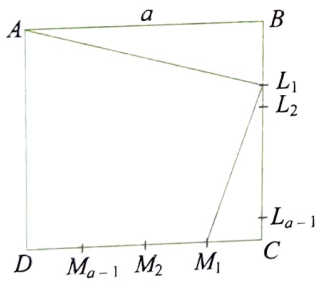
Subtracting, we get

$$S_1(1 - \omega) = 1 + \omega + \omega^2 + \ldots + \omega^{n-1} - n\omega^n = -n \qquad (\text{as } \omega^n = 1)$$

$$\Rightarrow \quad S_1 = \frac{n}{\omega - 1}$$

$$\therefore \quad S = \frac{na}{\omega - 1}$$

58. (3)



$$(AL_1)^2 + (L_1M_1)^2 = (a^2 + 1^2) + \{(a-1)^2 + 1^2\}$$

$$(AL_2)^2 + (L_2M_2)^2 = (a^2 + 2^2) + \{(a-2)^2 + 2^2\}$$

$$\vdots$$

$$(AL_{a-1})^2 + (L_{a-1}M_{a-1})^2 = \{a^2 + (a-1)^2\} + \{1^2 + (a-1)^2\}$$

Therefore, the required sum is

$$(a-1)a^2 + \{1^2 + 2^2 + \cdots + (a-1)^2\} + 2\{1^2 + 2^2 + \cdots + (a-1)^2\}$$

$$= (a-1)a^2 + 3\,\frac{(a-1)a(2a-1)}{6}$$

$$= a(a-1)\left(a + \frac{2a-1}{2}\right)$$

$$= \frac{1}{2}a(a-1)(4a-1)$$

59. (1) Reciprocals of the terms of the series are 2/5, 13/20, 9/10, 23/20, ... or 8/20, 13/20, 18/20, 23/20, ... Its nth term is

$$\frac{8 + (n-1)5}{20} = \frac{5n+3}{20}$$

Therefore, the 15th term is $\frac{20}{78} = \frac{10}{39}$.

60. (3) $a_1, a_2, \ldots, a_n$ are in H.P.

$$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_n} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a_1 + a_2 + a_3 + \cdots + a_n}{a_1}, \frac{a_1 + a_2 + a_3 + \cdots + a_n}{a_2}, \ldots,$$

$$\frac{a_1 + a_2 + a_3 + \cdots + a_n}{a_n} \text{ are in A.P.}$$

$$\Rightarrow \quad 1 + \frac{a_2 + a_3 + \cdots + a_n}{a_1}, 1 + \frac{a_1 + a_3 + \cdots + a_n}{a_2}, \ldots,$$

$$1 + \frac{a_1 + a_2 + \cdots + a_{n-1}}{a_n} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{a_2 + a_3 + \cdots + a_n}{a_1}, \frac{a_1 + a_3 + \cdots + a_n}{a_2}, \ldots, \frac{a_1 + a_2 + \cdots + a_{n-1}}{a_n}$$

are in A.P.

$$\Rightarrow \quad \frac{a_1}{a_2 + a_3 + \cdots + a_n}, \frac{a_2}{a_1 + a_3 + \cdots + a_n}, \ldots, \frac{a_n}{a_1 + a_2 + \cdots a_{n-1}}$$

are in H.P.

61. (3) $f(k) + a_k = \sum_{r=1}^{n} a_r = \lambda$ (say)

$$\therefore \quad f(k) = \lambda - a_k$$

$$\Rightarrow \quad \frac{f(k)}{a_k} = \frac{\lambda}{a_k} - 1$$

$$\therefore \quad \frac{f(1)}{a_1}, \frac{f(2)}{a_2}, \ldots, \frac{f(n)}{a_n} \text{ are in A.P.}$$

So $\quad \frac{a_1}{f(1)}, \frac{a_2}{f(2)}, \ldots, \frac{a_n}{f(n)}$ are in H.P.

62. (4) a, b, and c are in A.P. Hence,

$$2b = a + c \qquad (1)$$

$$\frac{a}{bc} + \frac{2}{b} = \frac{a+2c}{bc} \neq \frac{2}{c}$$

$$\Rightarrow \quad \frac{a}{bc}, \frac{1}{c}, \frac{2}{b} \text{ are not in A.P.}$$

$$\frac{bc}{a} + \frac{b}{2} = \frac{2bc+ab}{2a} \neq c$$

Hence, the given numbers are not in H.P. Again,

$$\frac{a}{bc}\,\frac{2}{b} = \frac{2a}{b^2c} \neq \frac{1}{c^2}$$

Therefore, the given numbers are not in G.P.

63. (2) $\frac{p}{r} + \frac{r}{p} = \frac{p^2 + r^2}{pr} = \frac{(p+r)^2 - 2pr}{pr}$

$$= \frac{\frac{4p^2r^2}{q^2} - 2pr}{pr} \qquad \left[\because p, q, r \text{ are in H.P. } \therefore q = \frac{2pr}{p+r}\right]$$

$$= \frac{4pr}{q^2} - 2 = \frac{4b^2}{ac} - 2$$

$$[\because ap, bq, cr \text{ are in A.P.} \Rightarrow b^2q^2 = acpr]$$

$$= \frac{(a+c)^2}{ac} - 2 \qquad [a, b, c \text{ are in A.P.} \Rightarrow 2b = a + c]$$

$$= \frac{a}{c} + \frac{c}{a}$$

64. (3) $2b = a + c,\ c = \frac{2bd}{b+d}$

$$\Rightarrow \quad 2bd = c(b+d)$$

$\Rightarrow (a+c)d = c(b+d)$ [as $2b = a+c$]

$\Rightarrow ad + cd = bc + cd$

$\Rightarrow bc = ad$

65. (3) $\frac{a_r - a_{r+1}}{a_r\, a_{r+1}} = k$ (constant)

$\Rightarrow \frac{1}{a_{r+1}} - \frac{1}{a_r} = k$

$\Rightarrow \frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}$ are in A.P.

$\Rightarrow a_1, a_2, a_3, \ldots,$ are in H.P.

66. (3) Let $a = 1, b = 2, c = 4$ Then,

$a + b = 3,\ 2b = 4,\ b + c = 6$

$\Rightarrow \frac{1}{4} - \frac{1}{3} = -\frac{1}{12}$ and $\frac{1}{6} - \frac{1}{4} = -\frac{1}{12}$

Hence, $a + b, 2b, b + c$ are in H.P.

67. (2) x is A.M. of a and b, y is G.M. of a and b, z is H.M. of a and b.

$y^2 = xz$

Also given,

$x = 9z$

$\Rightarrow x = 9y^2/x$

$\Rightarrow 9y^2 = x^2$

$\Rightarrow x = 3|y|$

68. (4) $A = \frac{25+n}{2}$, $G = 5\sqrt{n}$, $H = \frac{50n}{25+n}$

As A, G, H are natural numbers, n must be odd perfect square. Now, H will be a natural number, if we take $n = 225$.

69. (3) A.M. $= \frac{100+n}{2}$, G.M. $= 10\sqrt{n}$, H.M. $= \frac{200n}{100+n}$

For A.M. to be integer, n must be even

For G.M. to be integer, n must be a perfect square

So $n = 4k^2$, $HM = \frac{200n}{100+n} = \frac{200 \cdot 4k^2}{100+4k^2} = \frac{200k^2}{25+k^2}$

for H.M. to be integer, k^2 must be divisible by 25 and also $100 < n \le 500$

$k^2 = 50, 75, 100, 125$

Hence, $n = 400$ only

70. (3) $\frac{H_1+2}{H_1-2} + \frac{H_{20}+3}{H_{20}-3} = \frac{\frac{1}{2}+\frac{1}{H_1}}{\frac{1}{2}-\frac{1}{H_1}} + \frac{\frac{1}{3}+\frac{1}{H_{20}}}{\frac{1}{3}-\frac{1}{H_{20}}}$

$= \frac{\frac{1}{2}+\frac{1}{2}+d}{\frac{1}{2}-d-\frac{1}{2}} + \frac{\frac{1}{3}+\frac{1}{3}-d}{\frac{1}{3}+d-\frac{1}{3}}$

$= \frac{1+d}{-d} + \frac{\frac{2}{3}-d}{d}$

$= \frac{\frac{2}{3}-1}{d} - 2$

$= 2 \times 21 - 2$ [as also, $\frac{1}{3} = \frac{1}{2} + 21d$]

$= 40$

71. (2) If t_r be the rth term of the A.P., then

$t_r = S_r - S_{r-1}$

$= cr(r-1) - c(r-1)(r-2)$

$= c(r-1)(r-r+2) = 2c(r-1)$

We have,

$t_1^2 + t_2^2 + \cdots + t_n^2 = 4c^2 (0^2 + 1^2 + 2^2 + \cdots + (n-1)^2)$

$= 4c^2 \frac{(n-1)n(2n-1)}{6}$

$= \frac{2}{3} c^2 n (n-1)(2n-1)$

72. (4) $\Sigma a_i b_i = \Sigma a_i (1 - a_i)$

$= na - \Sigma a_i^2$

$= na - \Sigma (a_i - a + a)^2$

$= na - \Sigma[(a_i - a)^2 + a^2 + 2a(a_i - a)]$

$= na - \Sigma(a_i - a)^2 - \Sigma a^2 - 2a\,\Sigma(a_i - a)$

$\Rightarrow \Sigma a_i b_i + \Sigma(a_i - a)^2 = na - na^2 - 2a(na - na)$

$= na(1-a) = nab$ [$\because \Sigma b_i = \Sigma 1 - \Sigma a_i$; $\therefore nb = n - na$; or $a + b = 1$]

73. (3) Here the successive differences are 2, 4, 8, 16, ... which are in G.P.

$S = 1 + 3 + 7 + 15 + 31 + \cdots + T_{100}$

$= (2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^{100} - 1)$

$= (2 + 2^2 + 2^3 + \cdots + 2^{100}) - 100$

$= 2\left(\frac{2^{100} - 1}{2-1}\right) - 100$

$= 2^{101} - 102$

74. (3) Let the 1025th term fall is in the nth group. Then

$1 + 2 + 4 + \cdots + 2^{n-1} < 1025 \le 1 + 2 + 4 + \cdots + 2^n$

$\Rightarrow 2^n < 1026 \le 2^{n+1}$

$\Rightarrow n = 10$

Hence, 1025th term is 2^{10}.

75. (3) The sum equals $\frac{n(n+1)(n+2)}{6} = 220$

which is true for $n = 10$

76. (1) $S = (1)(2003) + (2)(2002) + (3)(2001) + \cdots + (2003)(1)$

$= \sum_{r=1}^{2003} r(2003 - (r-1))$

$= \sum_{r=1}^{2003} r(2004 - r)$

$= \sum_{r=1}^{2003} 2004r - \sum_{r=1}^{2003} r^2$

$= \frac{2004 \times 2003 \times 2004}{2} - 2003 \times 4007 \times 334$

$= 2003 \times 334 \times (6012 - 4007)$

$= 2003 \times 334 \times 2005$

Hence, $x = 2005$.

77. (4) $2 + 3 + 6 + 11 + 18 + \cdots$

$= (0^2 + 2) + (1^2 + 2) + (2^2 + 2) + (3^2 + 2) + \cdots$

Hence, $t_{50} = 49^2 + 2$.

78. (2) $\sum_{r=0}^{n}(-1)^r(n+2r)^2$

$= n^2 - (n+2)^2 + (n+4)^2 - \ldots - (3n)^2$

$= (n-n-2)(n+n+2) + (n+4-n-6)(n+4+n+6) + \ldots$

$= (-2)[(2n+2) + (2n+10) + (2n+18) + \ldots + n/2 \text{ terms}]$

$= (-2)[2n(n/2) + (2+10+18+\ldots+n/2 \text{ terms})]$

$= (-2)\left[n^2 + \frac{n}{4}\left(4 + \left(\frac{n}{2}-1\right)8\right)\right]$

$= (-2)[n^2 + n + n^2 - 2n]$

$= -4n^2 + 2n$

79. (4) $(1^2 - t_1) + (2^2 - t_2) + \cdots + (n^2 - t_n) = \frac{1}{3}n(n^2-1)$

$\Rightarrow \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 - \{t_1 + t_2 + \cdots + t_n\} = \frac{1}{3}n(n^2-1)$

$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6} - S_n = \frac{1}{3}n(n^2-1)$

$\Rightarrow \quad S_n = \frac{n(n+1)}{6}[2n+1-2(n-1)]$

$= \frac{n(n+1)}{6}[2n+1-2n+2]$

$= \frac{n(n+1)}{2}$

$\Rightarrow \quad S_{n-1} = \frac{n(n-1)}{2}$

$\Rightarrow \quad t_n = S_n - S_{n-1} = n$

80. (2) We know that $1 + 3 + 5 + \cdots + (2k-1) = k^2$. Thus, the given equation can be written as

$$\left(\frac{p+1}{2}\right)^2 + \left(\frac{q+1}{2}\right)^2 = \left(\frac{r+1}{2}\right)^2$$

$\Rightarrow \quad (p+1)^2 + (q+1)^2 = (r+1)^2$

As $p > 6$, $p+1 > 7$, we may take $p+1 = 8$, $q+1 = 6$, $r+1 = 10$. Hence,

$p + q + r = 21$

81. (2) $S_n = 1 + \frac{3}{2} + \frac{5}{3} + \cdots + \frac{99}{50}$

$= (2-1) + \left(2 - \frac{1}{2}\right) + \left(2 - \frac{1}{3}\right) + \cdots + \left(2 - \frac{1}{50}\right)$

$= 100 - H_{50}$

82. (1) Let T_r be the rth term of the given series. Then,

$$T_r = \frac{2r+1}{1^2 + 2^2 + \cdots + r^2} = \frac{6(2r+1)}{(r)(r+1)(2r+1)}$$

$$= 6\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

So, sum is given by

$$\sum_{r=1}^{50} T_r = 6\sum_{r=1}^{50}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= 6\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{50} - \frac{1}{51}\right)\right]$$

$$= 6\left[1 - \frac{1}{51}\right] = \frac{100}{17}$$

83. (2) $S = \frac{4}{19} + \frac{44}{19^2} + \frac{444}{19^3} + \cdots$ (1)

$\Rightarrow \quad \frac{1}{19}S = \frac{4}{19^2} + \frac{44}{19^3} + \cdots$ (2)

Subtracting (2) from (1), we get

$$\frac{18}{19}S = \frac{4}{19} + \frac{40}{19^2} + \frac{400}{19^3} + \cdots$$

$$= \frac{\frac{4}{19}}{1 - \frac{10}{19}} = \frac{4}{9}$$

$\Rightarrow \quad S = \frac{38}{81}$

84. (1) We have

$$\frac{\pi}{4} = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{9} - \frac{1}{11}\right) + \cdots$$

$$= \frac{2}{1\times3} + \frac{2}{5\times7} + \frac{2}{9\times11} + \cdots$$

$\Rightarrow \quad \frac{1}{1\times3} + \frac{1}{5\times7} + \frac{1}{9\times11} + \cdots = \frac{\pi}{8}$

85. (1) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \cdots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$= \frac{\pi^2}{6} - \frac{1}{4}\left(\frac{\pi^2}{6}\right)$$

$$= \frac{\pi^2}{8}$$

86. (3) $T(r) = \frac{r}{1\times3\times5\times\cdots\times(2r+1)}$

$$= \left(\frac{2r+1-1}{2(1\times3\times5\cdots(2r+1))}\right)$$

$$= \frac{1}{2}\left(\frac{1}{1\times3\times5\cdots(2r-1)} - \frac{1}{1\times3\times5\cdots(2r+1)}\right)$$

$$= -\frac{1}{2}[V(r) - V(r-1)]$$

$\Rightarrow \quad \sum_{r=1}^{n} T(r) = -\frac{1}{2}(V(n) - V(0))$

$$= \frac{1}{2}\left(1 - \frac{1}{1\times3\times5\times\cdots\times(2n+1)}\right)$$

$\Rightarrow \quad \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r}{1\times3\times5\times7\times9\times\cdots\times(2r+1)}$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{1\times3\times5\times\ldots\times(2n+1)}\right) = \frac{1}{2}$$

87. (4) $r \times r! = (r+1-1) \times r!$

$= (r+1)! - r!$

$= V(r) - V(r-1)$

$\Rightarrow \sum_{r=1}^{30} r(r!) = V(31) - V(0) = (31)! - 1$

$\Rightarrow 1 + \sum_{r=1}^{30} r(r!) = 31!$

which is divisible by 31, which are prime numbers.

88. (2) $I(2n) = 1^4 + 2^4 + 3^4 + \cdots + (2n-1)^4 + (2n)^4$

$= [(1^4 + 3^4 + 5^4 + \cdots + (2n-1)^4] + 2^4(1^4 + 2^4 + 3^4 + 4^4 + \cdots n^4)$

$= \sum_{r=1}^{n} (2r-1)^4 + 16 \times I(n)$

$\Rightarrow \sum_{r=1}^{n} (2r-1)^4 = I(2n) - 16\, I(n)$

89. (3) Consider the first product,

$$P = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)$$

$$= \frac{\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{\left(1-\frac{1}{3^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{\left(1-\frac{1}{3^4}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\cdots\left(1+\frac{1}{3^{2^n}}\right)}{\left(1-\frac{1}{3}\right)}$$

$$= \frac{1}{\left(1-\frac{1}{3}\right)}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$= \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right)$$

$$\Rightarrow \left(1+\frac{1}{3}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{3^4}\right)\left(1+\frac{1}{3^8}\right)\ldots \text{ infinity}$$

$$= \lim_{n\to\infty} \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{2^{n+1}}\right) = \frac{3}{2}$$

90. (1) Clearly, $\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_{20}}$ will be in A.P. Hence,

$$\frac{1}{x_2} - \frac{1}{x_1} = \frac{1}{x_3} - \frac{1}{x_2} = \ldots = \frac{1}{x_{r+1}} - \frac{1}{x_r} = \cdots = \lambda \text{ (say)}$$

$$\Rightarrow \frac{x_r - x_{r+1}}{x_r x_{r+1}} = \lambda$$

or $x_r x_{r+1} = -\frac{1}{\lambda}(x_{r+1} - x_r)$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = -\frac{1}{\lambda}\sum_{r=1}^{19}(x_{r+1} - x_r) = -\frac{1}{\lambda}(x_{20} - x_1)$$

Now, $\frac{1}{x_{20}} = \frac{1}{x_1} + 19\lambda$

or $\frac{x_1 - x_{20}}{x_1 x_{20}} = 19\lambda$

$$\Rightarrow \sum_{r=1}^{19} x_r x_{r+1} = 19 x_1 x_{20} = 19 \times 4 = 76$$

($\because$ x_1, 2, x_{20} are in G.P., then $x_1 x_{20} = 4$)

91. (2) $T_r = r(-a)^r + (r+1)\, a(-a)^r$

$= r(-a)^r - (r+1)(-a)^{r+1}$

$= v(r) - v(r+1)$

So, $\sum_{r=0}^{n} T_r = \sum_{r=0}^{n} (v(r) - v(r+1))$

$= v_0 - v_{n+1}$

$= -(n+1)(-a)^{n+1}$

92. (1) The general term of the given series is

$$t_n = \frac{x^{2^{n-1}}}{1-x^{2^n}} = \frac{1 + x^{2^{n-1}} - 1}{\left(1+x^{2^{n-1}}\right)\left(1-x^{2^{n-1}}\right)}$$

$$= \frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}$$

Now, $S_n = \sum_{n=1}^{n} t_n$

$$= \left[\left\{\frac{1}{1-x} - \frac{1}{1-x^2}\right\} + \left\{\frac{1}{1-x^2} - \frac{1}{1-x^4}\right\} + \cdots + \left\{\frac{1}{1-x^{2^{n-1}}} - \frac{1}{1-x^{2^n}}\right\}\right]$$

$$= \frac{1}{1-x} - \frac{1}{1-x^{2^n}}$$

Therefore, the sum to infinite terms is

$$\lim_{n\to\infty} S_n = \frac{1}{1-x} - 1$$

$$= \frac{x}{1-x} \qquad \left[\because \lim_{n\to\infty} x^{2^n} = 0, \text{ as } |x| < 1\right]$$

93. (2) $T_r = (-1)^r \frac{r^2 + r + 1}{r!}$

$$= (-1)^r\left[\frac{r}{(r-1)!} + \frac{1}{(r-1)!} + \frac{1}{r!}\right]$$

$$= (-1)^r\left[\frac{1}{(r-2)!} + \frac{1}{(r-1)!} + \frac{1}{(r-1)!} + \frac{1}{r!}\right]$$

$$= \left[\frac{(-1)^r}{r!} + \frac{(-1)^r}{(r-1)!}\right] + \left[\frac{(-1)^r}{(r-1)!} + \frac{(-1)^r}{(r-2)!}\right]$$

$$= \left[\frac{(-1)^r}{r!} - \frac{(-1)^{r-1}}{(r-1)!}\right] - \left[\frac{(-1)^{r-1}}{(r-1)!} - \frac{(-1)^{r-2}}{(r-2)!}\right]$$

$= V(r) - V(r-1)$

$$\therefore \quad \sum_{r=1}^{n} T_r = V(n) - V(0) = \left[\frac{(-1)^n}{n!} - \frac{(-1)^{n-1}}{(n-1)!}\right] - 1$$

Therefore, the sum of 20 terms is

$$\left[\frac{1}{20!} - \frac{-1}{19!}\right] - 1 = \frac{21}{20!} - 1$$

Multiple Correct Answers Type

1. (1), (3), (4)

$$a_1 + a_3 + a_5 = -12$$
$$a + a + 2d + a + 4d = -12 \quad (d > 0)$$
$$a + 2d = -4 \qquad (1)$$
$$a_1 a_3 a_5 = 80$$
$$a(a + 2d)(a + 4d) = 80$$

or $(-4 - 2d)(-4 + 2d) = -20$ or $d = \pm 3$

Since A.P. is increasing, so $d = +3$; $a = -10$. Hence,

$$\left.\begin{aligned} a_1 &= -10;\ a_2 = -7 \\ a_3 &= a + 2d = -10 + 6 = -4 \\ a_5 &= a + 4d = -10 + 12 = 2 \end{aligned}\right\}$$

2. (1), (3), (4)

$$S_n = \frac{n}{2}[2a' + (n-1)d] = a + bn + cn^2$$

$$\Rightarrow \quad na' + \frac{n(n-1)}{2}d = a + bn + cn^2$$

$$\Rightarrow \quad \left(a' - \frac{d}{2}\right)n + \frac{n^2 d}{2} = a + bn + cn^2$$

On comparing,

$$a = 0,\ b = a' - \frac{d}{2},\ c = \frac{d}{2} \quad \Rightarrow d = 2c$$

3. (1), (3)

Let $b = a + p$, $c = a + 2p$, $d = a + 3p$ (where p is common difference). Then,

$$\frac{\frac{1}{a} + \frac{1}{d}}{\frac{1}{b} + \frac{1}{c}} = \frac{\frac{1}{a} + \frac{1}{a+3p}}{\frac{1}{a+p} + \frac{1}{a+2p}}$$

$$= \frac{(a+p)(a+2p)}{a(a+3p)}$$

$$= \frac{a^2 + 3ap + 2p^2}{a^2 + 3ap} > 1$$

$$\therefore \quad \frac{1}{a} + \frac{1}{d} > \frac{1}{b} + \frac{1}{c}$$

$$\left(\frac{1}{b} + \frac{1}{c}\right)(a + d) = \left(\frac{1}{a+p} + \frac{1}{a+2p}\right)(a + a + 3p)$$

$$= \frac{(2a+3p)^2}{a^2 + 3ap + 2p^2}$$

$$= 4 + \frac{p^2}{a^2 + 3ap + 2p^2} > 4$$

4. (1), (2), (3)

Let a, b, c be pth, qth, and rth terms of A.P. Then

$$a = A + (p-1)D,\ b = A + (q-1)D,\ c = A + (r-1)D$$

$$\Rightarrow \quad \frac{r-q}{q-p} = \frac{c-b}{b-a} \text{ is rational number.}$$

Now for 1, 6, 19, $\frac{r-q}{q-p} = \frac{19-6}{6-1}$ is a rational number.

For $\sqrt{2}, \sqrt{50}, \sqrt{98}$, $\frac{r-q}{q-p} = \frac{\sqrt{98} - \sqrt{50}}{\sqrt{50} - \sqrt{2}} = \frac{7\sqrt{2} - 5\sqrt{2}}{5\sqrt{2} - \sqrt{2}}$

$$= \frac{1}{2} \text{ is a rational number.}$$

For log2, log16, log128,

$$\frac{r-q}{q-p} = \frac{\log 128 - \log 16}{\log 16 - \log 2} = \frac{7\log 2 - 4\log 2}{4\log 2 - \log 2} = 1$$

is a rational number.

But for $\sqrt{2}, \sqrt{3}, \sqrt{7}$, $\frac{r-q}{q-p}$ is not a rational number.

5. (2), (3), (4)

Let S_n denote the sum of first n terms of A.P..

According to the question,

$$r = \frac{S_n}{S_{2n} - S_n} = \frac{1}{\frac{S_{2n}}{S_n} - 1} \text{ is independent of } n.$$

Therefore, $\frac{S_{2n}}{S_n}$ is independent of n.

Now, $\frac{S_{2n}}{S_n} = \frac{\frac{2n}{2}(2\alpha + (2n-1)\beta)}{\frac{n}{2}(2\alpha + (n-1)\beta)} = \frac{2(2\alpha - \beta + 2n\beta)}{(2\alpha - \beta + n\beta)}$

This ratio is independent of n if

$$2\alpha - \beta = 0$$
$$\therefore \quad \alpha : \beta = 1 : 2$$
$$\therefore \quad r = \frac{1}{4-1} = \frac{1}{3}$$

α and β are the roots of the equation $ax^2 + bx + c = 0$.

$$\therefore \quad 2\alpha + \alpha = -\frac{b}{a} \text{ and } 2\alpha \times \alpha = \frac{c}{a}$$

Solving these, we get

$$2\left(-\frac{b}{3a}\right)^2 = \frac{c}{a}$$

$$\Rightarrow \quad 2b^2 = 9ac$$

$$1 + r + r^2 + \ldots = \frac{1}{1-r} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

$\alpha = 1$ then $\beta = 2$

$$\therefore \quad \text{Sum of 10 terms A.P.} = \frac{10}{2}(2 \times 1 + (10-1) \times 2) = 100$$

6. (1), (2)

$$a^2 + 2bc,\ b^2 + 2ca,\ c^2 + 2ab$$
$$\therefore \quad ab + bc + ca - a^2 - 2bc,\ ab + bc + ca - b^2 - 2ca,$$
$$ab + bc + ca - c^2 - 2ab \text{ are in A.P.}$$
$$\Rightarrow \quad (c-a)(a-b),\ (a-b)(b-c),\ (b-c)(c-a) \text{ are in A.P.}$$

$$\frac{(c-a)(a-b)}{(a-b)(b-c)(c-a)}, \frac{(a-b)(b-c)}{(a-b)(b-c)(c-a)},$$
$$\frac{(b-c)(c-a)}{(a-b)(b-c)(c-a)} \text{ are in A.P.}$$

$$\Rightarrow \quad b-c,\ c-a,\ a-b \text{ are in H.P.}$$

7. **(2), (3)**

We have

$$\frac{p}{1-1/p} = \frac{9}{2}$$

or $2p^2 - 9p + 9 = 0$

or $p = 3/2, 3$

8. **(1), (2), (3)**

Given that $a = 4$, $T_3 - T_5 = 32/81$. Hence,

$a(r^2 - r^4) = 32/81$

or $r^4 - r^2 + 8/81 = 0$

or $81r^4 - 81r^2 + 8 = 0$

or $(9r^2 - 8)(9r^2 - 1) = 0$

$\therefore$ $r^2 = 8/9, 1/9$

Therefore, the value of r is to be positive since all the terms are positive.

For $r = 1/3$,

$$S_\infty = \frac{a}{1-r} = \frac{4}{1-\frac{1}{3}} = \frac{4\times 3}{2} = 6$$

Similarly, we can find S_∞ when $r = 2\sqrt{2}/3$.

9. **(2), (4)**

Given

$3a_1 + 7a_2 + 3a_3 - 4a_5 = 0$

$\Rightarrow$ $7(a_1 + a_2 + a_3) = 4(a_1 + a_3 + a_5)$

$\Rightarrow$ $7(1 + r + r^2) = 4(1 + r^2 + r^4)$

$\Rightarrow$ $7 = 4(r^2 - r + 1)$

$\Rightarrow$ $4r^2 - 4r + 1 = 4$

$\Rightarrow$ $(2r - 1)^2 = 4$

$\Rightarrow$ $2r - 1 = \pm 2$

$\Rightarrow$ $r = 3/2, -1/2$

10. **(1), (4)**

$$p(x) = \left(\frac{1-x^{2n}}{1-x^2}\right)\left(\frac{1-x}{1-x^n}\right) = \frac{1+x^n}{1+x}$$

As $p(x)$ is a polynomial, $x = -1$ must be a zero of $1 + x^n$. Hence, $1 + (-1)^n = 0$. So, n must be odd.

11. **(1), (2), (3)**

$$a = \frac{n^{64}-1}{n-1}$$

$$= (n + 1)(n^2 + 1)(n^4 + 1)(n^8 + 1)(n^{16} + 1)(n^{32} + 1)$$

12. **(1), (4)**

$x, x^2 + 2, x^3 + 10$ are in G.P. Hence,

$x(x^3 + 10) = (x^2 + 2)^2 = x^4 + 4x^2 + 4$

$\Rightarrow$ $4x^2 - 10x + 4 = 0$

or $2x^2 - 5x + 2 = 0$

or $x = 2, \frac{1}{2}$

The 4th term of G.P. is

$$(x^3 + 10)r = (x^3 + 10)\left(\frac{x^2+2}{x}\right)$$

$$= \begin{cases} 54 & \text{when } x = 2 \\ \frac{729}{16} & \text{when } x = \frac{1}{2} \end{cases}$$

13. **(1), (4)**

$S = 1 + 2x + 3x^2 + 4x^3 + \cdots \infty$

$\Rightarrow$ $xS = x + 2x^2 + 3x^3 + 4x^4 + \cdots \infty$

$\Rightarrow$ $(1 - x)S = 1 + x + x^2 + \cdots \infty = \frac{1}{1-x}$

$\Rightarrow$ $S = \frac{1}{(1-x)^2}$

Now, $S \geq 4$

$\Rightarrow$ $\frac{1}{(1-x)^2} > 4$

$\Rightarrow$ $(x-1)^2 \leq \frac{1}{4}$

$\Rightarrow$ $-\frac{1}{2} \leq x-1 \leq \frac{1}{2}$

$\Rightarrow$ $\frac{1}{2} \leq x \leq \frac{3}{2}$. Also $0 < |x| < 1$

$\Rightarrow$ $\frac{1}{2} \leq x < 1$

14. **(2), (3), (4)**

We have, length of a side of S_n is equal to the length of a diagonal of S_{n+1}. Hence,

Length of a side of $S_n = \sqrt{2}$ (Length of a side of S_{n+1})

or $\frac{\text{Length of a side of } S_{n+1}}{\text{Length of side of } S_n} = \frac{1}{\sqrt{2}}$, for all $n \geq 1$

Hence, sides of $S_1, S_2, \ldots, S_n$ form a G.P. with common ratio $1/\sqrt{2}$ and first term 10.

$\therefore$ Side of $S_n = 10\left(\frac{1}{\sqrt{2}}\right)^{n-1} = \frac{10}{2^{\frac{n-1}{2}}}$

$\Rightarrow$ Area of S_n = (side)$^2 = \left(\frac{10}{2^{\frac{n-1}{2}}}\right)^2 = \frac{100}{2^{n-1}}$

Now, area of $S_n < 1$

$\Rightarrow$ $2^{n-1} > 100$

$\Rightarrow$ $n > 8$

15. **(1), (3)**

a, b, c are in G.P. Hence,

$b^2 = ac$

x is A.M. of a and b. Hence, (1)

$2x = a + b$

y is A.M. of b and c. Hence, (2)

$2y = b + c$ (3)

$\therefore$ $\frac{a}{x} + \frac{c}{y} = a \times \frac{2}{a+b} + c \times \frac{2}{b+c}$ [Using (2) and (3)]

$$= 2\left[\frac{ab + ac + ac + bc}{ab + ac + b^2 + bc}\right]$$

$= 2$ [Using (1)]

Again,

$$\frac{1}{x} + \frac{1}{y} = \frac{2}{a+b} + \frac{2}{b+c}$$

$$= \frac{2(a + c + 2b)}{ab + ac + b^2 + bc}$$

$$= \frac{2(a + c + 2b)}{ab + 2b^2 + bc} \quad (\because b^2 = ac)$$

$$= \frac{2(a + c + 2b)}{b(a + c + 2b)} = \frac{2}{b}$$

16. (1), (3)

Given $a_1 = 2$; $\frac{a_n}{a_{n-1}} = \frac{a_{n-1}}{a_{n-2}}$

Hence, $a_1, a_2, a_3, a_4, a_5, \ldots$ in G.P.

Let $a_2 = x$. Then for $n = 3$, we have

$$\frac{a_3}{a_2} = \frac{a_2}{a_1}$$

$$\Rightarrow \quad a_2^2 = a_1 a_3$$

$$\Rightarrow \quad a_3 = \frac{x^2}{2}$$

i.e. $2, x, \frac{x^2}{2}, \frac{x^3}{4}, \frac{x^4}{8}, \ldots$ with common ratio $r = \frac{x}{2}$

Given $\frac{x^4}{8} \le 162$

$$\Rightarrow \quad x^4 \le 1296 \Rightarrow \quad x \le 6$$

Also x and $\frac{x^4}{8}$ are integers.

So if x is even, then only $\frac{x^4}{8}$ will be an integer.

Hence, possible values of x is 4 and 6. ($x \neq 2$ as terms are distinct)

Hence, possible value of $a_5 = \frac{x^4}{8}$ is $\frac{4^4}{8}, \frac{6^4}{8}$

17. (3), (4)

$$4 = 1 + (n-1)d,\ 16 = 1 + (m-1)d$$

$$\Rightarrow \quad \frac{15}{3} = \frac{m-1}{n-1} \text{ or}$$

$$\frac{n-1}{1} = \frac{m-1}{5} = p = \text{positive integer.}$$

$\therefore \quad n = p+1, m = 5p+1$. So, n, m have infinite pairs of values.

Also, $4 = 1 \cdot r^n,\ 16 = 1 \cdot r^m$

$$\Rightarrow \quad r^{2n} = r^m \quad \Rightarrow 2n = m$$

$$\therefore \quad \frac{m}{2} = \frac{n}{1} = q = \text{Positive integer.}$$

So, m, m have infinite pairs of values.

18. (1), (3), (4)

$$S_\infty = \frac{a}{1-r} = 162$$

$$S_n = \frac{a(1-r^n)}{1-r} = 160$$

Solving these equations, we get

$$1 - r^n = \frac{160}{162} = \frac{80}{81}$$

$$\Rightarrow \quad 1 - \frac{80}{81} = r^n$$

$$\Rightarrow \quad r^n = \frac{1}{81}$$

or $\left(\frac{1}{r}\right)^n = 81$ ...(1)

Now, it is given that $\frac{1}{r}$ is an integer and n is also an integer.

Hence, the relation (1) implies that $\frac{1}{r} = 3, 9$, or 81 so that $n =$ 4, 2 or 1 accordingly.

Therefore, (i) $a = 162\left(1 - \frac{1}{3}\right) = 108$

(ii) $a = 162\left(1 - \frac{1}{9}\right) = 144$

(iii) $a = 162\left(1 - \frac{1}{81}\right) = 160$

19. (1), (2), (3), (4)

We have $b = \frac{2ac}{a+c}$ and $b^2 = -2ac$

$$\therefore \quad b = \frac{-b^2}{a+c}$$

$$\Rightarrow \quad a + b + c = 0$$

$$\Rightarrow \quad 2b = -2a - 2c$$

Therefore, $-2a, b, -2c$ are in A.P..

Also, $a^3 + b^3 + c^3 = 3abc$.

Since $b^2 = -2ac$, $b^4 = 4a^2c^2$.

So, $a^2, b^2, 4c^2$ are in G.P..

Since $a + b + c = 0$, equation $ax^2 + bx + c = 0$ has one root $x = 1$.

So, equation has real roots.

20. (1), (2), (4)

Let the first term be 'a' and common ratio be 'r'.

According to the question, we have

$$a + ar = 1 \text{ and } \frac{a}{1-r} = 2$$

Solving these, we get

$$\frac{1}{(1-r)(1+r)} = 2$$

$$\Rightarrow \quad 1 - r^2 = \frac{1}{2}$$

$$\Rightarrow \quad r^2 = \frac{1}{2}$$

$$\Rightarrow \quad r = \pm\frac{1}{\sqrt{2}}$$

If $r = \frac{1}{\sqrt{2}}$, $a = 2 - \sqrt{2}$

If $r = -\frac{1}{\sqrt{2}}$, $a = 2 + \sqrt{2}$

21. (2), (3)

We have for $0 < \phi < \pi/2$,

$$x = \sum_{n=0}^{\infty} \cos^{2n}\phi = 1 + \cos^2\phi + \cos^4\phi + \ldots \infty = \frac{1}{1 - \cos^2\phi}$$

$$\Rightarrow \quad x = \frac{1}{\sin^2\phi} \qquad \ldots(1)$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\phi = 1 + \sin^2\phi + \sin^4\phi + \ldots \infty = \frac{1}{1 - \sin^2\phi}$$

$$\Rightarrow \quad y = \frac{1}{\cos^2\phi} \qquad \ldots(2)$$

$$z = \sum_{n=0}^{\infty} \cos^{2n}\phi \sin^{2n}\phi$$

$$= 1 + \cos^2\phi \sin^2\phi + \cos^4\phi \sin^4\phi + \ldots \infty$$

$$\Rightarrow \quad z = \frac{1}{1 - \cos^2\phi \sin^2\phi} \qquad \ldots (3)$$

Substituting the values of $\cos^2\phi$ and $\sin^2\phi$ in (3), form (1) and (2), we get

$$z = \frac{1}{1 - \frac{1}{x} \times \frac{1}{y}}$$

$$\Rightarrow \quad z = \frac{xy}{xy - 1}$$

$$\Rightarrow \quad xyz - z = xy$$

$$\Rightarrow \quad xyz = xy + z$$

Also, $x + y + z = \dfrac{1}{\sin^2\phi} + \dfrac{1}{\cos^2\phi} + \dfrac{1}{1 - \cos^2\phi \sin^2\phi}$

$$= \frac{\cos^2\phi (1 - \cos^2\phi \sin^2\phi) + \sin^2\phi (1 - \cos^2\phi \sin^2\phi) + \cos^2\phi \sin^2\phi}{\cos^2\phi \sin^2\phi (1 - \cos^2\phi \sin^2\phi)}$$

$$= \frac{(\sin^2\phi + \cos^2\phi)(1 - \cos^2\phi \sin^2\phi) + \cos^2\phi \sin^2\phi}{\cos^2\phi \sin^2\phi (1 - \cos^2\phi \sin^2\phi)}$$

$$= \frac{1}{\cos^2\phi \sin^2\phi (1 - \cos^2\phi \sin^2\phi)} = xyz$$

22. (1), (3)

$$S = 1 + \frac{1}{(1+3)}(1+2)^2 + \frac{1}{(1+3+5)}(1+2+3)^2 + \frac{1}{(1+3+5+7)}(1+2+3+4)^2 + \ldots$$

The r^{th} term is given by

$$T_r = \frac{1}{r^2}(1+2+\cdots+r)^2 = \frac{1}{r^2}\left\{\frac{r(r+1)}{2}\right\}^2 = \frac{r^2+2r+1}{4}$$

$\therefore \quad T_7 = 16$

and $S_{10} = \sum_{r=1}^{10} T_r$

$$= \frac{1}{4}\left\{\frac{(10)(10+1)(20+1)}{6} + (10)(10+1) + 10\right\}$$

$$= \frac{505}{4}$$

23. (1), (2), (3), (4)

$$an^4 + bn^3 + cn^2 + dn + e$$

$$= 2\sum_{r=1}^{n} r(r+1)(r+2) - \sum_{r=1}^{n} r(r+1)$$

$$= \frac{2}{4}n(n+1)(n+2)(n+3) - \frac{1}{3}n(n+1)(n+2)$$

$$= \frac{1}{6}(3n^4 + 16n^3 + 27n^2 + 14n)$$

24. (1), (2), (3), (4)

Clearly, nth term of the given series is negative or positive accordingly as n is even or odd, respectively.

Case I: When n is even: In this case, the given series is

$$S_n = 1^2 - 2^2 + 3^2 - 4^2 + \cdots + (n-1)^2 - n^2$$

$$= (1^2 - 2^2) + (3^2 - 4^2) + \cdots + ((n-1)^2 - n^2)$$

$$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-1)-(n))(n-1+n)$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n-1) + n)$$

$$= -\frac{n(n+1)}{2} \qquad (1)$$

Case II: When n is odd: In this case, the given series is

$$S_n = (1^2 - 2^2) + (3^2 - 4^2) + \cdots + \{(n-2)^2 - (n-1)^2\} + n^2$$

$$= (1-2)(1+2) + (3-4)(3+4) + \cdots + ((n-2) - (n-1)) \times ((n-2) + (n-1)) + n^2$$

$$= -(1 + 2 + 3 + 4 + \cdots + (n-2) + (n-1)) + n^2$$

$$= -\frac{(n-1)(n-1+1)}{2} + n^2 = \frac{n(n+1)}{2} \qquad (2)$$

$\Rightarrow \quad S_{40} = -820$ [Using (1)]

$S_{51} = 1326$ [Using (2)]

Also, $S_{2n} > S_{2n+2}$ [From (1)]

$S_{2n+1} > S_{2n-1}$ [From (2)]

25. (1), (2), (3)

$$\frac{1}{\sqrt{2}+\sqrt{5}} + \frac{1}{\sqrt{5}+\sqrt{8}} + \frac{1}{\sqrt{8}+\sqrt{11}} + \cdots n \text{ terms}$$

$$= \frac{\sqrt{5}-\sqrt{2}}{3} + \frac{\sqrt{8}-\sqrt{5}}{3} + \ldots + \frac{\sqrt{5+(n-1)3} - \sqrt{2+(n-1)3}}{3}$$

$$= \frac{\sqrt{3n+2} - \sqrt{2}}{3}$$

$$= \frac{3n+2-2}{3(\sqrt{3n+2} + \sqrt{2})}$$

$$= \frac{n}{\sqrt{3n+2} + \sqrt{2}}$$

$$= \frac{n}{\sqrt{2+3n} + \sqrt{2}} < \frac{n}{\sqrt{3n}} < n$$

26. (1), (2), (3)

Last term in nth row is

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1) \qquad (1)$$

As terms in the nth row forms an A.P. with common difference 1, so

First term = Last term $-(n-1)(1)$

$$= \frac{1}{2}n(n+1) - n + 1$$

$$= \frac{1}{2}(n^2 - n + 2) \qquad (2)$$

$$\text{Sum of terms} = \frac{1}{2}n\left[\frac{1}{2}(n^2 - n + 2) + \frac{1}{2}(n^2 + n)\right]$$

$$= \frac{1}{2}n(n^2 + 1) \qquad (3)$$

Now, put $n = 20$ in (1), (2), (3) to get required answers.

27. (1), (2), (4)

$$x + y + z = 3\left(\frac{a+b}{2}\right)$$

$$\Rightarrow \quad 15 = 3\frac{(a+b)}{2}$$

$$\Rightarrow \quad a+b=10 \qquad (1)$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3\left(\frac{1}{a}+\frac{1}{b}\right)}{2}$$

$$\Rightarrow \quad \frac{5}{3}=\frac{3(a+b)}{2ab}=\frac{3\times 10}{2ab}$$

$$\Rightarrow \quad ab=9 \qquad (2)$$

From (1) and (2), $a=9$, $b=1$ or $a=1$ and $b=9$. Hence, G.M. $=\sqrt{ab}=3$, $a+2b=11$ or 19.

28. (1), (2)

$$\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$$

$$=\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{b}+\frac{1}{c}\right)\left(\frac{1}{c}+\frac{1}{b}-\frac{1}{c}\right)$$

$$=\left(\frac{2}{c}-\frac{1}{b}\right)\frac{1}{b}=\frac{2}{bc}-\frac{1}{b^2}$$

Also by eliminating b, we get the given expression $\frac{(a+c)(3a-c)}{4a^2c^2}$.

29. (1), (2), (3)

If p, q, r are in A.P., then pth, qth and rth, terms are equidistant terms which are always in the same series of which they are terms.

30. (1), (2)

$$x^2+9y^2+25z^2=15yz+5zx+3xy$$

$$\Rightarrow \quad (x)^2+(3y)^2+(5z)^2-(x)(3y)-(3y)(5z)-(x)(5z)=0$$

$$\Rightarrow \quad \frac{1}{2}[(x-3y)^2+(3y-5z)^2+(x-5z)^2]=0$$

$$\Rightarrow \quad x-3y=0,\ 3y-5z=0,\ x-5z=0$$

$$x=3y=5z$$

$$\Rightarrow \quad x:y:z=\frac{1}{1}:\frac{1}{3}:\frac{1}{5}$$

Therefore, $1/x$, $1/y$, and $1/z$ are in A.P. and x, y, and z are in H.P.

31. (1), (2), (3)

Since A_1, A_2 are two arithmetic means between a and b, therefore, a, A_1, A_2, b are in A.P. with common difference d given by

$$d=\frac{b-a}{2+1}=\frac{b-a}{3} \qquad \left[\text{using } d=\frac{b-a}{n+1}\right]$$

Now, $A_1=a+d=a+\frac{b-a}{3}=\frac{2a+b}{3}$

and $A_2=a+2d=a+2\left(\frac{b-a}{3}\right)=\frac{a+2b}{3}$

It is given that G_1, G_2 are two geometric means between a and b. Therefore, a, G_1, G_2, b are in G.P. with common ratio r given by

$$r=\left(\frac{b}{a}\right)^{\frac{1}{2+1}}=\left(\frac{b}{a}\right)^{1/3} \qquad \left[\because r=\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]$$

Now, $G_1=ar=a\left(\frac{b}{a}\right)^{1/3}=a^{2/3}\,b^{1/3}$

and $G_2=ar^2=a\left(\frac{b}{a}\right)^{2/3}=a^{1/3}\,b^{2/3}$

It is also given that H_1, H_2 are two harmonic means between a and b, therefore, a, H_1, H_2, b are in H.P. Hence, $1/a$, $1/H_1$, $1/H_2$, $1/b$, are in A.P. with common difference D given by

$$D=\frac{a-b}{(2+1)ab}=\frac{a-b}{3ab} \qquad \left[\because D=\frac{a-b}{(n+1)ab}\right]$$

Now, $\frac{1}{H_1}=\frac{1}{a}+D=\frac{1}{a}+\frac{a-b}{3ab}=\frac{a+2b}{3ab}$

or $H_1=\frac{3ab}{a+2b}$

$$\frac{1}{H_2}=\frac{1}{a}+2D$$

$$=\frac{1}{a}+\frac{2(a-b)}{3ab}$$

$$=\frac{2a+b}{3ab}$$

or $H_2=\frac{3ab}{2a+b}$

We have,

$$A_1H_2=\frac{2a+b}{3}\times\frac{3ab}{2a+b}=ab,$$

$$A_2H_1=\frac{a+2b}{3}\times\frac{3ab}{a+2b}=ab,$$

$$G_1G_2=(a^{2/3}\,b^{1/3})(a^{1/3}\,b^{2/3})=ab$$

$\therefore \quad A_1H_2=A_2H_1=G_1G_2=ab$

32. (1), (3)

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$$

or $\frac{1}{b-a}-\frac{1}{c}=\frac{1}{a}-\frac{1}{b-c}$

or $\frac{c-b+a}{c(b-a)}=\frac{b-c-a}{a(b-c)}$

$\Rightarrow \quad c-b+a=0$ or $\frac{1}{c(b-a)}=\frac{1}{a(c-b)}$

$\Rightarrow \quad b=a+c$ or $bc-ac=ac-ab$

$\Rightarrow \quad b=a+c$ or $b=\frac{2ac}{a+c}$

33. (3), (4)

Let $a=\frac{A}{r}$, $b=A$ and $C=Ar$.

Now b, c, a are in A.P..

$\therefore \quad 2c=a+b$

$\Rightarrow \quad 2Ar=\frac{A}{r}+A$

$\Rightarrow \quad 2r^2-r-1=0$

$\Rightarrow \quad r=1$ or $r=-\frac{1}{2}$

Sine a, b, c are distinct, $r=-\frac{1}{2}$

$\therefore \quad a=-2A,\ b=A,\ c=-\frac{A}{2}$

a, bc, abc are in H.P.

$\therefore \quad \frac{1}{a}, \frac{1}{bc}, \frac{1}{abc}$ are in A.P.

$\Rightarrow \quad \frac{2}{bc}=\frac{1}{abc}+\frac{1}{a}$

$$\Rightarrow \quad -\frac{4}{A^2}=\frac{1}{A^3}-\frac{1}{2A}$$

$$\Rightarrow \quad -\frac{4}{A^2}=\frac{2-A^2}{2A^3}$$

$$\Rightarrow \quad -8A=2-A^2$$

$$\Rightarrow \quad A^2-8A-2=0$$

$$\Rightarrow \quad A=4\pm 3\sqrt{2}$$

34. **(1), (3)**

Given that a, b, c are in A.P.

$$\Rightarrow \quad 2b=a+c \qquad \ldots (1)$$

and a^2, b^2, c^2 are in H.P.

$$\Rightarrow \quad \frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{c^2}-\frac{1}{b^2}$$

$$\Rightarrow \quad \frac{(a-b)(a+b)}{b^2a^2}=\frac{(b-c)(b+c)}{b^2c^2}$$

$$\Rightarrow \quad ac^2+bc^2=a^2b+a^2c \qquad [\because a-b=b-c]$$

$$\Rightarrow \quad ac(c-a)+b(c-a)(c+a)=0$$

$$\Rightarrow \quad (c-a)(ab+bc+ca)=0$$

$$\Rightarrow \quad c-a=0 \text{ or } ab+bc+ca=0$$

For $c=a$, from (1), $a=b=c$

For $(a+c)b+ca=0$, from (1),

$$2b^2+ca=0$$

$$\Rightarrow \quad b^2=a\left(\frac{-c}{2}\right)$$

This implies that $a, b, -c/2$ are in G.P..

35. **(2), (4)**

Let x be the first term and y be the $(2n-1)$th term of A.P., G.P. and H.P. whose nth terms are a, b, c, respectively.

According to the property of A.P., G.P. and H.P.,

x, a, y are in A.P.

x, b, y are in G.P.

x, c, y are in H.P.

Using this information, we get

$$a=\frac{x+y}{2}=\text{A.M.}$$

$$b=\sqrt{xy}=\text{G.M.}$$

and $$c=\frac{2xy}{x+y}=\text{H.M.}$$

We know that A.M, G.M. and H.M. are in G.P..

$$\therefore \quad b^2=ac$$

Also, A.M. $\geq$ G.M. $\geq$ H.M.

$$\Rightarrow \quad a\geq b\geq c$$

36. **(1), (2), (4)**

$$E<1+\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\cdots$$

$$=\quad 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots=2$$

$$E>1+\frac{1}{(2)(3)}+\frac{1}{(3)(4)}+\cdots$$

$$=\quad 1+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots=\frac{3}{2}$$

Linked Comprehension Type

1. (3), 2. (2), 3. (4)

Let the odd integers be $2m+1, 2m+3, 2m+5, \ldots$ and let their number be n. Then,

$$57^2-13^2=(n/2)[2(2m+1)+(n-1)\times 2]$$

$$=n(2m+n)$$

$$=2mn+n^2$$

$$\Rightarrow \quad (n+m)^2-m^2=57^2-13^2$$

$$\Rightarrow \quad m=13 \text{ and } n+m=57$$

$$\Rightarrow \quad n=57-13=44$$

Hence, the required odd integers are 27, 29, 31, ..., 113.

4. (3) a, b, c are in G.P. Hence, a, ar, ar^2 are in G.P. So,

$$\frac{a^2+a^2r^2+a^2r^4}{(a+ar+ar^2)^2}=\frac{t^2}{\alpha^2t^2}=\frac{1}{\alpha^2}$$

$$\alpha^2=\frac{r^2+r+1}{r^2-r+1}$$

Let $\alpha^2=y$.

$$\therefore \quad y=\frac{r^2+r+1}{r^2-r+1}$$

$$(y-1)r^2-r(y+1)+(y-1)=0$$

For real r,

$$(y+1)^2-4(y-1)^2\geq 0$$

$$\Rightarrow \quad \frac{1}{3}\leq y\leq 3$$

But $y\neq 1/3, 1, 3$ $\qquad (\because r\neq 1, -1, 0)$

$$\therefore \quad \frac{1}{3}<y<3 \quad \text{and} \quad y\neq 1$$

$$\alpha^2\in\left(\frac{1}{3},3\right)-\{1\}$$

5. (2) $S=r+\frac{1}{r}$

$$S\in(-\infty,-2)\cup(2,\infty)$$

6. (4) Let $b=ar, c=ar^2$ and $r>0$.

As the sum of two sides is more than the third side, we have

$$r\in\left(\frac{\sqrt{5}-1}{2},\frac{\sqrt{5}+1}{2}\right)-\{1\}$$

$$\Rightarrow \quad r+\frac{1}{r}-1\in(1,\sqrt{5}-1)$$

As $$\alpha^2=\frac{r^2+r+1}{r^2-r+1}=1+\frac{2}{r+\frac{1}{r}-1}$$

$$\therefore \quad \alpha^2\in\left(\frac{\sqrt{5}+3}{2},3\right)$$

7. (4), 8. (2), 9. (4)

Let a be the first term and r the common ratio of the given G.P.

Further, let there be n terms in the given G.P. Then

$$a_1+a_n=66\Rightarrow a+ar^{n-1}=66 \qquad (1)$$

$$a_2\times a_{n-1}=128$$

$$\Rightarrow \quad ar\times ar^{n-2}=128$$

or $$a^2r^{n-1}=128$$

or $$a\times(ar^{n-1})=128 \text{ or } ar^{n-1}=\frac{128}{a}$$

Putting this value of ar^{n-1} in (1), we get

$$a + \frac{128}{a} = 66$$

or $a^2 - 66a + 128 = 0$

or $(a-2)(a-64) = 0$

or $a = 2, 64$

Putting $a = 2$ in (1), we get

$$2 + 2 \times r^{n-1} = 66 \text{ or } r^{n-1} = 32$$

Putting $a = 64$ in (1), we get

$$64 + 64r^{n-1} = 66 \text{ or } r^{n-1} = \frac{1}{32}$$

For an increasing G.P., $r > 1$. Now,

$$S_n = 126$$

$$\Rightarrow \quad 2\left(\frac{r^n - 1}{r-1}\right) = 126$$

or $\dfrac{r^n - 1}{r-1} = 63$

or $\dfrac{r^{n-1} \times r - 1}{r-1} = 63$

or $\dfrac{32r - 1}{r-1} = 63$

or $r = 2$

$\therefore \quad r^{n-1} = 32$

$\Rightarrow \quad 2^{n-1} = 32 = 2^5$

$\Rightarrow \quad n - 1 = 5$

$\Rightarrow \quad n = 6$

For decreasing G.P., $a = 64$ and $r = 1/2$. Hence, the sum of infinite terms is $64/\{1 - (1/2)\} = 128$.

For $a = 2$, $r = 2$, terms are 2, 4, 8, 16, 32, 64. For $a = 64$, $r = 1/2$ terms are 64, 32, 16, 8, 4, 2. Hence, difference is 62.

10. (3), 11. (4), 12. (1)

Let the four integers be $a - d$, a, $a + d$, and $a + 2d$, where a and d are integers and $d > 0$. Now,

$$a + 2d = (a-d)^2 + a^2 + (a+d)^2$$

or $2d^2 - 2d + 3a^2 - a = 0$ (1)

$\therefore \quad d = \dfrac{1}{2}\left[1 \pm \sqrt{1 + 2a - 6a^2}\right]$ (2)

Since d is a positive integer, so

$$1 + 2a - 6a^2 > 0$$

or $6a^2 - 2a - 1 < 0$

or $\dfrac{1 - \sqrt{7}}{6} < a < \dfrac{1 + \sqrt{7}}{6}$

or $a = 0$ ($\because a$ is an integer)

Hence, from (2),

$$d = 1 \text{ or } 0$$

But since $d > 0$,

$\therefore \quad d = 1$

Hence, the four numbers are $-1, 0, 1, 2$.

13. (4), 14. (1) 15. (2)

1, 2, 2, 3, 3, 3, 4, 4, 4, 4, ...

Let us write the terms in the groups as follows: 1, (2, 2), (3, 3, 3), (4, 4, 4, 4), ... consisting of 1, 2, 3, 4, ... terms. Let 2000th term fall in nth group. Then,

$$\frac{(n-1)n}{2} < 2000 \le \frac{n(n+1)}{2}$$

or $n(n-1) < 4000 \le n(n+1)$

Let us consider,

$$n(n-1) < 4000$$

or $n^2 - n - 4000 < 0$

or $n < \dfrac{1 + \sqrt{16001}}{2}$ or $n < 64$

We have

$$n(n+1) \ge 4000 \text{ or } n^2 + n - 4000 \ge 0 \text{ or } n \ge 63$$

That means 2000th term falls in 63rd group, which means that the 2000th term is 63. Now, the total number of terms up to 62nd group is $(62 \times 63)/2 = 1953$. Hence, the sum of first 2000 terms is

$$1^2 + 2^2 + \cdots + 62^2 + 63(2000 - 1953)$$

$$= \frac{62(63)125}{6} + 63 \times 47 = 84336$$

Sum of the remaining terms $= 63 \times 16 = 1008$.

16. (2), 17. (1), 18. (3)

Let numbers in set A be $a - D$, a, $a + D$ and those in set B be $b - d$, b, $b + d$. Now,

$$3a = 3b = 15$$

or $a = b = 5$

Set $A = \{5 - D, 5, 5 + D\}$

Set $B = \{5 - d, 5, 5 + d\}$

where $D = d + 1$

Also,

$$\frac{p}{q} = \frac{5(25 - D^2)}{5(25 - d^2)} = \frac{7}{8}$$

$$\Rightarrow \quad 25(8 - 7) = 8(d+1)^2 - 7d^2$$

$$\Rightarrow \quad d = -17, 1 \text{ but } d > 0 \Rightarrow d = 1$$

So, the numbers in set A are 3, 5, 7 and the numbers in set B are 4, 5, 6.

Now, sum of product of numbers in set A taken two at a time is $3 \times 5 + 3 \times 7 + 5 \times 7 = 71$. The sum of product of numbers in set B taken two at a time is $4 \times 5 + 5 \times 6 + 6 \times 4 = 74$. Also,

$$p = 3 \times 5 \times 7 = 105 \text{ and } q = 4 \times 5 \times 6 = 120$$

$$\Rightarrow \quad q - p = 15$$

19. (3) $G_1 G_2 \cdots G_n = (\sqrt{1 \times 1024})^n = 2^{5n}$

Given, $2^{5n} = 2^{45} \Rightarrow n = 9$

Hence, $r = (1024)^{\frac{1}{9+1}} = 2$

$$\Rightarrow \quad G_1 = 2, r = 2$$

$$\Rightarrow \quad G_1 + G_2 + \cdots + G_9 = \frac{2 \times (2^9 - 1)}{2 - 1} = 1024 - 2 = 1022$$

20. (2) $A_1 + A_2 + A_3 + \cdots + A_{m-1} + A_m = 1025 \times 771$

$$\Rightarrow \quad m\left(\frac{-2 + 1027}{2}\right) = 1025 \times 171$$

or $m = 342$

21. (1) We have

$$A_{171} + A_{172} = -2 + 1027 = 1025$$

or $\dfrac{2A_{171} + 2A_{172}}{2} = 1025$

Also,

$$G_5 = 1 \times 2^5 = 32$$

$\Rightarrow \quad G_5^2 = 1024$

or $\quad G_5^2 + 1 = 1025$

Hence, $2A_{171}, G_5^2 + 1, 2A_{172}$ are in A.P.

22. **(1), 23. (3), 24. (2)**

Let m and $(m+1)$ be the removed numbers from 1, 2, ..., n.

Then, the sum of the remaining numbers is $n(n+1)/2 - (2m+1)$.

From given condition,

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2m+1)}{(n-2)}$$

or $\quad 2n^2 - 103n - 8m + 206 = 0$

Since n and m are integers, so n must be even. Let $n = 2k$. Then,

$$m = \frac{4k^2 + 103(1-k)}{4}$$

Since m is an integer, then $1 - k$ must be divisible by 4. Let $k = 1 + 4t$. Then we get $n = 8t + 2$ and $m = 16t^2 - 95t + 1$. Now,

$1 \le m < n$

$\Rightarrow \quad 1 \le 16t^2 - 95t + 1 < 8t + 2$

Solving, we get $t = 6$. Hence,

$n = 50$ and $m = 7$

Hence, the removed numbers are 7 and 8. Also, sum of all numbers is $50(50 + 1)/2 = 1275$.

25. **(3), 26. (2), 27. (1)**

Let the first term be a and common difference be d for the first A.P. and the first term be b and common difference be e for the second A.P. and let the number of terms be n. Then

$$\frac{a + (n-1)d}{b} = \frac{b + (n-1)e}{a} = 4 \qquad (1)$$

$$\frac{\frac{n}{2}[2a + (n-1)d]}{\frac{n}{2}[2b + (n-1)e]} = 2 \qquad (2)$$

From (1) and (2), we get

$a - 4b + (n-1)d = 0$ (3)

$b - 4a + (n-1)e = 0$ (4)

$2a - 4b + (n-1)d - 2(n-1)e = 0$ (5)

$4 \times (3) + (4)$ gives

$-15b + 4(n-1)\,d + (n-1)e = 0$ (6)

$(4) + 2 \times (5)$ gives

$-7b + 2(n-1)d - 3(n-1)e = 0$ (7)

Further, $15 \times (7) - 7 \times (6)$ gives

$2(n-1)d - 52(n-1)e = 0$

or $\quad d = 26e \qquad (\because n > 1)$

$\therefore \quad d/e = 26$

Putting $d = 26e$ in (3) and solving it with (4), we get

$a = 2(n-1)e,\ b = 7(n-1)e$

Then, the ratio of their n^{th} terms is

$$\frac{2(n-1)e + (n-1)26e}{7(n-1)e + (n-1)e} = \frac{7}{2}$$

28. **(4), 29. (3) 30. (2)**

We have,

$a + b + c = 25$ (1)

$2a = b + 2$ (2)

$c^2 = 18b$ (3)

Eliminating a from (1) and (2), we have

$$b = 16 - \frac{2c}{3}$$

Then from (3),

$$c^2 = 18\left(16 - \frac{2c}{3}\right)$$

or $\quad c^2 + 12c - 18 \times 16 = 0$

or $\quad (c-12)(c+24) = 0$

Now, $c = -24$ is not possible since it does not lie between 2 and 18. Hence, $c = 12$. Then from (3), $b = 8$ and finally from (2), $a = 5$.

Thus, $a = 5$, $b = 8$ and $c = 12$. Hence, $abc = 5 \times 8 \times 12 = 480$.

Also, equation $ax^2 + bx + c = 0$ is $5x^2 + 8x + 12 = 0$, which has imaginary roots.

If a, b, c are roots of the equation $x^3 + qx^2 + rx + s = 0$, then the sum of product of roots taken two at a time is $r = 5 \times 8 + 5 \times 12 + 8 \times 12 = 196$.

Matrix Match Type

1. **a → q; b → r; c → s; d → p.**

We have, $\alpha + \beta = 8$ and $\alpha\beta = 4$.

a. If $\alpha, a_1, a_2, a_3, a_4, a_5, \beta$ are in A.P., then

$$\sum_{i=1}^{5} a_i = 5\left(\frac{\alpha+\beta}{2}\right) = 5\left(\frac{8}{2}\right) = 20$$

b. If $\alpha, a_1, a_2, a_3, a_4, a_5, \beta$ are in G.P., then

$$\sum_{i=1}^{5} \log_2 a_i = \log_2 a_1 + \log_2 a_2 + \log_2 a_3 + \log_2 a_4 + \log_2 a_5$$

$$= \log_2(a_1 a_2 a_3 a_4 a_5)$$

$$= \log_2(\sqrt{\alpha\beta})^5 = 5\log_2\sqrt{4} = 5$$

c. If $\alpha^2, (a_1)^2, (a_2)^2, (a_3)^2, (a_4)^2, (a_5)^2, \beta^2$ are in A .P., then

$$\sum_{i=1}^{5} a_i^2 = 5\left(\frac{\alpha^2+\beta^2}{2}\right)$$

$$= 5\left(\frac{(\alpha+\beta)^2 - 2\alpha\beta}{2}\right)$$

$$= 5\left(\frac{8^2 - 2\times 4}{2}\right) = 140$$

d. If $\alpha, a_1, a_2, a_3, a_4, a_5, \beta$ are in H.P., then

$$\sum_{i=1}^{5} \frac{1}{a_i} = \frac{1}{5}\left(\frac{\frac{1}{\alpha}+\frac{1}{\beta}}{2}\right) = \frac{\alpha+\beta}{10\alpha\beta} = \frac{8}{10\times 4} = \frac{1}{5}$$

$$\therefore \quad \left(\frac{1}{\sum_{i=1}^{5}\frac{1}{a_i}} + 1\right) = 5 + 1 = 6$$

2. **a → q; b → s; c → p; d → r.**

a. $(x+1)(x+3)\ldots(x+19)$

Here, number of factors = 10

$\therefore$ The coefficient of $x^{19} = 1 + 3 + 5 + \ldots + 19$

$$= \frac{10}{2}(1+19) = 100$$

b. Coefficient of x^8 = Sum of product of numbers 1, 3, 5, 7, …, 19 taken two at a time

Let the coefficient be S. Then

$(1+3+5+\ldots+19)^2 = 1^2+3^2+5^2+7^2+\ldots+19^2+2\times S$

$$\therefore \quad S=\frac{\left(\frac{10}{2}(1+19)\right)^2-\sum_{r=1}^{10}(2r-1)^2}{2}$$

$$=\frac{10000-1330}{2}=4335$$

c. Coefficient of x^9 in $(x+1)(x+2)(x+2^2)\ldots(x+2^9)$

$$=1+2+2^2+\ldots+2^9$$

$$=1\left(\frac{2^{10}-1}{2-1}\right)$$

$$=2^{10}-1=1023$$

d. Coefficient of x^8 = Sum of product of numbers $1, 2, 2^2, 2^3, \ldots, 2^9$ taken two at a time

Let the coefficient be S. Then

$$(1+2+2^2+\ldots+2^9)^2$$

$$=(1)^2+(2)^2+(2^2)^2+\ldots+(2^9)^2+2\times S$$

$$\therefore \quad S=\frac{\left(\frac{2^{10}-1}{2-1}\right)^2-\frac{(2^2)^{10}-1}{2^2-1}}{2}$$

$$=\frac{2^{20}-3\cdot 2^{10}+2}{3}$$

$$=\frac{1021\times 2^{10}+2}{3}$$

3. (3) a. Multiply the given expression by 2 and rewrite it as

$$(2x-3y)^2+(3y-4z)^2+(4z-2x)^2=0$$

$$\Rightarrow \quad 2x=3y=4z$$

So, $\frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A.P..

Therefore, x, y, z are in H.P..

b. Since, $21=1^2+2^2+4^2$, we have

$$(1^2+2^2+4^2)(x^2+y^2+z^2)=(x+2y+4z)^2$$

$$\Rightarrow \quad (1^2+2^2+4^2)(x^2+y^2+z^2)-(1\times x+2\times y+4\times z)^2=0$$

$$\Rightarrow \quad (y-2x)^2+(2z-4y)^2+(4z-x)^2=0$$

This is possible only when

$$y-2x=0,\ 2z-4y=0 \text{ and } 4x-z=0$$

or $\frac{y}{x}=2,\frac{z}{y}=2,\frac{z}{x}=4$

$$\Rightarrow \quad \frac{y}{x}=\frac{z}{y}\Rightarrow y^2=zx$$

Therefore, x, y, z are in G.P..

c. We have $216x^3+64y^3+27z^3=72xyz$

$$\therefore \quad (6x+4y+3z)(36x^2+16y^2+9z^2-24xy-12yz-18xz)=0$$

Since $x, y, z > 0$

$$36x^2+16y^2+9z^2-24xy-12yz-18xz=0$$

$$\therefore \quad 72x^2+32y^2+18z^2-48xy-24yz-36xz=0$$

$$\Rightarrow \quad (6x-4y)^2+(4y-3z)^2+(3z-6x)^2=0$$

$$\Rightarrow \quad 6x=4y=3z=\lambda \text{ (say)}$$

$$\Rightarrow \quad x=\frac{\lambda}{6},\ y=\frac{\lambda}{4},\ z=\frac{\lambda}{3}$$

$$\Rightarrow \quad x+z=\frac{\lambda}{6}+\frac{\lambda}{3}=\frac{\lambda}{2}=2y$$

Therefore, x, y, z are in A.P..

d. $ax^2+2px+b=0$ has root -1.

$$\therefore \quad a-2p+b=0$$

$$\Rightarrow \quad a+b=2p$$

So, a, p, b are in A.P..

$ax^2+2qx+b=0$ has equal roots.

$$\therefore \quad 4q^2-4ab=0$$

$$\Rightarrow \quad q^2=ab$$

Therefore, a, q, b are in G.P..

Line $\frac{x}{a}+\frac{y}{b}=\frac{2}{r}$ passes through the point $(1, 1)$.

$$\therefore \quad \frac{1}{a}+\frac{1}{b}=\frac{2}{r}$$

Therefore, a, r, b are in H.P..

Therefore p, q, r are in G.P..

Numerical Value Type

1. (16) Since a, b, c, d are in A.P., we have

$$b-a=c-b=d-c=D \text{ (let common difference)}$$

or $d=a+3D$

$$\Rightarrow \quad a-d=-3D \text{ and } d=b+2D$$

or $b-d=-2D$

Also $c=a+2D$ or $c-a=2D$

$\therefore$ Given equation $2(a-b)+k(b-c)^2+(c-a)^3$

$$=2(a-d)+(b-d)^2+(c-d)^3$$

becomes $-2D+kD^2+(2D)^3=-6D+4D^2-D^3$

$$\Rightarrow \quad 9D^2+(k-4)D+4=0$$

Since D is real, we have $(k-4)^2-4(4)(9)\geq 0$

or $k^2-8k-128\geq 0$ or $(k-16)(k+8)\geq 0$

$\therefore$ $k\in(-\infty,-8]\cup[16,\infty)$

Hence, the smallest positive value of $k = 16$.

2. (9) Given that $T_4=a+3d=6$

and $T_m=a+(m-1)d=18$

$$\Rightarrow \quad (m-4)\,d=12$$

Since terms are integers, we have

$$(m-4)=\pm1,\pm2,\pm3,\pm4,\pm6,\pm12$$

But $m>0$.

$\therefore$ $(m-4)=\pm1,\pm2,\pm3,4,6,12$.

So, m has nine possible values.

3. (3) Let $a, ar, ar^2, ar^3, \ldots$ are in G.P.

Now $ar^4=7!$ and $ar^7=8!$

On dividing, we get $r^3=8$ or $r=2$

Hence, $a\times 2^4=5040$

or $a=\frac{5040}{16}=315$

So 315, 630, 1260, ... are in G.P.

$$\therefore \quad S_3=2205$$

$$\Rightarrow \quad n=3$$

4. (1) Let a be the first term and r be the common ratio of G.P. Then

$$\frac{a(1-r^{201})}{1-r} = 625 \quad (1)$$

Now $\sum_{i=1}^{201} \frac{1}{a_i} = \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{201}}\right)$

$$= \frac{1}{a} + \frac{1}{ar} + \cdots + \frac{1}{ar^{200}}$$

$$= \frac{\frac{1}{a}\left(\left(\frac{1}{r}\right)^{201} - 1\right)}{\left(\frac{1}{r} - 1\right)}$$

$$= \frac{1}{a}\left(\frac{1-r^{201}}{1-r}\right)\frac{1}{r^{200}}$$

$$= \frac{1}{a} \times \frac{625}{a} \times \frac{1}{r^{200}} \quad [\text{from (1)}]$$

$$= \frac{625}{(ar^{100})^2}$$

$$= \frac{625}{(a_{101})^2}$$

$$= \frac{625}{625} = 1$$

5. (6) For the given A.P., we have

$$2(2a+b) = (5a-b) + (a+2b)$$

$$\Rightarrow \quad b = 2a \quad (1)$$

Also for the given G.P., we have

$$(ab+1)^2 = (a-1)^2 (b+1)^2 \quad (2)$$

Putting $b = 2a$ from (1) in (2), we get $a = 0, -2$, or $\frac{1}{4}$

But $a > 0$, so $a = \frac{1}{4}$ and $b = 2a = \frac{1}{2}$

Hence, $(a^{-1} + b^{-1}) = 2 + 4 = 6$.

6. (1) $\frac{a}{1-r_1} = r_1$ and $\frac{a}{1-r_2} = r_2$

Hence, r_1 and r_2 are the roots of $\frac{a}{1-r} = r$

$\Rightarrow \quad r^2 - r + a = 0$

$\Rightarrow \quad r_1 + r_2 = 1$

7. (6) Let $\frac{\alpha}{r}, \alpha, \alpha r$ be the roots. Then

$$\alpha^3 = -216 \quad (1)$$

Again $\frac{\alpha^2}{r} + \alpha^2 r + \alpha^2 = b$

$$\alpha^2\left(1 + r + \frac{1}{r}\right) = b \quad (2)$$

and $\alpha\left(1 + r + \frac{1}{r}\right) = -a \quad (3)$

On dividing (2) by (3), we get

$$\alpha = -\frac{b}{a}$$

or $\alpha^3 = -\frac{b^3}{a^3} \quad (4)$

From (1) and (4), $\left(\frac{b}{a}\right)^3 = 216$

or $\frac{b}{a} = 6$

8. (32) For the G.P. $a, ar, ar^2, \ldots$

$$P_n = a(ar)(ar^2)\ldots(ar^{n-1}) = a^n r^{n(n-1)/2}$$

$$\therefore \quad S = \sum_{n=1}^{\infty} \sqrt[n]{P_n} = \sum_{n=1}^{\infty} ar^{(n-1)/2}$$

Now, $\sum_{n=1}^{\infty} ar^{(n-1)/2} = a[1 + \sqrt{r} + r + r\sqrt{r} + \ldots\infty] = \frac{a}{1-\sqrt{r}}$

Given, $a = 16$ and $r = 1/4$

$$\therefore \quad S = \frac{16}{1-(1/2)} = 32$$

9. (–1) Let $a_1 = a - d$; $a_2 = a$; $a_3 = a + d$

$\therefore \quad 3a = 18$ or $a = 6$

Hence, the number in A.P.

$6 - d, d, 6 + d$

$a_1 + 1, a_2, a_3 + 2$ in G.P.

i.e., $7 - d, 6, 8 + d$ in G.P.

$\therefore \quad 36 = (7 - d)(8 + d)$

$36 = 56 - d - d^2$

$d^2 + d - 20 = 0$

Hence, the sum of all possible common different is –1.

10. (4) Let $\frac{a}{r}, a, ar$ be three terms in G.P.

$\therefore \quad$ Product of terms $= a^3 = -1$ (Given)

$\Rightarrow \quad a = -1$

Now, sum of terms $= \frac{a}{r} + a + ar = \frac{13}{12}$ (Given)

$\Rightarrow \quad \frac{-1}{r} - 1 - r = \frac{13}{12}$

or $12r^2 + 25r + 12 = 0$

or $(3r + 4)(4r + 3) = 0$

or $r = \frac{-4}{3}, \frac{-3}{4}$

But $r \neq \frac{-4}{3}$

$$\therefore \quad |S| = \left|\frac{a}{1-r}\right| = \left|\frac{-1}{1-\left(\frac{-3}{4}\right)}\right| = \left|\frac{-1}{1+\frac{3}{4}}\right|$$

$$= \left|\frac{-4}{7}\right| = \frac{4}{7}$$

11. (27) $369 = \frac{9}{2}[2 + (9-1)d]$

$\Rightarrow \quad 82 = 2 + 8d$

$\Rightarrow \quad d = 10$

Now, $ar^8 = a + 8d$

$\Rightarrow \quad 1 \times r^8 = 1 + 8 \times 10$

$\Rightarrow \quad r^8 = 81$

$\Rightarrow \quad r = \sqrt{3}$

$\Rightarrow \quad ar^{(7-1)} = 1 \times (\sqrt{3})^6 = 27$

12. (60) The ball is dropped from the height, $h = 80$ m.

So, initially it travels the distance of 80 m.

Then rebounds to x% of its previous height.

So, it rebounds by the distance of $\frac{80x}{100}$ m and from the same height it drops again.

It continues like this till it comes to rest.

So, total distance travelled by the ball is

$$320 = 80 + \left(2 \times \frac{80x}{100} + 2 \times \frac{80x^2}{100^2} + \ldots\right)$$

$$\therefore \quad 4 = 1 + \left(\frac{2x}{100} + \frac{2x^2}{100^2} + \ldots\right)$$

$$\Rightarrow \quad 3 = \frac{\frac{2x}{100}}{1 - \frac{x}{100}}$$

$$\Rightarrow \quad 3 = \frac{2x}{100 - x}$$

$$\Rightarrow \quad x = 60$$

13. (1) $$\frac{10^{2013} - 1}{10^{33} - 1} = \frac{(10^{33})^{61} - 1}{10^{33} - 1}$$

$$= ((10^{33})^{60} + (10^{33})^{59} + (10^{33})^{58} + \ldots + (10^{33}) + 1)$$

Thus, digit in unit place is 1

14. (7) 6, a, b in H.P.

$$\Rightarrow \quad \frac{1}{6}, \frac{1}{a}, \frac{1}{b} \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{2}{a} = \frac{1}{6} + \frac{1}{b}$$

$$\text{or} \quad \frac{1}{b} = \frac{2}{a} - \frac{1}{6}$$

$$\text{or} \quad \frac{1}{b} = \frac{12 - a}{6a}$$

$$\text{or} \quad b = \frac{6a}{12 - a}$$

$a \in \{3, 4, 6, 8, 9, 10, 11\}$

15. (9) $10x^3 - nx^2 - 54x - 27 = 0$ has roots in H.P.

put $x = 1/t$

$$27t^3 + 54t^2 + nt - 10 = 0$$

This equation has roots in A.P. Let the roots be $a - d$, a, and $a + d$

$$\therefore \quad 3a = -\frac{54}{27} \text{ or } a = -\frac{2}{3}$$

Also $(a - d)\, a\, (a + d) = \frac{10}{27}$

$$\therefore \quad \frac{2}{3}\left(\frac{4}{9} - d^2\right) = -\frac{10}{27} \text{ or } \left(\frac{4}{9} - d^2\right) = -\frac{5}{9}$$

$$\therefore \quad d^2 = 1 \text{ or } d = \pm 1$$

So, roots are $\frac{1}{3}, -\frac{2}{3}, -\frac{5}{3}$

$$\therefore \quad \frac{n}{27} = \frac{10}{9} - \frac{5}{9} - \frac{2}{9} \text{ or } \frac{n}{27} = \frac{3}{9}$$

or $\quad n = 9$

16. (0) a, b, c are in A.P. $\Rightarrow b = \frac{a + c}{2}$ (1)

b, c, d are in G.P.

$\Rightarrow \quad c^2 = bd$ (2)

and c, d, e are in H.P.

$\Rightarrow \quad d = \frac{2ce}{c + e}$ (3)

Now $c^2 = bd$

or $\quad c^2 = \left(\frac{a + c}{2}\right)\left(\frac{2ce}{c + e}\right)$ [using (1) and (3)]

$\therefore \quad c^2 + ce = ae + ce$

or $\quad c^2 = ae$

Now given $a = 2$ and $e = 18$

$\therefore \quad c^2 = ae$

or $\quad c^2 = 2 \times 18 = 36$

$\Rightarrow \quad c = 6$ or -6

17. (65) $S_k = \frac{k}{1 - \frac{1}{k+1}} = k + 1$

$$\sum_{k=1}^{10} S_k = \sum_{k=1}^{10} (k + 1) = \frac{10 \times 11}{2} + 10 = 65$$

18. (115) Given sum is

$$1\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{20}\right) + 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{20}\right) + 3\left(\frac{1}{3} + \ldots + \frac{1}{20}\right) + 20\left(\frac{1}{20}\right)$$

$$= \frac{1}{1}(1) + \frac{1}{2}(1 + 2) + \frac{1}{3}(1 + 2 + 3) + \ldots + \frac{1}{20}(1 + 2 + 3 + \ldots + 20)$$

$$= \sum_{r=1}^{20} \frac{1}{r} \cdot \frac{r(r + 1)}{2}$$

$$= \frac{1 + 1}{2} + \frac{2 + 1}{2} + \frac{3 + 1}{2} + \ldots + \frac{20 + 1}{2}$$

$$= \frac{20 \times 21}{4} + 10 = 115$$

19. (9) $$\left[\frac{k(k+1)}{2}\right]^2 - \frac{k(k+1)}{2} = 1980$$

$$\text{or} \quad \frac{k(k+1)}{2}\left[\frac{k(k+1)}{2} - 1\right] = 1980$$

or $\quad k(k + 1)(k^2 + k - 2) = 1980 \times 4$

or $\quad (k - 1)\, k(k + 1)(k + 2) = 8 \times 9 \times 10 \times 11$

$\therefore \quad k - 1 = 8 \Rightarrow k = 9$

20. (6) We have $S = 3 + \sum_{n=1}^{\infty} \frac{2n + 3}{3^n} = 3 + \underbrace{\sum_{n=1}^{\infty} \frac{1}{3^{n-1}}}_{S_1} + \underbrace{\sum_{n=1}^{\infty} \frac{2n}{3^n}}_{S_2}$

Now $S_1 = \sum_{n=1}^{\infty} \frac{1}{3^{n-1}} = 1 + \frac{1}{3} + \frac{1}{3^2} + \cdots \infty = \frac{1}{1 - (1/3)} = \frac{3}{2}$

$$S_2 = \sum_{n=1}^{\infty} \frac{2n}{3^n}$$

$\therefore \quad S_2 = \frac{2}{3} + \frac{4}{3^2} + \frac{6}{3^3} + \frac{8}{3^4} + \cdots\infty$

$\frac{S_2}{3} = + \frac{2}{3^2} + \frac{4}{3^3} + \frac{6}{3^4} + \cdots\infty$

Now, $\frac{2S_2}{3} = \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots\infty\right]$ [On subtracting]

$\therefore \quad S_2 = \frac{1}{1-(1/3)} = \frac{3}{2}$

Hence, $S = 3 + \frac{3}{2} + \frac{3}{2} = 6.$

21. (27) Let AGP be
$a, (a + d)r, (a + 2d)r^2, \ldots$

Comparing with given progression, we get
$a = 3, (3 + d)r = 4, (3 + 2d)r^2 = 4$

$\therefore \quad \frac{4}{r} - 3 = d = \left(\frac{4}{r^2} - 3\right)\frac{1}{2}$

$\Rightarrow \quad 3r^2 - 8r + 4 = 0$

$\Rightarrow \quad (3r - 2)(r - 2) = 0$

Since we have infinite AGP, $r = \frac{2}{3}$.

$\therefore \quad (3 + d)\frac{2}{3} = 4$

$\Rightarrow \quad d = 3$

$\therefore$ Sum of infinite AGP is, $S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2} = 27$

22. (25) Let $I = \sum_{r=1}^{50} \frac{r^2}{r^2 + (11-r)^2}$ ...(1)

Writing the series in reverse order, we get

$I = \sum_{r=1}^{50} \frac{(11-r)^2}{r^2 + (11-r)^2}$ ...(2)

Adding (1) and (2), we get

$2I = 50$

$\Rightarrow \quad I = 25$

23. (50) $S = \frac{1}{2^0 + 2^{50}} + \frac{1}{2^1 + 2^{50}} + \ldots + \frac{1}{2^{100} + 2^{50}}$

$S = \frac{1}{2^{100} + 2^{50}} + \frac{1}{2^{99} + 2^{50}} + \ldots + \frac{1}{2^0 + 2^{50}}$

$2S = \frac{1}{2^{50}} + \frac{1}{2^{50}} + \ldots 100 \text{ times}$

$\therefore \quad S = \frac{50}{2^{50}}$

24. (–40) Given that $a_1 + a_3 + \ldots + a_{99} = 1000$

$\Rightarrow \quad 25(a_1 + a_{99}) = 1000$

$\Rightarrow \quad a_1 + a_{99} = 40$

Now, $S = \sum_{r=1}^{50} (-1)^{\frac{r(r+1)}{2}} \cdot a_{2r-1}$

$= -a_1 - a_3 + a_5 + a_7 - a_9 - a_{11} + \ldots + a_{93} + a_{95} - a_{97} - a_{99}$

There are 26 negative terms and 24 positive terms.

$\therefore \quad S = -a_1 - a_{99} = -40$

25. (20) Let $\frac{2n+1}{2n-1} = r$

Then the given series is

$S = r + 3r^2 + 5r^3 + 7r^4 + \ldots + (2n-1)r^n$

$rS = r^2 + 3r^3 + 5r^4 + \ldots + (2n-3)r^n + (2n-1)r^{n+1}$

Subtracting, we get

$(1 - r)S = r + 2r^2 + 2r^3 + \ldots + 2r^n - (2n-1)r^{n+1}$

$\Rightarrow \quad 820(1 - r) = r + \frac{2r^2(1 - r^{n-1})}{1-r} - (2n-1)r^{n+1}$

$\Rightarrow \quad 820(1 - r)^2 = r - r^2 + 2r^2 - 2r^{n+1} - (2n-1)r^{n+1} + (2n-1)r^{n+2}$

$\Rightarrow \quad 820(1 - r)^2 = r + r^2 - (2n+1)r^{n+1} + (2n-1)r^{n+2}$

$\Rightarrow \quad 820(1 - r)^2 = r + r^2 - (2n-1)\left[\frac{2n+1}{2n-1}r^{n+1} - r^{n+2}\right]$

$\Rightarrow \quad 820(1 - r)^2 = r + r^2 - (2n-1)\left[r \cdot r^{n+1} - r^{n+2}\right]$

$\Rightarrow \quad 820(1 - r)^2 = r(1 + r)$

$\Rightarrow \quad 820\left[1 - \frac{2n+1}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[1 + \frac{2n+1}{2n-1}\right]$

$\Rightarrow \quad 820\left[\frac{-2}{2n-1}\right]^2 = \frac{2n+1}{2n-1}\left[\frac{4n}{2n-1}\right]$

$\Rightarrow \quad 820 = n(2n+1)$

$\Rightarrow \quad n = 20$

26. (9) Given $S = \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)}$

$= \sum_{n=1}^{9999} \frac{1}{\left(\sqrt{n} + \sqrt{n+1}\right)\left(\sqrt[4]{n} + \sqrt[4]{n+1}\right)} \left(\frac{\sqrt[4]{n} - \sqrt[4]{n+1}}{\sqrt[4]{n} - \sqrt[4]{n+1}}\right)$

$= \sum_{n=1}^{9999} \left((n+1)^{1/4} - n^{1/4}\right)$

$= \left(\left(2^{\frac{1}{4}} - 1\right) + \left(3^{\frac{1}{4}} - 2^{\frac{1}{4}}\right) + \left(4^{\frac{1}{4}} - 3^{\frac{1}{4}}\right) + \ldots + \left((9999+1)^{\frac{1}{4}} - (9999)^{\frac{1}{4}}\right)\right)$

$= (10^4)^{\frac{1}{4}} - 1$

$= 9$

27. (2) Let $S = \sum_{r=1}^{\infty} \frac{r+2}{2^{r+1} \cdot r \cdot (r+1)}$

$= \sum_{r=1}^{\infty} \frac{2(r+1) - r}{2^{r+1} \cdot r \cdot (r+1)}$

$= \sum_{r=1}^{\infty} \frac{1}{2^{r+1}}\left(\frac{2}{r} - \frac{1}{r+1}\right)$

$= \sum_{r=1}^{\infty} \left(\frac{1}{2^r \cdot r} - \frac{1}{2^{r+1}(r+1)}\right)$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{2^1\times1}-\frac{1}{2^2\times2}\right)+\left(\frac{1}{2^2\times2}-\frac{1}{2^3\times3}\right)\right]$$
$$+\left(\frac{1}{2^3\times3}-\frac{1}{2^4\times4}\right)+\cdots+\left(\frac{1}{2^n\cdot n}-\frac{1}{2^{n+1}\cdot(n+1)}\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{2^{n+1}(n+1)}\right)$$

$\therefore \quad S=\frac{1}{2}$

Hence, $S^{-1}=2$.

28. (85) $S=\frac{7}{2^2\times5^2}+\frac{13}{5^2\times8^2}+\frac{19}{8^2\times11^2}+\ldots 10\ terms$

$$=\sum_{r=1}^{10}\frac{6r+1}{(3r-1)^2(3r+2)^2}$$
$$=\frac{1}{3}\sum_{r=1}^{10}\frac{(3r+2)^2-(3r-1)^2}{(3r-1)^2(3r+2)^2}$$
$$=\frac{1}{3}\sum_{r=1}^{10}\left(\frac{1}{(3r-1)^2}-\frac{1}{(3r+2)^2}\right)$$
$$=\frac{1}{3}\left(\frac{1}{2^2}-\frac{1}{2^{10}}\right)$$
$$=\frac{1}{3}\left(\frac{2^8-1}{2^{10}}\right)$$
$$=\frac{85}{1024}$$

Archives

JEE Main

Single Correct Answer Type

1. (2) Let $S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+\cdots$ (1)

$\therefore \quad \frac{1}{3}S=\frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+\cdots$ (2)

Subtracting (2) from (1), we get

$$S\left(1-\frac{1}{3}\right)=1+\frac{1}{3}+\frac{4}{3^2}+\frac{4}{3^3}+\frac{4}{3^4}+\cdots$$
$$\Rightarrow \quad \frac{2}{3}S=\frac{4}{3}+\frac{4}{3^2}\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)$$
$$=\frac{4}{3}+\frac{4}{3^2}\left(\frac{1}{1-\frac{1}{3}}\right)=\frac{4}{3}+\frac{4}{3^2}\frac{3}{2}=\frac{4}{3}+\frac{2}{3}=\frac{6}{3}$$
$$\Rightarrow \quad S=3$$

2. (3) Till 10^{th} minute, the number of counted notes is 1500.

$\therefore \quad 3000=\frac{n}{2}[2\times148+(n-1)(-2)]$

$=n[148-n+1]$

$\Rightarrow \quad n^2-149n+3000=0$

$\therefore \quad n=125, 24$

Since $n=125$ is not possible, total time required is 24 + 10 = 34 min.

3. (1) Total savings = 200 + 200 + 200 + 240 + 280 + ... to n months

Given, total savings = 11040

$$\Rightarrow \quad 400+\frac{n-2}{2}(400+(n-3)\cdot40)=11040$$
$$\Rightarrow \quad (n-2)(140+20n)=10640$$
$$\Rightarrow \quad 20n^2+100n-280=10640$$
$$\Rightarrow \quad n^2+5n-546=0$$
$$\Rightarrow \quad (n-21)(n+26)=0$$
$$\Rightarrow \quad n=21 \text{ as } n\neq-26$$

4. (2) $T_n=(n-1)^2+(n-1)n+n^2$

$$=\frac{((n-1)^3-n^3)}{(n-1)-n}=n^3-(n-1)^3$$

$T_1=1^3-0^3$

$T^2=2^3-1^3$

$\vdots \quad \vdots \quad \vdots$

$T_{20}=20^3-19^3$

Adding, $S_{20}=20^3-0^3=8000$

5. (4) Given $100(a+99d)=50(a+49d)$

$\therefore \quad 2a+198d=a+49d$

$\therefore \quad a+149d=0$

$\therefore \quad T_{150}=a+149d=0$

6. (3) $0.7+0.77+0.777+\cdots+0.777\cdots7$

$$=\frac{7}{9}[0.9+0.99+0.999+\cdots+0.999\cdots9]$$
$$=\frac{7}{9}[(1-0.1)+(1-0.01)+(1-0.001)+\cdots+(1-0.000\cdots1)]$$
$$=\frac{7}{9}\left[20-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ldots+\frac{1}{10^{20}}\right)\right]$$
$$=\frac{7}{9}\left[20-\frac{1}{10}\cdot\frac{1-\frac{1}{10^{20}}}{1-\frac{1}{10}}\right]=\frac{7}{9}\left[20-\frac{1}{9}\cdot\left(\frac{10^{20}-1}{10^{20}}\right)\right]$$
$$=\frac{7}{81}\left[180-\left(1-\frac{1}{10^{20}}\right)\right]=\frac{7}{81}[179+10^{-20}]$$

7. (3) $S=10^9+2\cdot11^1\cdot10^8+\ldots+10\cdot11^9$

$\therefore \quad \frac{11}{10}\cdot S=11^1\cdot10^8+\ldots+9\cdot11^9+11^{10}$

Subtracting

$$\Rightarrow \quad -\frac{1}{10}\cdot S=10^9+11^1\cdot10^8+11^2\cdot10^7+\ldots+11^9-11^{10}$$
$$\Rightarrow \quad -\frac{1}{10}S=10^9\left(\frac{\left(\frac{11}{10}\right)^{10}-1}{\frac{11}{10}-1}\right)-11^{10}$$
$$\Rightarrow \quad -\frac{1}{10}S=11^{10}-10^{10}-11^{10}$$
$$\Rightarrow \quad S=10^{11}$$
$$\Rightarrow \quad S=100\cdot10^9$$
$$\Rightarrow \quad k=100$$

8. (2) $m=\frac{l+n}{2}$

and l, G_1, G_2, G_3, n are in G.P.

$\therefore \quad r=\left(\frac{n}{l}\right)^{1/4}$

$\therefore \quad G_1 = l\left(\frac{n}{l}\right)^{1/4}; \quad G_2 = l\left(\frac{n}{l}\right)^{1/2}; \quad G_3 = l\left(\frac{n}{l}\right)^{3/4}$

$(G_1)^4 + 2(G_2)^4 + (G_3)^4$

$\therefore \quad = l^4\left(\frac{n}{l}\right) + 2l^4\left(\frac{n}{l}\right)^2 + l^4\left(\frac{n}{l}\right)^3$

$= l^3n + 2l^2n^2 + ln^3$

$= nl(l^2 + 2nl + n^2)$

$= nl(l+n)^2$

$= 4m^2nl$

9. (2) $T_n = \frac{1^3 + 2^3 + 3^3 \ldots + n^3}{1+3+5+\ldots+(2n-1)} = \frac{\frac{n^2(n+1)^2}{4}}{\frac{n}{2}(1+2n-1)}$

$\Rightarrow \quad T_n = \frac{\frac{n^2(n+1)^2}{4}}{n^2}$

$\therefore \quad T_n = \frac{1}{4}(n+1)^2$

$T_n = \frac{1}{4}[n^2 + 2n + 1]$

$S_n = \sum_{n=1}^{n} T_n$

$S_n = \frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6} + n(n+1) + n\right]$

Putting $n = 9$, we get

$S_9 = \frac{1}{4}\left[\frac{9 \times 10 \times 19}{6} + 9 \times 10 + 9\right]$

$= \frac{1}{4}[285 + 90 + 9] = \frac{384}{4} = 96$

10. (1) Given $a + d$, $a + 4d$, $a + 8d$ are in G.P.

$\therefore \quad (a+4d)^2 = (a+d)(a+8d) = a^2 + 9ad + 8d^2$

$\Rightarrow \quad 8d^2 = ad \quad \Rightarrow \quad a = 8d$

$\therefore \quad 9d, 12d, 16d$ are G.P.

Common ratio $r = \frac{12}{9} = \frac{4}{3}$

11. (1) $S_n = \left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + \left(\frac{20}{5}\right)^2 + \left(\frac{24}{5}\right)^2 + \ldots$

$= \frac{8^2}{5^2} + \frac{12^2}{5^2} + \frac{16^2}{5^2} + \frac{20^2}{5^2} + \frac{24^2}{5^2} + \ldots\ldots$

$\therefore \quad T_n = \frac{(4n+4)^2}{5^2}$

$\therefore \quad S_n = \frac{1}{5^2}\sum_{n=1}^{10} 16(n+1)^2$

$= \frac{16}{25}\left[\left(\sum_{n=1}^{11} n^2\right) - 1\right]$

$= \frac{16}{25}\left[\frac{11 \times 12 \times 23}{6} - 1\right]$

$= \frac{16}{25} \times 505 = \frac{16}{5}m$

$\Rightarrow \quad m = 101$

12. (1) We have

$x(x+1) + (x+1)(x+2) + \ldots + (x + \overline{n-1})(x+n) = 10n$

Roots of above equation are consecutive integers.

Let roots be α and $\alpha + 1$.

$\Rightarrow \quad \alpha(\alpha+1) + (\alpha+1)(\alpha+2) + \ldots + (\alpha + \overline{n-1})(\alpha+n) = 10n \quad (1)$

and $(\alpha+1)(\alpha+2) + (\alpha+2)(\alpha+3) + \ldots + (\alpha+n)(\alpha+n+1) = 10n \quad (2)$

Subtracting (2) from (1), we get

$\alpha(\alpha+1) - (\alpha+n)(\alpha+n+1) = 0$

$\Rightarrow \quad \alpha^2 + \alpha - \alpha^2 - (2n+1)\alpha - n(n+1) = 0$

$\Rightarrow \quad \alpha = -\frac{n+1}{2}$

Putting this value in the original equation, i.e.,

$nx^2 + x\sum_{r=1}^{n}(2r-1) + \sum_{r=1}^{n}(r-1)r = 10n$, we get

$n\frac{(n+1)^2}{4} - \frac{(n+1)}{2}n^2 + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = 10n$

$\Rightarrow \quad 3(n+1)^2 - 6n(n+1) + 2(n+1)(2n+1) - 6(n+1) = 120$

$\Rightarrow \quad n^2 = 121$

$\Rightarrow \quad n = 11$

13. (3) $(15a)^2 + (3b)^2 + (5c)^2 - (15a)(5c) - (15a)(3b) - (3b)(5c) = 0$

$\Rightarrow \quad \frac{1}{2}[(15a-3b)^2 + (3b-5c)^2 + (5c-15a)^2] = 0$

$\Rightarrow \quad 15a = 3b = 5c$

$\Rightarrow \quad b = \frac{5c}{3}, a = \frac{c}{3}$

$\Rightarrow \quad a + b = 2c$

Thus, b, c and a are in A.P.

14. (2) $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 3$ (given)

Now $f(x+y) = f(x) + f(y) + xy$

Putting $y = 1$, we get

$f(x+1) = f(x) + f(1) + x = f(x) + x + 3$

Putting $x = 1$, we get $f(2) = f(1) + 1 + 3 = 7$

Putting $x = 2$, we get $f(3) = f(2) + 2 + 3 = 12$ and so on.

Now $S_{10} = 3 + 7 + 12 + \ldots + t_n \quad \ldots(i)$

$S_{10} = \quad 3 + 7 + \ldots + t_{n-1} + t_n \quad \ldots(ii)$

Subtracting (ii) from (i), we get

$t_n = 3 + 4 + 5 + \ldots$ upto n terms

$\therefore \quad t_n = \frac{(n+2)(n+3)}{2} - 3$

$= \frac{n^2 + 5n}{2}$

$S_n = \sum t_n = \sum \frac{n^2 + 5n}{2}$

$= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}\right]$

$= \frac{n(n+1)(n+8)}{6}$

$\therefore \quad S_{10} = 330$

15. (3) Let the given series have n terms, where n is even.

$$\therefore \quad S_n = 1^2 + 2\times 2^2 + 3^2 + 2\times 4^2 + 5^2 + 2\times 6^2 + \ldots$$

$$= (1^2 + 2^2 + 3^2 + \ldots + n^2) + (2^2 + 4^2 + 6^2 + \ldots + n^2)$$

$$= \frac{n(n+1)(2n+1)}{6} + 2^2(1^2 + 2^2 + 3^2 + \ldots + (n/2)^2)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{4\times\frac{n}{2}\times\left(\frac{n}{2}+1\right)(n+1)}{6}$$

$$= \frac{n(n+1)}{6}[2n+1+n+2]$$

$$= \frac{n(n+1)^2}{2}$$

Given that $B = S_{40}$ and $A = S_{20}$

Also, $100\lambda = B - 2A$

$$= \frac{40\times(41)^2}{2} - 2\times\frac{20\times(21)^2}{2}$$

$$= 20\times[(41)^2 - (21)^2] = 20\times 20\times 62$$

$\therefore \quad \lambda = 248$

16. (4) Let the first term of A.P. be 'a' and the common difference be 'd'.

$$\sum_{k=0}^{12} a_{4k+1} = 416$$

$$\Rightarrow \quad a_1 + a_5 + a_9 + \ldots + a_{49} = 416$$

$$\Rightarrow \quad \frac{13}{2}\times(a_1 + a_{49}) = 416$$

$$\Rightarrow \quad a + a + 48d = 64$$

$$\therefore \quad a + 24d = 32 \qquad \ldots(i)$$

$$a_9 + a_{43} = 66$$

$$\Rightarrow \quad a + 8d + a + 42d = 66$$

$$\Rightarrow \quad a + 25d = 33 \qquad \ldots(ii)$$

Solving (i) and (ii), we get

$$d = 1,\ a = 8$$

Now, $140m = a_1^2 + a_2^2 + \ldots + a_{17}^2$

$$= 8^2 + 9^2 + 10^2 + \ldots + 24^2$$

$$= (1^2 + 2^2 + \ldots + 7^2 + 8^2 + \ldots + 24^2) - (1^2 + 2^2 + \ldots + 7^2)$$

$$= \frac{24\times 25\times 49}{6} - \frac{7\times 8\times 15}{6}$$

$$= 4900 - 140$$

$$= 140(35-1)$$

$$= 140\times 34$$

$\therefore \quad m = 34$

JEE ADVANCED

Single Correct Answer Type

1. (4) $a_1, a_2, a_3, \ldots$ are in H.P.

$$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \text{ are in A.P.}$$

$$\Rightarrow \quad \frac{1}{a_n} = \frac{1}{a_1} + (n-1)\,d < 0,$$

where $\dfrac{\frac{1}{25} - \frac{5}{25}}{19} = d = \left(\dfrac{-4}{19\times 25}\right)$

$$\Rightarrow \quad \frac{1}{5} + (n-1)\left(\frac{-4}{19\times 25}\right) < 0$$

$$\text{or} \quad \frac{4(n-1)}{19\times 5} > 1$$

$$\text{or} \quad n - 1 > \frac{19\times 5}{4}$$

$$\text{or} \quad n > \frac{19\times 5}{4} + 1$$

$$\text{or} \quad n \geq 25.$$

2. (3) $$2\sum_{k=1}^{13} \frac{\sin\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

$$= 2\sum \frac{\sin\left\{\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\right\}}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right).\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

$$= 2\sum_{k=1}^{13}\left\{\cot\left(\frac{\pi}{4} + \frac{(k-1)\,\pi}{6}\right) - \cot\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)\right\}$$

$$= 2\left[\cot\left(\frac{\pi}{4}\right) - \cot\left(\frac{\pi}{4} + \frac{13\pi}{6}\right)\right]$$

$$= 2[1 - (2 - \sqrt{3})]$$

$$= 2(\sqrt{3} - 1)$$

3. (2) We have $\log b_2 - \log b_1 = \log(2)$

$$\Rightarrow \quad \frac{b_1}{b_2} = 2$$

$\Rightarrow \quad b_1, b_2 \ldots$ are in are in G.P. with common ratio 2

$$\therefore \quad t = b_1 + 2b_1 + \cdots + 2^{50} b_1 = b_1(2^{51} - 1)$$

$$s = a_1 + a_2 + \cdots + a_{51}$$

$$= \frac{51}{2}(a_1 + a_{51})$$

$$= \frac{51}{2}(b_1 + b_{51})$$

$$= \frac{51}{2} b_1(1 + 2^{50})$$

$$\Rightarrow \quad s - t = b_1\left(\frac{51}{2} + 51\times 2^{49} - 2^{51} + 1\right)$$

$$= b_1\left(\frac{53}{2} + 2^{49}\times 47\right) > 0$$

$$\Rightarrow \quad s > t$$

$$b_{101} = 2^{100} b_1$$

$$a_{101} = a_1 + 100\,d$$

$$= 2(a_1 + 50d) - a_1$$

$$= 2a_{51} - a_1$$

$$= 2b_{51} - b_1$$

$$= (2\times 2^{51} - 1)\,b_1$$

$$= (2^{51} - 1)\,b_1$$

$\therefore \quad b_{101} > a_{101}$

Multiple Correct Answer Type

1. (1), (4)

$$S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$$

$$= \sum_{r=0}^{(n-1)} ((4r+4)^2 + (4r+3)^2 - (4r+2)^2 - (4r+1)^2)$$

$$= \sum_{r=0}^{(n-1)} (2(8r+6) + 2(8r+4))$$

$$= \sum_{r=0}^{(n-1)} (32r + 20)$$

$$= 16(n-1)n + 20n$$

$$= 4n(4n+1)$$

$$= \begin{cases} 1056 \text{ for } n = 8 \\ 1332 \text{ for } n = 9 \end{cases}$$

Matching Column Type

1. (d) – (q), (t)

$$\frac{2ab}{a+b} = 4$$

$\Rightarrow \quad ab = 2a + 2b \qquad (1)$

Now, $q = 10 - a$

and $2q = 5 + b$

$\Rightarrow \quad 20 - 2a = 5 + b$

$\Rightarrow \quad 15 = 2a + b \qquad (2)$

From (1) and (2),

$a(15 - 2a) = 2a + 2(15 - 2a)$

$\Rightarrow \quad 15a - 2a^2 = -2a + 30$

$\Rightarrow \quad 2a^2 - 17a + 30 = 0$

$\Rightarrow \quad a = 6, \frac{5}{2}$ accordingly $q = 4, \frac{15}{2}$

$\Rightarrow \quad |q - a| = 2, 5$

Note: Solutions of the remaining parts are given in their respective chapters.

Numerical Value Type

1. (3) $S_k = \dfrac{\frac{k-1}{k!}}{1 - \frac{1}{k}} = \dfrac{1}{(k-1)!}$

$$\sum_{k=2}^{100} \left| (k^2 - 3k + 1) \frac{1}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{(k-1)^2 - k}{(k-1)!} \right|$$

$$= \sum_{k=2}^{100} \left| \frac{k-1}{(k-2)!} - \frac{k}{(k-1)!} \right|$$

$$= \left| \frac{1}{0!} - \frac{2}{1!} \right| + \left| \frac{2}{1!} - \frac{3}{2!} \right| + \left| \frac{3}{2!} - \frac{4}{3!} \right| + \cdots$$

$$= \frac{2}{1!} - \frac{1}{0!} + \frac{2}{1!} - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + \cdots + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}$$

2. (0) $a_k = 2a_{k-1} - a_{k-2} \Rightarrow a_1, a_2, \ldots, a_{11}$ are in A.P.

$$\therefore \quad \frac{a_1^2 + a_2^2 + \cdots + a_{11}^2}{11} = \frac{11a^2 + 35 \times 11d^2 + 110ad}{11} = 90$$

$\Rightarrow \quad 225 + 35d^2 + 150d = 90$

$35d^2 + 150d + 135 = 0 \quad \Rightarrow \quad -3, -9/7$

Given $a_2 < \frac{27}{2}$, we get $d = -3$ and $d = -9/7$

$$\Rightarrow \quad \frac{a_1 + a_2 + \cdots + a_{11}}{11} = \frac{11}{2}[30 - 10 \times 3] = 0$$

3. (6) $a_1, a_2, a_3, \ldots, a_{100}$ is an A.P.

$$a_1 = 3, S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$$

$$\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}(6 + (5n-1)d)}{\frac{n}{2}(6 + (n-1)d)}$$

$\frac{S_m}{S_n}$ is independent of n of $6 - d = 0 \quad \Rightarrow \quad d = 6.$

4. (5) Clearly, $1 + 2 + 3 + \ldots + n - 2 \le 1224 \le 3 + 4 + \ldots n$

$$\Rightarrow \quad \frac{(n-2)(n-1)}{2} \le 1224 \le \frac{(n-2)}{2}(3+n)$$

$\Rightarrow \quad n^2 - 3n - 2446 \le 0$ and $n^2 + n - 2454 \ge 0$

$\Rightarrow \quad 49 < n < 51 \Rightarrow n = 50$

$$\therefore \quad \frac{n(n+1)}{2} - (2k+1) = 1224$$

or $\quad k = 25$

or $\quad k - 20 = 5$

5. (4) According to the question

$$\frac{b}{a} = \frac{c}{b} = \text{(integer)}$$

$$\Rightarrow \quad b^2 = ac \Rightarrow c = \frac{b^2}{a}$$

Also given $\frac{a+b+c}{3} = b + 2$

$\Rightarrow \quad a + b + c = 3b + 6$

$\Rightarrow \quad a - 2b + c = 6$

$$\Rightarrow \quad a - 2b + \frac{b^2}{a} = 6$$

$$\Rightarrow \quad 1 - \frac{2b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$$

$$\Rightarrow \quad \left(\frac{b}{a} - 1\right)^2 = \frac{6}{a} \qquad \left(\because \frac{b}{a} \text{ is an integer}\right)$$

$\Rightarrow \quad a = 6$ only

$$\Rightarrow \quad \frac{a^2 + a - 14}{a+1} = 4$$

6. (9) Given $\frac{S_7}{S_{11}} = \frac{6}{11}$

$$\Rightarrow \quad \frac{\frac{7}{2}[2a+6d]}{\frac{11}{2}[2a+10d]} = \frac{6}{11}$$

$$\Rightarrow \quad \frac{7(a+3d)}{11(a+5d)} = \frac{6}{11}$$

$$\Rightarrow \quad 7a + 21d = 6a + 30d$$

$$\Rightarrow \quad a = 9d$$

$$\Rightarrow \quad T_7 = a + 6d = 15d$$

Given $130 < T_7 < 140$

$$\Rightarrow \quad 130 < 15d < 140$$

$$\Rightarrow \quad d = 9$$

7. (6) Let the sides be $a - d$, a, $a + d$, such that $d > 0$.

By Pythagoras theorem, we have

$$a^2 + (a-d)^2 = (a+d)^2$$

$$\Rightarrow \quad a = 4d$$

Thus, sides are $3d$, $4d$ and $5d$.

Given that area $= 24 = \frac{1}{2}(3d)(4d)$

$$\Rightarrow \quad d = 2$$

Thus, sides are 6, 8 and 10.

Hence, the smallest side is 6.

8. (3748)

$X = \{1, 6, 11, 16, \ldots, 2018 \text{ terms}\}$

$Y = \{9, 16, 23, \ldots, 2018 \text{ terms}\}$

$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$

For $n(X \cap Y)$, we have to find common terms in two A.P.s.

Common terms will form an A.P. with common difference as L.C.M. of common differences of the given A.P.s.

Common difference of set X is 5 and that of set Y is 7.

So, common difference of A.P. of common terms is $5 \times 7 = 35$.

So, common terms are 16, 51, 86, ...

Now, last term in set X is $1 + (2018 - 1) \times 5 = 10086$

Last term in set Y is $9 + (2018 - 1) \times 7 = 14128$

So, last term of the common term series is

$$16 + (n-1) \times 35 \le 10086, \quad n \le 288.7.$$

So, there will be 288 common terms.

$$\therefore \quad n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$$

$$= 2018 + 2018 - 288$$

$$= 3748$$

Chapter 6

Concept Application Exercises

Exercise 6.1

1. $\frac{1}{b}+\frac{1}{c} \geq 2\left[\frac{1}{b}\frac{1}{c}\right]^{1/2}$

$\frac{1}{a}+\frac{1}{b} \geq 2\left[\frac{1}{a}\frac{1}{b}\right]^{1/2}$

$\frac{1}{a}+\frac{1}{c} \geq 2\left[\frac{1}{a}\frac{1}{c}\right]^{1/2}$

Adding, we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{\sqrt{(bc)}}+\frac{1}{\sqrt{(ca)}}+\frac{1}{\sqrt{(ab)}}$$

2. Put $a = y + z$, $b = z + x$, $c = x + y$, so that $x + y + z = 1$.

As $x = 1 - a, y = 1 - b, z = 1 - c$ and $0 < a < 1, 0 < b < 1, 0 < c < 1$ it follows that $x, y, z > 0$. Now,

$$\frac{a}{1-a}\frac{b}{1-b}\frac{c}{1-c} = \frac{[(y+z)(z+x)(x+y)]}{xyz} \qquad (1)$$

Now, A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{y+z}{2} \geq \sqrt{yx}, \frac{z+x}{2} \geq \sqrt{zx}, \frac{x+y}{2} \geq \sqrt{xy}$$

Multiplying, we get

$$\frac{[(y+z)(z+x)(x+y)]}{(xyz)} \geq 8 \qquad (2)$$

From (1) and (2), we get

$$\frac{a}{1-a}\frac{b}{1-b}\frac{c}{1-c} \geq 8$$

3. Using A.M. $\geq$ G.M., we have

$$\frac{1+x^{2n}}{2} \geq (1 \times x^{2n})^{1/2}$$

or $$\frac{1+x^{2n}}{2} \geq x^n \qquad (1)$$

and $$\frac{1+y^{2m}}{2} \geq (1 \times y^{2m})^{1/2}$$

or $$\frac{1+y^{2m}}{2} \geq y^m \qquad (2)$$

Multiplying (1) and (2), we get

$$\frac{(1+x^{2n})(1+y^{2m})}{4} \geq x^n y^m$$

$$\Rightarrow \quad \frac{x^n y^m}{(1+x^{2n})(1+y^{2m})} \leq \frac{1}{4}$$

4. $\sqrt{a_1 a_2} < \frac{a_1 + a_2}{2}$, $(\because$ G.M. < A.M.)

$$\sqrt{a_1 a_3} < \frac{a_1 + a_3}{2}$$

$$\vdots$$

$$\sqrt{a_{n-1} a_n} < \frac{a_{n-1} + a_n}{2}$$

Adding, we get

$$\sqrt{a_1 a_2} + \sqrt{a_1 a_3} + \cdots < \frac{1}{2}(n-1)(a_1 + a_2 + \ldots a_n)$$

since in all these inequalities in R.H.S. $a_1, a_2, \ldots, a_n$ all occur $n - 1$ times each.

5. Using A.M. $\geq$ G.M., we have

$$2^{\sin x} + 2^{\cos x} \geq 2\sqrt{2^{\sin x}\, 2^{\cos x}}$$

$$= 2\sqrt{2^{\sin x + \cos x}}$$

Now we know that

$$\sin x + \cos x \geq -\sqrt{2}$$

$$\Rightarrow \quad 2^{\sin x} + 2^{\cos x} \geq 2\sqrt{2^{-\sqrt{2}}}$$

Hence, the minimum value of $2^{\sin x} + 2^{\cos x}$ is $2^{1-\frac{1}{\sqrt{2}}}$.

6. $\log_{10}(x^3 + y^3) - \log_{10}(x^2 + y^2 - xy) \leq 2$

$\therefore \quad \log_{10}(x + y)(x^2 + y^2 - xy) - \log_{10}(x^2 + y^2 - xy) \leq 2$

$\therefore \quad \log_{10}(x + y) \leq 2$

$\therefore \quad x + y \leq 100$

Now $$\frac{x+y}{2} \geq (xy)^{1/2}$$

or $$xy \leq \left(\frac{x+y}{2}\right)^2 \leq \left(\frac{100}{2}\right)^2$$

$\therefore \quad xy \leq 2500$

$\therefore \quad xy_{\max.} = 2500$

7. Using A.M. $\geq$ G.M., we get

$$\frac{a^2x^4 + b^2y^4}{2} \geq (a^2x^4b^2y^4)^{\frac{1}{2}}$$

$\therefore \quad \frac{c^6}{2} \geq (a^2x^4b^2y^4)^{\frac{1}{2}}$

$\Rightarrow \quad \frac{c^6}{2} \geq abx^2y^2$

$\Rightarrow \quad xy \leq \frac{c^3}{\sqrt{2ab}}$

Hence, $(xy)_{\max} = \frac{c^3}{\sqrt{2ab}}$

8. $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right) = \frac{1+x+y+xy}{xy} = \frac{9+xy}{xy} = \frac{9}{xy} + 1.$

Now, using A.M. $\geq$ G.M., we have

$$\frac{x+y}{2} \geq \sqrt{xy}$$

$\Rightarrow \quad 4 \geq \sqrt{xy}$

$\Rightarrow \quad 16 \geq xy$

$\Rightarrow \quad \frac{1}{xy} \geq \frac{1}{16}$

$\Rightarrow \quad \frac{9}{xy} + 1 \geq \frac{25}{16}$

Therefore, the minimum value of $\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)$ is $\frac{25}{16}$.

Exercise 6.2

1. We have $\dfrac{1+2+3+4+\ldots+n}{n} > (1\cdot 2\cdot 3\cdot 4\ldots\cdot n)^{1/n}$

$\therefore \quad \dfrac{\frac{n(n+1)}{2}}{n} > (n!)^{1/n}$

$\Rightarrow \quad \left(\dfrac{n+1}{2}\right)^n > n!$

2. Since A.M. > G.M., we have

$$\frac{1}{n}\left(\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\cdots+\frac{a_{n-1}}{a_n}+\frac{a_n}{a_1}\right) > \left(\frac{a_1}{a_2}\frac{a_2}{a_3}\frac{a_3}{a_4}\cdots\frac{a_{n-1}}{a_n}\frac{a_n}{a_1}\right)^{1/n}$$

or $\left(\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\dfrac{a_3}{a_4}+\cdots+\dfrac{a_{n-1}}{a_n}+\dfrac{a_n}{a_1}\right) > n(1)^n$

or $\left(\dfrac{a_1}{a_2}+\dfrac{a_2}{a_3}+\dfrac{a_3}{a_4}+\cdots+\dfrac{a_{n-1}}{a_n}\right) > n$

3. In ΔABC,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Also,

$$\frac{\tan A+\tan B+\tan C}{3} \geq \sqrt[3]{\tan A \tan B \tan C}$$

[$\because$ A.M. $\geq$ G.M.]

or $\tan A \tan B \tan C \geq \sqrt[3]{\tan A \tan B \tan C}$

or $\tan^2 A \tan^2 B \tan^2 C \geq 27$ [cubing both sides]

or $\tan A \tan B \tan C \geq 3\sqrt{3}$

or $\tan A + \tan B + \tan C \geq 3\sqrt{3}$

4. We know that,

$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + b^{n-1})$

$\therefore \quad 3^n - 2^n = 3^{n-1} + 3^{n-2}2 + 3^{n-3}2^2 + \ldots + 2^{n-1}$

Using A.M $\geq$ G.M., we get

$$\frac{3^{n-1}+3^{n-2}\cdot 2+\ldots+2^{n-1}}{n} \geq \left[(3\cdot 3^2\cdot\ldots\cdot 3^{n-1})(2\cdot 2^2\cdot\ldots\cdot 2^{n-1})\right]^{1/n}$$

$$= 3^{\frac{n-1}{2}}\cdot 2^{\frac{n-1}{2}} = 6^{\frac{n-1}{2}}$$

$\Rightarrow \quad 3^n \geq 2^n + n\cdot 6^{\frac{n-1}{2}}$

5. Given, $abcd = 1$

We know that, $AM \geq GM$

$$\therefore \quad \frac{a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd}{10} \geq [a^2\times b^2\times c^2\times d^2\times (ab)\times(ac)\times(ad)\times(bc)\times(bd)\times(cd)]^{1/10}$$

$$= (a^5\times b^5\times c^5\times d^5)^{1/10}$$

$$= (abcd)^{1/2}$$

$$= (1)^{1/2} \qquad [\because abcd = 1]$$

$$\Rightarrow \quad \frac{a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd}{10} \geq 1$$

$\Rightarrow \quad a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd \geq 10$

6. Using A.M. $\geq$ H.M., we have

$$\frac{(x+y)+(y+z)+(z+x)}{3} \geq \frac{3}{\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}}$$

$$\Rightarrow \quad 2(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right) \geq 9$$

$$\Rightarrow \quad 2\left(\frac{x+y+z}{x+y}+\frac{x+y+z}{y+z}+\frac{x+y+z}{z+x}\right) \geq 9$$

$$\Rightarrow \quad 2\left[\left(1+\frac{z}{x+y}\right)+\left(1+\frac{x}{y+z}\right)+\left(1+\frac{y}{x+z}\right)\right] \geq 9$$

$$\Rightarrow \quad 2\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right) \geq 3$$

$$\Rightarrow \quad 2\left(\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}\right) \geq 3 \qquad (\because x+y+z=1)$$

Exercise 6.3

1. $$\frac{[a+a+\cdots a \text{ times}]+[b+b+\cdots b \text{ times}]}{a+b} \geq \left[a^a b^b\right]^{\frac{1}{a+b}} \geq \frac{a+b}{\left(\frac{1}{a}+\frac{1}{a}+\cdots a \text{ times}\right)+\left(\frac{1}{b}+\frac{1}{b}+\cdots b \text{ times}\right)}$$

or $\dfrac{a^2+b^2}{a+b} \geq \left[a^a b^b\right]^{\frac{1}{a+b}} \geq \dfrac{a+b}{1+1}$

or $\left(\dfrac{a^2+b^2}{a+b}\right)^{a+b} \geq a^a b^b \geq \left(\dfrac{a+b}{2}\right)^{a+b}$

or $\left[\dfrac{a^2+b^2}{a+b}\right]^{a+b} > a^a b^b > \left\{\dfrac{a+b}{2}\right\}^{a+b}$

2. Consider p quantities each equal to a and q quantities each equal to b. We know that

A.M. > G.M.

$$\therefore \quad \frac{(a+a+\cdots \text{to } p \text{ terms})+(b+b+\cdots \text{to } q \text{ terms})}{p+q} > [(a\times a\cdots \text{to } p \text{ factors})(b\times b\cdots \text{to } q \text{ factors})]^{\frac{1}{p+q}}$$

$$\Rightarrow \quad \frac{ap+bq}{p+q} > (a^p b^q)^{\frac{1}{p+q}}$$

$$\Rightarrow \quad \left(\frac{ap+bq}{p+q}\right)^{p+q} > (a^p b^q)$$

3. We know that

Weighted A.M. > weighed G.M.

$$\Rightarrow \quad \frac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > \left[(x^{q-r})^p (x^{r-p})^q (x^{p-q})^r\right]^{\frac{1}{p+q+r}}$$

or $\dfrac{px^{q-r}+qx^{r-p}+rx^{p-q}}{p+q+r} > 1$

or $px^{q-r} + qx^{r-p} + rx^{p-q} > p + q + r$

In case either $p = q = r$ or $x = 1$, inequality becomes equality.

4. Using weighted A.M., G.M. inequality, we have

$$\frac{ca + ab + bc}{a + b + c} \geq (a^c b^a c^b)^{\frac{1}{a+b+c}}$$

or $ca + ab + bc \geq (a^c b^a c^b)$

Similarly,

$$\frac{ba + cb + ac}{b + c + a} \geq (a^b b^c c^a)^{\frac{1}{a+b+c}} \text{ or } ba + cb + ac \geq (a^b b^c c^a)$$

$$\frac{aa + bb + cc}{a + b + c} \geq (a^a b^b c^c)^{\frac{1}{a+b+c}} \text{ or } aa + bb + cc \geq (a^a b^b c^c)$$

Adding together, we get

$$(a + b + c)(a + b + c) \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b$$

$$\Rightarrow \quad 1 \geq a^a b^b c^c + a^b b^c c^a + a^c b^a c^b$$

5. Using A.M. $\geq$ G.M., we get

$$\frac{\frac{a^2}{2} + \frac{a^2}{2} + b^2}{3} \geq \left(\frac{a^2}{2} \cdot \frac{a^2}{2} \cdot b^2\right)^{\frac{1}{3}}$$

$$\Rightarrow \quad \frac{4}{3} \geq \frac{(a^2 b)^{\frac{2}{3}}}{2^{\frac{2}{3}}}$$

$$\Rightarrow \quad (a^2 b) \leq \left(\frac{4}{3}\right)^{\frac{3}{2}} \times 2 = \frac{16}{3\sqrt{3}}$$

Therefore, maximum value of $a^2 b$ is $\frac{16}{3\sqrt{3}}$.

6. We have

$$\frac{2\frac{x^2}{2} + 3\frac{y^2}{3} + 4\frac{z^2}{4}}{9} \geq \left(\left(\frac{x^2}{2}\right)^2 \left(\frac{y^2}{3}\right)^3 \left(\frac{z^2}{4}\right)^4\right)^{\frac{1}{9}}$$

$$\Rightarrow \quad \frac{1}{9^9} \geq \frac{x^4 y^6 z^8}{2^2 3^3 4^4}$$

$$\Rightarrow \quad x^4 y^6 z^8 \leq \frac{2^2 3^3 4^4}{9^9}$$

$$\Rightarrow \quad x^2 y^3 z^4 \leq 2^5 \times 3^{-\frac{15}{2}}$$

Exercise 6.4

1. $\frac{a^4 + b^4 + c^4}{3} > \left(\frac{a + b + c}{3}\right)^4$

$$= \left(\frac{a + b + c}{3}\right)\left(\frac{a + b + c}{3}\right)^3$$

Now,

$$\frac{a + b + c}{3} > (abc)^{1/3} \text{ or } \left(\frac{a + b + c}{3}\right)^3 > abc$$

$$\Rightarrow \quad \frac{a^4 + b^4 + c^4}{3} > \left(\frac{a + b + c}{3}\right) abc$$

or $a^4 + b^4 + c^4 > abc(a + b + c)$

2. A.M. of (1/2)th powers < (1/2)th power of A.M.

$$\therefore \quad \frac{(C_1)^{\frac{1}{2}} + (C_2)^{\frac{1}{2}} + \cdots + (C_n)^{\frac{1}{2}}}{n} < \left(\frac{C_1 + C_2 + \cdots + C_n}{n}\right)^{1/2}$$

$$\text{or} \quad \frac{\sqrt{C_1} + \sqrt{C_2} + \cdots + \sqrt{C_n}}{n} < \left(\frac{2^n - 1}{n}\right)^{1/2}$$

$$\text{or} \quad \sqrt{C_1} + \sqrt{C_2} + \cdots + \sqrt{C_n} < \frac{n\sqrt{(2^n - 1)}}{\sqrt{n}}$$

Hence,

$$\sqrt{C_1} + \sqrt{C_2} + \cdots + \sqrt{C_n} < \sqrt{\left[n(2^n - 1)\right]}$$

3. We know that A.M. of mth power > mth power of A.M.

$$\therefore \quad \frac{\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2}{2} > \left[\frac{\left(a + \frac{1}{a}\right) + \left(b + \frac{1}{b}\right)}{2}\right]^2, \text{ here } m = 2$$

$$\text{or} \quad \left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 > \frac{1}{2}\left[(a + b) + \left(\frac{1}{a} + \frac{1}{b}\right)\right]^2 \qquad (1)$$

Also, $\frac{a^{-1} + b^{-1}}{2} > \left(\frac{a + b}{2}\right)^{-1}$

$$\text{or} \quad \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) > \frac{2}{a + b}$$

$$\text{or} \quad \frac{1}{a} + \frac{1}{b} > \frac{4}{a + b}$$

$$\text{or} \quad \frac{1}{a} + \frac{1}{b} > 4$$

Hence, from (1),

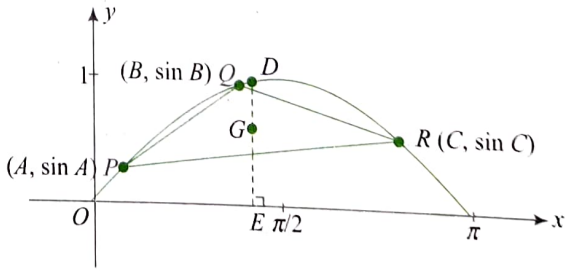
From (i), $\left(a + \frac{1}{a}\right)^2 + \left(b + \frac{1}{b}\right)^2 > \frac{1}{2}(1 + 4)^2 = \frac{25}{2}$

4. In ΔABC, we have $0 < A, B, C < \pi$.

So, consider the graph of function $y = \sin x$ for $x \in (0, \pi)$.

The graph of function is concave downward.

Now, consider three points on the graph $P(A, \sin A)$, $Q(B, \sin B)$ and $R(C, \sin C)$.



Join these points to get triangle PQR.

Now, centroid of the triangle PQR is

$$G\left(\frac{A + B + C}{3}, \frac{\sin A + \sin B + \sin C}{3}\right).$$

Through point G, draw a line parallel to y-axis meeting graph at point D and x-axis at point E.

The abscissa of point D is $\dfrac{A+B+C}{3}$.

So, coordinates of point D are $\left(\dfrac{A+B+C}{3}, \sin\left(\dfrac{A+B+C}{3}\right)\right)$.

Clearly, $DE \geq GE$

$$\therefore \quad \sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \quad \sin\frac{\pi}{3} \geq \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \quad \sin A+\sin B+\sin C \leq \frac{3\sqrt{3}}{2}$$

Exercises

Single Correct Answer Type

1. (2) Using A.M. $\geq$ G.M., we have

$$x^4+y^4 \geq 2x^2y^2 \text{ and } 2x^2y^2+z^2 \geq \sqrt{8}xyz$$

$$\Rightarrow \quad \frac{x^4+y^4+z^2}{xyz} \geq \sqrt{8}$$

2. (4) $a^2+b^2=k^2$

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 = a^2+b^2+4+\frac{1}{a^2}+\frac{1}{b^2}$$

$$= k^2+4+\frac{k^2}{a^2b^2}$$

$$\geq k^2+4+\frac{k^2}{\left(\frac{a^2+b^2}{2}\right)^2}$$

$$= k^2+4+\frac{4}{k^2}$$

3. (2) Applying A.M. $\geq$ G.M. in $6\tan^2\phi$, $54\cot^2\phi$, 18, we get

$$\frac{6\tan^2\phi+54\cot^2\phi+18}{3} \geq (6\times 54\times 18)^{1/3} \geq 18$$

Now equality holds when

$$6\tan^2\phi = 54\cot^2\phi = 18$$

$$\Rightarrow \quad \tan^2\phi = 3 \text{ and } \cot^2\phi = \frac{1}{3}$$

Hence, the statements I and II are correct.

4. (2) A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{a+b+c}{3} \geq (abc)^{1/3}$$

$$\text{or} \quad a+b+c \geq 3(abc)^{1/3} \qquad (1)$$

But given $ab^2c^3, a^2b^3c^4, a^3b^4c^5$ are in A.P. ($\because abc \neq 0$). Hence,

$$2abc = 1+a^2b^2c^2$$

$$\Rightarrow \quad (abc-1)^2 = 0$$

$$\therefore \quad abc = 1$$

Now from Eq. (1), we get

$$a+b+c \geq 3(1)^{1/3}$$

$$\text{or} \quad (a+b+c) \geq 3$$

Hence, minimum value of $a+b+c$ is 3.

5. (3) $y = \dfrac{3^x}{3}+\dfrac{1}{3^x 3}$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{\frac{3^x}{3}+\frac{1}{3^x 3}}{2} \geq \sqrt{\frac{3^x}{3}\frac{1}{3^x 3}}$$

$$\text{or} \quad 3^{x-1}+3^{-x-1} \geq \frac{2}{3}$$

6. (4) We have A.M. $\geq$ G.M. Therefore,

$$\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c} \geq 6$$

$$\text{or} \quad \frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c} \geq 6$$

Hence, the least value is 6.

7. (4) Let $a_1, a_2, \ldots, a_n$ be n positive numbers such that $a_1a_2 \ldots a_n = n^n$.
Since A.M. $\geq$ G.M., we have

$$\frac{a_1+a_2+\cdots+a_n}{n} \geq (a_1a_2 \ldots a_n)^{1/n}$$

$$\text{or} \quad \frac{a_1+a_2+\cdots+a_n}{n} \geq n$$

$$\text{or} \quad a_1+a_2+\ldots+a_n \geq n^2$$

8. (1) A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{bcx+cay+abz}{3} \geq (a^2b^2c^2xyz)^{1/3}$$

$$\text{or} \quad bcx+cay+abz \geq 3xyz$$

$$\text{or} \quad bcx+acy+abz \geq 3abc$$

9. (3) Consider $1/l$, $2/m$, $3/n$ and use A.M. $\geq$ G.M. Then,

$$\frac{1}{3}\left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right) \geq \left(\frac{1}{l}\frac{2}{m}\frac{3}{n}\right)^{1/3} = \left(\frac{6}{lmn}\right)^{1/3}$$

But $lmn = 48$.

$$\therefore \quad \frac{1}{3}\left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right) \geq \left(\frac{6}{48}\right)^{1/3} = \frac{1}{2}$$

$$\therefore \quad \left(\frac{1}{l}+\frac{2}{m}+\frac{3}{n}\right)_{\min} = \frac{3}{2}$$

10. (3) a, b, c are in H.P. Hence, H.M. of a and c is b.

$$\therefore \quad \sqrt{ac} > b \qquad (\because \text{G.M.} > \text{H.M.})$$

Since A.M. > G.M., so

$$\frac{a^{101}+c^{101}}{2} > (\sqrt{ac})^{101} > b^{101} \qquad (\because \sqrt{ac} > b)$$

$$\text{or} \quad 2b^{101}-a^{101}-c^{101} < 0$$

Let $f(x) = x^2-kx+2b^{101}-a^{101}-c^{101}$

$$\therefore \quad f(0) = 2b^{101}-a^{101}-c^{101} < 0$$

Hence, equation $f(x) = 0$ has one root in $(-\infty, 0)$ and other in $(0, \infty)$.

12. (4) $a=\dfrac{x^2+4}{|x|}-3$

$=|x|+\dfrac{4}{|x|}-3$

$\geq 2\sqrt{|x|\times\dfrac{4}{|x|}}-3$ ($\because$ A.M. $\geq$ G.M.)

$\Rightarrow \quad a\geq 1$

13. (1) $2E=\dfrac{2a}{b+c-a}+\dfrac{2b}{c+a-b}+\dfrac{2c}{a+b-c}$

$=\dfrac{2a}{b+c-a}+1+\dfrac{2b}{c+a-b}+1+\dfrac{2c}{a+b-c}+1-3$

$=(a+b+c)\left(\dfrac{1}{b+c-a}+\dfrac{1}{c+a-b}+\dfrac{1}{a+b-c}\right)-3$

Using A.M. $\geq$ H.M. we have

$$\frac{\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}}{3}\geq\frac{3}{a+b+c}$$

or $\quad (a+b+c)\left(\dfrac{1}{b+c-a}+\dfrac{1}{c+a-b}+\dfrac{1}{a+b-c}\right)\geq 9$

or $\quad (a+b+c)\left(\dfrac{1}{b+c-a}+\dfrac{1}{c+a-b}+\dfrac{1}{a+b-c}\right)-3\geq 6$

$\Rightarrow \quad E\geq 3$

13. (1) A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{\log_d a+\log_c b+\log_a c+\log_b d}{4}\geq\sqrt[4]{\frac{\log a}{\log d}\times\frac{\log b}{\log c}\times\frac{\log c}{\log a}\times\frac{\log d}{\log b}}$$

or $\quad \log_d a+\log_b d+\log_a c+\log_c b\geq 4$

14. (1) Using A.M. and H.M. inequality, we get

$$\frac{2bc}{b+c}\leq\frac{b+c}{2},\ \frac{2ac}{a+c}\leq\frac{a+c}{2},\ \frac{2ab}{a+b}\leq\frac{a+b}{2}$$

Adding, $\dfrac{bc}{b+c}+\dfrac{ac}{a+c}+\dfrac{ab}{a+b}\leq\dfrac{1}{2}(a+b+c)$

15. (2) $\dfrac{a+b+c}{3}\geq\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$ ($\because$ A.M. $\geq$ H.M.)

or $\quad (a+b+c)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\right)\geq 9$

16. (4) $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)$

$=ab^2+ac^2+bc^2+ba^2+ca^2+cb^2$

Using A.M. $\geq$ G.M., we get

$$\frac{ab^2+ac^2+bc^2+ba^2+ca^2+cb^2}{6}\geq(a^6b^6c^6)^{1/6}$$

or $\quad a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)\geq 6abc$

17. (1) a, b, c, d are in H.P. Hence, b is H.M. of a and c; c is H.M. of b and d. Using A.M. $\geq$ H.M., we get

$$\frac{a+c}{2}>b \text{ or } a+c>2b$$

and $\dfrac{b+d}{2}>c$ or $b+d>2c$

or $\quad a+c+b+d>2b+2c$

or $\quad a+d>b+c$

18. (4) $a+b+c=18$

$$\Rightarrow \quad 2\times\frac{a}{2}+3\times\frac{b}{3}+4\times\frac{c}{4}=18$$

Using weighted A.M. and G.M. inequality, we get

$$\frac{2\times\frac{a}{2}+3\times\frac{b}{3}+4\times\frac{c}{4}}{9}\geq\left(\left(\frac{a}{2}\right)^2\left(\frac{b}{3}\right)^3\left(\frac{c}{4}\right)^4\right)^{1/9}$$

or $\quad 2^9\geq\dfrac{a^2}{2^2}\times\dfrac{b^3}{3^3}\times\dfrac{c^4}{4^4}$

or $\quad a^2b^3c^4\leq 3^3\times 2^{19}$

19. (1) Let

$x-3=t$

$\Rightarrow \quad x-2=(t+1)$ and $x-1=t+2$

$$\Rightarrow \quad f(x)=\frac{(x-2)(x-1)}{(x-3)}=\frac{(t+1)(t+2)}{t}=\frac{(t^2+3t+2)}{t}=t+\frac{2}{t}+3\geq 3+2\sqrt{2}$$

(using A.M. $\geq$ G.M., as $t>0$)

20. (3) We have

$$\frac{a^3+1}{2}\geq\sqrt{a^3\times 1}$$

and $\dfrac{a^2+a^1}{2}\geq\sqrt{a^2a^1}$

Adding, we get

$$\frac{a^3+a^2+a+1}{2}\geq 2\sqrt{a^3}$$

Multiple Correct Answers Type

1. (1), (2)

We have

$2s=a+b+c$

$A^2=s(s-a)(s-b)(s-c)$

Now,

A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{s+(s-a)+(s-b)+(s-c)}{4}\geq[s(s-a)(s-b)(s-c)]^{1/4}$$

$$\Rightarrow \quad \frac{4s-2s}{4}\geq[A^2]^{1/4}$$

$\Rightarrow \quad s/2\geq A^{1/2}$ or $A\leq s^2/4$

Also,

$$\frac{(s-a)+(s-b)+(s-c)}{3}\geq[(s-a)(s-b)(s-c)]^{1/3}$$

$$\Rightarrow \quad \frac{s}{3}\geq\left[\frac{A^2}{s}\right]^{1/3}$$

or $\frac{A^2}{s} \le \frac{s^3}{27}$

or $A \le \frac{s^2}{3\sqrt{3}}$

2. (1), (3)

A.M. of x and z is y. G.M. of x and z is $\sqrt{xz}$.

Now, A.M. $\ge$ G.M. $\Rightarrow y^2 \ge xz$

Also, A.M $\ge$ H.M. $\Rightarrow y \ge \frac{2xz}{x+z}$

$$\frac{x+y}{2y-x} = \frac{x+y}{x+z-x} = \frac{x+y}{z} \text{ and } \frac{y+z}{2y-z} = \frac{y+z}{x}$$

$$\therefore \quad \frac{\frac{x+y}{2y-x} + \frac{y+z}{2y-z}}{2} \ge \sqrt{\frac{x+y}{z}\frac{y+z}{x}}$$

$$= \sqrt{1+\frac{y(x+y+z)}{xz}} \qquad [\because x+z=2y]$$

$$= \sqrt{1+\frac{3y^2}{xz}}$$

$$\therefore \quad \frac{x+y}{2y-x} + \frac{y+z}{2y-z} \ge 2\sqrt{1+3\frac{y^2}{xz}} \ge 4 \qquad [\because y^2 \ge xz]$$

3. (1), (2)

Using A.M. $\ge$ G.M. one can show that

$$(b+c)(c+a)(a+b) \ge 8abc$$

$$\Rightarrow \quad (p-a)(p-b)(p-c) \ge 8abc$$

Therefore, (2) holds. Also,

$$\frac{(p-a)+(p-b)+(p-c)}{3} \ge [(p-a)(p-b)(p-c)]^{1/3}$$

or $$\frac{3p-(a+b+c)}{3} \ge [(p-a)(p-b)(p-c)]^{1/3}$$

or $$\frac{2p}{3} \ge [(p-a)(p-b)(p-c)]^{1/3}$$

or $$(p-a)(p-b)(p-c) \le \frac{8p^3}{27}$$

Therefore, (1) holds. Again,

$$\frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}\right) \ge \sqrt{\left(\frac{bc}{a}\frac{ca}{b}\right)}$$

and so on. Adding the inequalities, we get

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \ge a+b+c = p$$

Therefore, (3) does not hold.

4. (3), (4)

Consider the A.P. Since a is equidistant from the first term α and the last term β of the A.P., therefore, α, a, β are in A.P. Hence, a is the A.M. of α and β. So,

$$a = \frac{\alpha+\beta}{2}$$

Similarly, b and c are the geometric and harmonic means, i.e.,

$$b = \sqrt{\alpha\beta} \text{ and } c = \frac{2\alpha\beta}{\alpha+\beta}$$

Since A.M., G.M., and H.M. are in G.P. and A.M. $\ge$ G.M. $\ge$ H.M., therefore, a, b, c are in G.P. and $a \ge b \ge c$.

5. (1), (2), (3), (4)

A.M. $\ge$ G.M.

$$\Rightarrow \quad \frac{\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{2a}{4}+\frac{b}{2}+\frac{b}{2}+\frac{3c}{2}+\frac{3c}{2}}{8}$$

$$\ge \sqrt[8]{\frac{2a}{4}\cdot\frac{2a}{4}\cdot\frac{2a}{4}\cdot\frac{2a}{4}\cdot\frac{b}{2}\cdot\frac{b}{2}\cdot\frac{3c}{2}\cdot\frac{3c}{2}}$$

$$\Rightarrow \quad \frac{2a+b+3c}{8} \ge \left(\frac{3^2}{2^8}a^4b^2c^2\right)^{1/8}$$

The greatest value takes place when A.M. = G.M. and

$$\frac{2a}{4} = \frac{b}{2} = \frac{3c}{2}$$

$$\Rightarrow \quad a = b = 3c = K$$

Now, $2a + b + 3c = 1$

$$\Rightarrow \quad 2K + K + K = 1$$

$$\Rightarrow \quad K = 1/4$$

$$\Rightarrow \quad a = b = 1/4 \text{ and } c = 1/12$$

Linked Comprehension Type

1. (3), 2. (1), 3. (2)

Let roots of equation $x^6 - 12x^5 + bx^4 + cx^3 + dx^2 + ex + 64 = 0$ be x_i, $i = 1, 2, \ldots, 6$. Now,

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 12$$

and $$x_1 x_2 x_3 x_4 x_5 x_6 = 64$$

Thus,

$$\frac{x_1 + x_2 + \cdots + x_6}{6} = 2 \text{ and } (x_1 x_2 x_3 x_4 x_5 x_6)^{1/6} = 2$$

$$\Rightarrow \quad \text{A.M.} = \text{G.M.}$$

$$\Rightarrow \quad x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 2$$

Hence, the given equation is equivalent to

$$(x-2)^6 = 0$$

or $$x^6 - 12x^5 + 60x^4 - 160x^3 + 240x^2 - 192x - 64 = 0$$

$$\therefore \quad f(1) = 1 - 12 + 60 - 160 + 240 - 192 + 64 = 1$$

4. (4), 5. (3), 6. (4)

Since, a, b, $c \ge 0$ roots must be negative. Let the roots be $x_1, x_2, x_3, x_4 (< 0)$. Then,

$$x_1 + x_2 + x_3 + x_4 = -a$$

$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = b$$

$$x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_1 + x_4x_1x_2 = -c$$

$$x_1x_2x_3x_4 = 1$$

Now,

$$\frac{(-x_1)+(-x_2)+(-x_3)+(-x_4)}{4} \ge \left[(-x_1)(-x_2)(-x_3)(-x_4)\right]^{1/4}$$

$(\because \text{A.M.} \ge \text{G.M.})$

$$\Rightarrow \quad \frac{a}{4} \ge 1$$

or $$a \ge 4$$

Hence, the minimum value of a is 4. Similarly,

$$\frac{x_1x_2 + x_1x_3 + \cdots + x_3x_4}{6} \ge \left[x_1^3x_2^3x_3^3x_4^3\right]^{1/4}$$

$$\Rightarrow \quad \frac{b}{6} \ge 1$$

or $$b \ge 6$$

Hence, the minimum value of b is 6. Finally,

$$\frac{-x_1x_2x_3 - x_2x_3x_4 - x_3x_4x_1 - x_4x_1x_2}{4} \geq \left[x_1^3x_2^3x_3^3x_4^3\right]^{1/4}$$

$$\Rightarrow \quad \frac{c}{4} \geq 1$$

or $\quad c \geq 4$

Hence, the minimum value of c is 4.

Numerical Value Type

1. **(2)** $f(x) = \dfrac{4x^2+8x+13}{6(1+x)}$

$$= \frac{4(x+1)^2+9}{6(1+x)}$$

$$= \frac{2}{3}(x+1) + \frac{3}{2(x+1)}$$

$$\geq 2\sqrt{\frac{2}{3}\cdot\frac{3}{2}} = 2 \qquad \text{(A.M.} \geq \text{G.M.)}$$

Therefore, minimum value of $f(x)$ is 2.

2. **(3)** $a + b = 3$

H.M. ≤ A.M. for 3 numbers $\frac{a}{2}, \frac{a}{2}, b$. We have

$$\frac{3}{\frac{2}{a}+\frac{2}{a}+\frac{1}{b}} \leq \frac{\frac{a}{2}+\frac{a}{2}+b}{3} = 1$$

$$\therefore \quad 1 \geq \frac{3}{\frac{2}{a}+\frac{2}{a}+\frac{1}{b}} \text{ or } \frac{2}{a}+\frac{2}{a}+\frac{1}{b} \geq 3$$

$$\therefore \quad \frac{4}{a}+\frac{1}{b} \geq 3$$

3. **(8)** Consider values xy, yz, zx

Now A.M. ≥ G.M.

$$\Rightarrow \quad \frac{xy+yz+zx}{3} \geq (x^2y^2z^2)^{1/3}$$

$$\Rightarrow \quad 4^{3/2} \geq xyz$$

or $\quad xyz \leq 8$

4. **(3)** $9a + 3b + c = 90$

$$\Rightarrow \quad 3a + b + \frac{c}{3} = 30$$

Now consider numbers $3a$, b, and $\frac{c}{3}$

$$\left(3a \times b \times \frac{c}{3}\right)^{\frac{1}{3}} \leq \frac{3a+b+\frac{c}{3}}{3} \qquad \text{(as GM} \leq \text{AM)}$$

or $\quad (abc)^{1/3} \leq \dfrac{30}{3} = 10$

or $\quad abc \leq 1000$

$\Rightarrow \quad \log a + \log b + \log c \leq 3$

5. **(96)** Using AM ≥ GM for $x^2, 2xy, 2xy, 4y^2, z^2, z^2$

$$\therefore \quad \frac{x^2+2xy+2xy+4y^2+z^2+z^2}{6} \geq [16(xyz)^4]^{1/6}$$

$$= [16(32)^4]^{1/6} = (2^{24})^{1/6} = 16$$

6. **(4)** We have $\dfrac{2\left(\frac{x}{2}\right)+y}{3} \geq \left(\left(\frac{x}{2}\right)^2 y\right)^{1/3}$

$$\Rightarrow \quad \left(\frac{3}{3}\right)^3 \geq \frac{x^2y}{4}$$

$$\Rightarrow \quad x^2y \leq 4$$

Therefore, maximum value of x^2y is 4.

7. **(2)** As $x, y \in R$ and $xy > 0$, so x and y will be of same sign.

Therefore, all the quantities $\frac{2x}{y^3}, \frac{x^3y}{3}, \frac{4y^2}{9x^4}$ are positive.

Now A.M. ≥ G.M.

$$\Rightarrow \quad \frac{2x}{y^3}+\frac{x^3y}{3}+\frac{4y^2}{9x^4} \geq 3\left(\left(\frac{2x}{y^3}\right)\left(\frac{x^3y}{3}\right)\left(\frac{4y^2}{9x^4}\right)\right)^{1/3}$$

$$= 3 \times \frac{2}{3} = 2$$

8. **(55)** $\dfrac{\left(a+\frac{b}{2}+\frac{b}{2}+\frac{c}{3}+\frac{c}{3}+\frac{c}{3}+\frac{d}{4}+\frac{d}{4}+\frac{d}{4}+\frac{d}{4}+\frac{e}{5}+\frac{e}{5}+\frac{e}{5}+\frac{e}{5}+\frac{e}{5}\right)}{15} = 1$

$$\therefore \quad \text{A.M.} = 1$$

$$\text{G.M.} = \left(\frac{ab^2c^3d^4e^5}{1\cdot 2^2\cdot 3^3\cdot 4^4\cdot 5^5}\right)^{1/15}$$

$$= \left(\frac{(120)^3\times 50}{(3\times 5\times 8)^3\times 2\times 5^2}\right)^{1/15} = 1$$

$$\therefore \quad \text{A.M.} = \text{G.M.}$$

$$\Rightarrow \quad a = \frac{b}{2} = \frac{c}{3} = \frac{d}{4} = \frac{e}{5} = \lambda$$

$$\Rightarrow \quad \lambda = 1$$

$$\therefore \quad a = 1, b = 2, c = 3, d = 4, e = 5$$

$$\therefore \quad a^2+b^2+c^2+d^2+e^2 = 1^2+2^2+3^2+4^2+5^2 = 55$$

9. **(1)** Using A.M. ≥ G.M., we get

$$x_2^2 + 1 \geq 2x_2$$

$$x_3^3 + 1 + 1 \geq 3x_3$$

$$x_4^4 + 1 + 1 + 1 \geq 4x_4$$

$$x_5^5 + 1 + 1 + 1 + 1 \geq 5x_5$$

Adding these, we get

$$x_1 + x_2^2 + x_3^3 + x_4^4 + x_5^5 + 10 \geq x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5$$

$$x_1 + x_2^2 + x_3^3 + x_4^4 + x_5^5 = 5$$

The equality holds if $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

So, there is only one possibility for $(x_1, x_2, x_3, x_4, x_5)$.

Archives

JEE Advanced

Singe Correct Answer Type

1. (3) Using A.M. $\geq$ G.M.

$$\frac{4\alpha x^2 + \frac{1}{2x} + \frac{1}{2x}}{3} \geq \sqrt[3]{4\alpha x^2 \cdot \frac{1}{4x^2}}$$

$$\Rightarrow \quad 4\alpha x^2 + \frac{1}{x} \geq 3\sqrt[3]{\alpha}$$

If this is true for all $x > 0$, then

$$3\sqrt[3]{\alpha} \geq 1 \Rightarrow \alpha \geq \frac{1}{27}$$

Numerical Value Type

1. (8) Using A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1}{8} \geq 1$$

or $\quad a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1 \geq 8$

or $\quad \left(a^{-5} + a^{-4} + 3a^{-3} + a^8 + a^{10} + 1\right)_{\min} = 8$

Chapter 7

Concept Application Exercises

Exercise 7.1

1. Since the man can go in 4 ways and come back in 3 ways, the total number of ways is $4 \times 3 = 12$.

2. Since each question can be answered in 4 ways, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

3. Total variables if only alphabet is used = 26

   Total variables if alphabets and digits both are used = 26×10

   So, total variables = $26(1 + 10) = 286$

4. For five persons to stand in a row, we create five places for them in a row.

1	2	3	4	5

   1st place can be occupied by any one of the five persons. So, there are five options for first place.

   2nd place can be occupied by any one of the remaining four persons. So, there are four options for second place.

   Similarly, for 3rd, 4th and 5th places there are 3, 2 and 1 options respectively.

   So, total number of ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$

5. First and second prizes can be given in Mathematics in (30×29) ways.

   First and second prizes can be given in Physics in (30×29) ways.

   First prize can be given in Chemistry in 30 ways.

   First prize can be given in English in 30 ways.

   Hence, the number of ways to give prizes in all the four subjects.

   $= (30 \times 29) \times (30 \times 29) \times 30 \times 30$

6. Let A_1, A_2, A_3, A_4, A_5 are five persons.

   (a) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$

   (b) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$

7. There are six straight lines $L_1, L_2, L_3, L_4, L_5, L_6$ in a plane.

   Since no two lines are parallel and no three of which pass through the same point, we have maximum points of intersection.

   Line L_1 intersect five lines, so there will be five points of intersection.

   Similarly, for each of the lines L_2, L_3, L_4, L_5, L_6, there will be five points of intersection.

   But there will be double counting.

   So, total number of points of intersection is $\frac{5 \times 6}{2} = 15$

8. When $x - y > 5$, we have following cases:

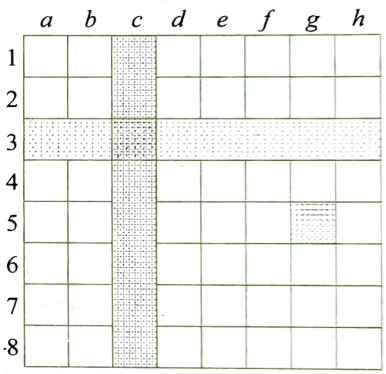
Value of x	Value of y	Number of cases
6	0	1
7	0, 1	2
8	0, 1, 2	3
9	0, 1, 2, 3	4
10	0, 1, 2, 3, 4	5
	Total	15

   Similarly, we have 15 cases for $x - y < -5$. Thus, there are 30 pairs.

9. First square can be selected in 64 ways. Second square can be selected in $65 - 15 = 49$ ways.

   But when first square, say $c3$ is selected, sometimes $g5$ is selected as second square.

   Similarly, when first square, say $g5$ is selected, sometimes $c3$ is selected as second square. Thus, each pair is selected twice. So, the total number of ways is $64 \times 49/2 = 1568$



10. Number of 1 digit number = 2

    Number of 2 digit number = $2 \times 2 = 2^2$

    Number of 3 digit number = $2 \times 2 \times 2 = 2^3$

    ...

    ...

    Number of 8 digit number = $2 \times 2 \times \cdots \times 2 = 2^8$

    Number of 9 digit number = $1 \times 2 \times 2 \times \cdots \times 2$(8 times) $= 2^8$

    $\therefore$ Total numbers = $2 + 2^2 + 2^3 + \cdots + 2^8 + 2^8$

    $$= \frac{2(2^8 - 1)}{2 - 1} + 2^8 = 2^9 - 2 + 2^8 = 766$$

11. Set must contain minimum two elements, such that sum of smallest and largest element is 13.

    If set contains smallest number 1 and largest number 12 then we can select other elements of subset from $\{2, 3, \ldots, 12\}$.

    So, number of subsets are 2^{10}.

    If set contains smallest number 2 and largest number 11 then we can select other elements of subset from $\{3, 4, \ldots, 10\}$.

    So, number of subsets are 2^8.

    Similarly, we have subsets $2^6, 2^4, 2^2, 2^0$.

    So, total number of subsets $1 + 2^2 + 2^4 + \ldots + 2^{10} = 1365$

12.

1	2	3

Case I: All the digits are even

Number of numbers = $4 \times 5 \times 5 = 100$

Case II: One digit is even and two digits are odd

If first digit is even then number of numbers = $4 \times 5 \times 5 = 100$

If second digit is even then number of numbers = $5 \times 5 \times 5 = 125$

If third digit is even then number of numbers = $5 \times 5 \times 5 = 125$

So, total number of numbers = $100 + 100 + 125 + 125 = 450$

13. Distinct n-digit numbers which can be formed using digits 2, 5 and 7 are 3^n since each place can be filled in 3 ways.

We have to find n so that $3^n \geq 900$

$\Rightarrow \quad 3^{n-2} \geq 100$

$\Rightarrow \quad n - 2 \geq 5$

$\Rightarrow \quad n \geq 7.$

So, the least value of n is 7.

14. We know that a number is divisible by 3 if the sum of its digits is divisible by 3.

Since the sum of six digits 0, 1, 2, 3, 4, 5 is 15 which is divisible by 5, we can omit digit either 0 or 3.

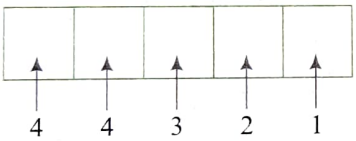
So, we have following two cases.

Case I: Digits are 1, 2, 3, 4, 5

Here first, second, third, fourth and fifth places can be filled in 5, 4, 3, 2 and 1 ways respectively.

So, number of numbers in this case are $5 \times 4 \times 3 \times 2 \times 1 = 120$

Case II: Digits are 0, 1, 2, 4, 5



We have four options to fill first place (1, 2, 3, 4).

For second place again, we have four options as now digit 0 can be considered.

For third, fourth, fifth places we have 3, 2 and 1 ways respectively.

So, number of numbers in this case are $4 \times 4 \times 3 \times 2 \times 1 = 96$

From cases I and II, total number of numbers = $120 + 96 = 216$.

Exercise 7.2

1. $$\frac{(2n)!}{n!} = \frac{1\times2\times3\times4\times\cdots\times(2n-2)\times(2n-1)2n}{n!}$$

$$= \frac{\{1\times3\times\cdots\times(2n-1)\}\{2\times4\times\cdots\times(2n)\}}{n!}$$

$$= \frac{\{1\times3\times\cdots\times(2n-1)\}2^n\{1\times2\times\cdots\times(n-1)n\}}{n!}$$

$$= \frac{\{1\times3\times5\times7\times\cdots\times(2n-1)\}2^n n!}{n!}$$

$$= \{1\times3\times5\times7\times\cdots\times(2n-1)\}2^n$$

2. We have,

$$1! + 2! + 3! + \cdots + n! = 1 + 2 + 6 + 24 + 5! + 6! + \cdots + n!$$
$$= 33 + 5! + 6! + \cdots + n!$$
$$= 33 + 10k,\ k \in I$$

This sum always ends with 3 or the digit in the unit's place of the sum is 3. Now, we know that square of any integer never ends with 3. Then the given sum cannot be a perfect square.

3. Let p be divisible by k and r be any natural number between 1 and k. If $p + r$ is divided by k, we obtain r as the remainder.

Now, $n! = 1 \times 2 \times 3 \times 4 \times \cdots \times (n-1)n$

Therefore, $n!$ is divisible by every natural number between 2 and n. So, $n! + 1$, when divided by any natural number between 2 and n, leaves 1 as the remainder. Hence, $n! + 1$ is not divisible by any natural number between 2 and n.

4. Let,

$$N = 1! + 2! + 3! + 4! + 5! + 6! + \cdots + n!$$

$$\Rightarrow \quad \frac{N}{15} = \frac{1!+2!+3!+4!+5!+\cdots+n!}{15}$$

$$= \frac{1!+2!+3!+4!}{15} + \frac{5!+6!+\cdots+n!}{15}$$

$$= \frac{33}{15} + \text{integer} \qquad (\text{as } 5!, 6!, \ldots \text{ are divisible by } 15)$$

Hence, remainder is 3.

5. $80 = 2^4 \times 5$

To find the exponent of 80 in 200!, we find the exponent of 2 and 5. Exponent of 2 is

$$\left[\frac{200}{2}\right] + \left[\frac{200}{2^2}\right] + \left[\frac{200}{2^3}\right] + \left[\frac{200}{2^4}\right] + \left[\frac{200}{2^5}\right] + \left[\frac{200}{2^6}\right] + \left[\frac{200}{2^7}\right]$$
$$= 100 + 50 + 25 + 12 + 6 + 3 + 1$$
$$= 197$$

Exponent of 5 is

$$\left[\frac{200}{5}\right] + \left[\frac{200}{5^2}\right] + \left[\frac{200}{5^3}\right] = 40 + 8 + 1 = 49$$

Now, exponent of 16 in 200! is [197/4] = 49. Hence, exponent of 80 is 49.

Exercise 7.3

1. $$^{n-1}P_r + r\,{}^{n-1}P_{r-1} = \frac{(n-1)!}{(n-1-r)!} + r\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-1-r)!}\left\{1 + r\frac{1}{n-r}\right\}$$

$$= \frac{(n-1)!}{(n-1-r)!}\left(\frac{n}{n-r}\right)$$

$$= \frac{n!}{(n-r)!} = {}^nP_r$$

2. $$\frac{n!}{(n-5)!} \times \frac{(n-3)!}{n!} = 20$$

or $\quad (n-3)(n-4) = 20$

or $\quad n = -1, 8$

But -1 is not acceptable.

3. There are 10 letters in the word LOGARITHMS. So, the numbe of 4-letter words is equal to the number of arrangements of letters, taken 4 at a time, i.e., ${}^{10}P_4 = 5040$.

4. **(a)** Given,

$$^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

or $$\frac{22!}{(21-r)!} : \frac{20!}{(18-r)} = 11:52$$

or $$\frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

or $$\frac{22\times21\times20}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

or $(21-r)(20-r)(19-r) = 2 \times 21 \times 52$

or $(21-r)(20-r)(19-r) = 2 \times 3 \times 7 \times 4 \times 13$

or $(21-r)(20-r)(19-r) = 12 \times 13 \times 14$

or $r = 7$

(b) ${}^{56}P_{r+6} : {}^{34}P_{r+3} = 30800 : 1$

or $\dfrac{56!(51-r)!}{(50-r)!54!} = 30800$

or $56 \times 55 \times (51-r) = 30800$

or $r = 41$

5. The number of 1-digit numbers is ${}^4P_1$. The number of 2-digit numbers is ${}^4P_2$. The number of 3-digit numbers is ${}^4P_3$. The number of 4-digit numbers is ${}^4P_4$. Hence, the required number is

$${}^4P_1 + {}^4P_2 + {}^4P_3 + {}^4P_4.$$

6. Extreme left place can be filled in 6 ways, the middle place can be filled in 6 ways and the extreme right place in only 3 ways. Since the number to be formed is odd, so the required number is $6 \times 6 \times 3 = 108$.

7. Given set of numbers is $\{1, 2, \ldots 11\}$ in which 5 are even and six are odd, which signifies that in the given product, it is not possible to arrange but only to subtract the even numbers from odd numbers. There must be at least one factor involving subtraction of an odd number from another odd number. So at least one of the factors is even. Hence, product is always even.

8. (i) **When repetition is not allowed.**

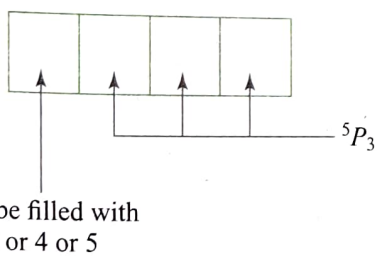
There are 6 digits 0, 1, 2, 3, 4, 5 and we can use any number of digits.

Number of digits	Numbers
1	$(\neq 0)\ 5 = 5$
2	$(\neq 0)\ 5 \times 5 = 25$
3	$(\neq 0)\ 5 \times 5 \times 4 = 100$
4	$(\neq 0)\ 5 \times 5 \times 4 \times 3 = 300$
5	$(\neq 0)\ 5 \times 5 \times 4 \times 3 \times 2 = 600$
6	$(\neq 0)\ 5 \times 5 \times 4 \times 3 \times 2 \times 1 = 600$
Total	1630

For numbers greater than 3000, number of digits is greater than or equal to 4.

All the 6- and 5-digit numbers are greater than 3000.

Now for 4-digit numbers > 3000:



Hence, number greater than 3000 having 4 digits is

$$3 \times {}^5P_3 = 180$$

So, total numbers > 3000 is $180 + 600 + 600 = 1380$.

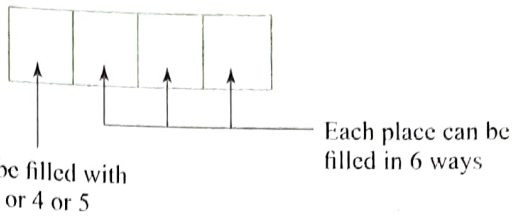
(ii) **When repetition is allowed.**

It is equivalent to permutation of 6 objects (digits) when any object is repeated any number of time, which is equal to 6^6. But this includes one number 0. Hence, total numbers are $6^6 - 1 = 46655$.

Now, number of 5-digit numbers is $5 \times 6 \times 6 \times 6 \times 6 = 6480$.

And, number of 6-digit numbers is $5 \times 6 \times 6 \times 6 \times 6 \times 6 = 38880$.

Again, 4-digit numbers are to be greater than 3000.



Hence, total number of 4-digit numbers is $3 \times 6 \times 6 \times 6 = 648$

But these 648 numbers have 3000 as a number which should be excluded. So, the number of 4-digit numbers > 3000 is $648 - 1 = 647$. Therefore, total numbers >3000 are $647 + 6480 + 38880 = 46007$.

9. First women choose the chairs from amongst the chairs 1 to 4.

Two women can be arranged in 4 chairs in ${}^4P_2$ ways.

In remaining 6 chairs 3 men can be arranged in ${}^6P_3$ ways.

$\therefore$ Total number of possible arrangements $= {}^4P_2 \times {}^6P_3$

$$= \frac{4!}{2!} \times \frac{6!}{3!}$$

$$= 4 \times 3 \times 6 \times 5 \times 4$$

$$= 1440$$

10. There are 26 English alphabets and 10 digits (0 to 9).

Since, it is given that each plate contains two different letters followed by three different digits.

$\therefore$ Arrangement of 26 letters, taken 2 at a time $= {}^{26}P_2$

$$= 26 \times 25$$

$$= 650$$

And arrangement of 10 digit taken three at a time $= {}^{10}P_3$

$$= 10 \times 9 \times 8$$

$$= 720$$

$\therefore$ Total number of license plates $= 650 \times 720 = 468000$

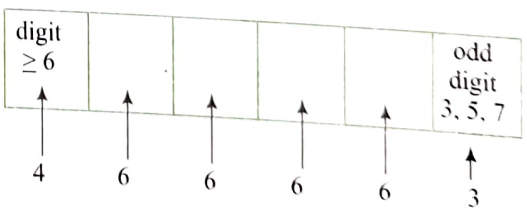
11. We have 6 digits, viz., 5, 6, 7, 8, 9, and 0 and we have to form numbers (integers) greater than 6,00,000, which are odd.

So the first place (lakh's position) should be ≥ 6 and the last position (i.e., unit) must be odd, i.e., 5, 7, or 9.

(a) **When repetitions is allowed.**

First place can be filled by 6, 7, 8, or 9 i.e., in 4 ways, last place can be filled by 5, 7, or 9 i.e., in 3 ways and each of the remaining 4 places (i.e., 2nd, 3rd, 4th, 5th) can be filled by any of the 6 digits in 6 ways. Hence, the total numbers will be $4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552$.

Note: The above discussion can also be shown as follows:



The total number of numbers that can be formed is

$$4 \times 6 \times 6 \times 6 \times 6 \times 3 = 15552.$$

(b) **When repetitions is not allowed.**

Since we have restrictions on first and last places and no digit can be repeated, we have the following algorithm: 1st place having numbers ≥ 6 and last place having odd numbers.

Digits in first place	Digits in last place
6	5, 7, 9
7	5, 9
8	5, 7, 9
9	5, 7
Total number of ways of filling last place	10 ways

Thus, the first and the last place can be filled in 10 ways, and the remaining 4 places can be filled by the remaining 4 digits in $4! = 24$ ways. Hence, the total number of numbers that can be formed is $10 \times 24 = 240$.

Exercise 7.4

1. Clearly, one of the odd digits from 1, 3, 5, 7, 9 will be repeated.
If digit '1' is repeated then number of ways of arranging 1, 1, 3, 5, 7, 9 is $\frac{5!}{2!}$.

So, total number of numbers is $5 \times \frac{5!}{2!} = 300$

2.

M	E	D	I	T	E	R	R	A	N	E	A	N
1	2	3	4	5	6	7	8	9	10	11	12	13

Consonants M, D, T, R, R, N, N can be arranged in positions 1, 3, 5, 7, 8, 10, 13 in $\frac{7!}{2!2!}$ ways.

Vowels E, I, E, A, E, A can be arranged in positions 2, 4, 6, 9, 11, 12 in $\frac{6!}{3!2!}$ ways.

So, total number of ways = $\frac{7!}{2!2!} \times \frac{6!}{3!2!}$.

3. We have letters (I, I, I), U, O, (N, N), S, (T, T, T)
Words start with consonant.
Case I: Words start with *N*
We have to arrange (I, I, I), U, O, N, S, (T, T, T)

Number of words are $\frac{10!}{3!3!}$

Case II: Words start with *S*
We have to arrange (I, I, I), U, O, (N, N), (T, T, T)

Number of words are $\frac{10!}{3!3!2!}$

Case III: Words start with *T*
We have to arrange (I, I, I), U, O, (N, N), S, (T, T)

Number of words are $\frac{10!}{3!2!2!}$

So, total number of words are $\frac{10!}{3!3!} + \frac{10!}{3!3!2!} + \frac{10!}{3!2!2!}$.

4. Total number of books = $a + 2b + 3c + d$
Total number of ways in which these books can be arranged in a shelf

$$= \frac{(a+2b+3c+d)!}{a!b!b!c!c!c!}$$

5. Dice is marked with numbers 1, 2, 3, 4, 5, 6
If the sum of dice in three throw is 11 then observations must be (1, 4, 6), (1, 5, 5), (2, 3, 6), (2, 4, 5), (3, 3, 5), (3, 4, 4).

We can get this observation in $3! + \frac{3!}{2!} + 3! + 3! + \frac{3!}{2!} + \frac{3!}{2!}$ = 27 ways.

Exercise 7.5

1. If the best and worst papers appear always together, the number of ways is $5! \times 2$. Therefore, the required number of ways is as follows.
Total number of ways without any restrictions – Number of ways when best and worst paper are together = $6! - 5! \times 2 = 480$.
2. There are 2 primary teachers. They can stand in a row together in $2! = 2$ ways.
There are 2 middle teachers. They can stand in a row together in $2! = 2$ ways.
There are 2 secondary teachers. They can stand in a row together in $2! = 2$ ways.
These three sets can be arranged in $3! = 6$ ways
Hence, the required number of ways = $2 \times 2 \times 2 \times 6 = 48$
3. Five boys can sit in 5! ways; in this case, there are 6 vacant places where the girls can sit in ${}^6P_3$ ways. Therefore, the required number of ways is ${}^6P_3 \times 5!$
4. The word MOBILE has three even places and three odd places. It has 3 consonants and 3 vowels. In three odd places, we have to fix up 3 consonants, which can be done in ${}^3P_3$ ways.
Now, in the remaining three places, we have to fix up the remaining three, which can be done in ${}^3P_3$ ways.
Hence, the total number of ways is ${}^3P_3 \times {}^3P_3 = 36$.
5. m men can be seated in $m!$ ways, creating $(m+1)$ places for ladies to sit.
n ladies in $(m+1)$ places can be arranged in ${}^{m+1}P_n$ ways.
$\therefore$ Total ways = $m! \times {}^{m+1}P_n$

$$= m! \times \frac{(m+1)!}{(m+1-n)!} = \frac{(m+1)!m!}{(m-n+1)!}$$

Exercise 7.6

1. ${}^{15}C_{3r} = {}^{15}C_{r+3}$
$\Rightarrow$ $3r = r + 3$ or $3r + r + 3 = 15$
$\Rightarrow$ $r = 3/2$ or $r = 3$
$\Rightarrow$ $r = 3$ (as r is positive integer)
2. We have

$$^{n+2}C_8 : {}^{n-2}P_4 = 57:16$$

or $$\frac{{}^{n+2}C_8}{{}^{n-2}P_4} = \frac{57}{16}$$

or $$\frac{(n+2)!}{8!(n-6)!} \times \frac{(n-6)!}{(n-2)!} = \frac{57}{16}$$

or $$(n+2)(n+1)n(n-1) = \frac{57}{16} \times 8!$$

$$= \frac{19 \times 3}{16} \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$\Rightarrow$ $(n-1)n(n+1)(n+2) = 18 \times 19 \times 20 \times 21$
or $n - 1 = 18$
or $n = 19$

3. Maximum value of ${}^{20}C_r$ is ${}^{20}C_{10}$ and maximum value of ${}^{25}C_r$ is ${}^{25}C_{12}$. Hence, the required ratio is ${}^{20}C_{10}/{}^{25}C_{12} = 143/4025$.

4. Number of cards exchanged is $2 \times$ (number of ways 2 students can be selected who will exchange cards), which is given by $2\ {}^{20}C_2$.

5. We have 10 men and 7 women and a committee of 6 containing at least 3 men and 2 women is to be formed.

Men (10)	Women (7)	Number of selections
3	3	${}^{10}C_3 \times {}^{7}C_3 = 4200$
3	2	${}^{10}C_4 \times {}^{7}C_2 = 4410$
4	Total	8610

Now, let us consider the case when 2 particular women are always there in the same committee. We have to make a selection of 4 from 10 men and 5 women. In this case, to comply with the initial condition of at least 3 men and at least 2 women, we have the following cases:

Men (10)	Women (5)	Number of selections
4	0	${}^{10}C_4 \times {}^{5}C_0 = 210$
3	1	${}^{10}C_3 \times {}^{5}C_1 = 600$
	Total	810

Hence, the number of committees when two particular women are never together is $8610 - 810 = 7800$.

6. Since $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$, therefore, $x_1, x_2 < 30$, i.e., x_1 and x_2 should come from tickets numbered from 1 to 29 and this may happen in ${}^{29}C_2$ ways. Now x_4, x_5 should come from 20 tickets numbered from 31 to 50 in ${}^{20}C_2$ ways. So, total number of ways is ${}^{29}C_2\ {}^{20}C_2$.

7. Four hotels can be chosen out of five hotels in ${}^{5}C_4$ ways and the four visitors can stay in one of four hotels in 4! ways (one visitor in one hotel).

Therefore, the total number of ways of accommodating the visitors is ${}^{5}C_4 \times 4! = 5 \times 24 = 120$.

8. Let there be r white and $(15 - r)$ black balls.

Then, total number of permutations of these balls is $15!/r!\,(15 - r)! = {}^{15}C_r$ since r white balls are alike and $(15 - r)$ black balls are alike.

Therefore, the number of arrangement is ${}^{15}C_r$ which is maximum, when $r = (15 - 1)/2$ or $(15 + 1)/2$, i.e., $r = 7$ or 8.

9. The total number of units to be covered is $3 + 7 + 11 = 21$. A person can choose 3 units in ${}^{21}C_3$ ways. A person can choose 7 units from remaining 18 in ${}^{18}C_7$ ways. The rest 11 units can be chosen in 1 way.

Therefore, the total number of ways is ${}^{21}C_3 \times {}^{18}C_7 \times 1$.

10. The selection scheme is as follows:

	Category I	Category II	Category III
Group A (4)	2	2	3
Group B (5)	2	3	2
Group C (6)	3	2	2
No. of ways of selections	${}^{4}C_2\ {}^{5}C_2\ {}^{6}C_3 = 1200$	${}^{4}C_2\ {}^{5}C_3\ {}^{6}C_2 = 900$	${}^{4}C_3\ {}^{5}C_2\ {}^{6}C_2 = 600$

Total number of different selections is $1200 + 900 + 600 = 2700$.

11. Required number of ways = Total number of ways – no of ways in which he does not select any question from any one section

$= {}^{12}C_5 - 3 \times {}^{8}C_5$

12. The number of three elements subsets containing a_1 is equal to the number of ways of selecting 2 elements out of $n - 1$ elements.

So, the required number of subsets is ${}^{n-1}C_2$.

13. **Case I:** When exactly one of C and E is selected

Number of ways of selecting one of C and E is ${}^{2}C_1$.

For any one of them selected there is only one way of arranging him with A, B, D.

So, number of arrangements are ${}^{2}C_1 \times 1 = 2$

Case II: When both are selected

When both are selected, D must be selected.

The remaining one may be A or B.

So, number of ways of arrangements are ${}^{2}C_1 \times 1 = 2$

So, total number of ways = 4

14. Let three numbers selected be 3^a, 3^b, 3^c which are in G.P.

$\therefore \quad (3^b)^2 = (3^a)(3^c)$

$\Rightarrow \quad 2b = a + c$

$\Rightarrow \quad a, b, c$ are in A.P.

Thus, selecting three number in G.P. from given set is equivalent to selecting 3 numbers from $\{1, 2, 3, \ldots, 101\}$ which are in A.P.

Now a, b, c are in A.P. if either a and c are odd or a and c are even.

Number of ways of selecting two odd numbers is ${}^{51}C_2$ and those of selecting two even numbers is ${}^{50}C_2$.

Once a and c are selected, b is fixed.

Hence total number of ways $= {}^{51}C_2 + {}^{50}C_2 = 1275 + 1225 = 2500$

15. The selection scheme is as follows:

Husband's Relatives		Wife's Relatives		Number of selections
Male (3)	Female (4)	Male (4)	Female (3)	
3	0	0	3	${}^{3}C_3 \times {}^{3}C_3 = 1$
2	1	1	2	${}^{3}C_2 \times {}^{4}C_1 \times {}^{4}C_1 \times {}^{3}C_2 = 144$
1	2	2	1	${}^{3}C_1 \times {}^{4}C_2 \times {}^{4}C_2 \times {}^{3}C_1 = 324$
0	3	3	0	${}^{4}C_3 \times {}^{4}C_3 = 16$
			Total :	485

16. Total number of points on sides of triangle is 12.

For triangle three points can be selected in ${}^{12}C_3$ ways.

But selection of three points from any of the sides will not form triangle.

So, number of triangles $= {}^{12}C_3 - {}^{3}C_3 - {}^{4}C_3 - {}^{5}C_3$

$= 220 - 15 = 205$

17. Suppose he answers x questions correctly and $10 - x$ question wrongly.

$\therefore \quad$ His score is $3x + (10 - x)(-1) = 4x - 10$

Now score of the candidate is from the set $\{15, 16, 17, 18, 19, 20\}$.

Only $4x - 10 = 18$ is possible.

$\therefore \quad x = 7$.

So, his 7 answer are correct and 3 answers are wrong.

Number of ways of selection of 7 questions from $10 = {}^{10}C_7 = {}^{10}C_3$

$$= \frac{10 \times 9 \times 8}{6} = 120$$

18. **(a)** A straight line can be formed by joining any two points. So number of straight lines (i.e., selection of two points from n) is ${}^{n}C_2$. But, selection of two points from m collinear points gives no extra line. Hence, the number of distinct straight lines is

$${}^nC_2 - ({}^mC_2 - 1) = \frac{1}{2}n(n-1) - \frac{1}{2}m(m-1) + 1 \qquad (1)$$

(b) Formation of triangles is equivalents to section of three points from n points. As m points are collinear, selection of three points from m collinear points gives no triangles. Hence, the number of triangles is

$${}^nC_3 - {}^mC_3 = \frac{1}{6}\left[n(n-1)(n-2) - m(m-1)(m-2)\right]$$

(c) Four points determine a quadrilateral. But of these four points, not more than two is to be selected from m collinear points. Now, the number of selections of four points from all n is ${}^nC_4$. The number of selections of three points from m collinear and one from rest is ${}^mC_3 \, {}^{n-m}C_1$. The number of selections of four points from m collinear is ${}^mC_4$.
Hence, Number of quadrilaterals

$$= {}^nC_4 - {}^mC_3 \times {}^{(n-m)}C_1 - {}^mC_4$$

Exercise 7.7

1. Number of arrangements with no two vowels together

= (Arranging P, R, M, T, T, N)
× (Selecting 5 gaps from 7 created for vowels)
× (Arranging E, U, A, I, O in five gaps)

$$= \frac{6!}{2!} \times {}^7C_5 \times 5!$$

Number of arrangements which contains 2T's together

= (Arranging P, R, M, (T, T), N)
× (Selecting 5 gaps from 6 created for vowels)
× (Arranging E, U, A, I, O in five gaps)

$$= 5! \times {}^6C_5 \times 5!$$

$\therefore$ Required arrangements $= \frac{6!}{2!} \times {}^7C_5 \times 5! - 5! \times {}^6C_5 \times 5!$

$$= 57 \times (5!)^2$$

2. 6 places can be selected in ${}^9C_6$ ways and 6 can be placed only at 5 places except the right most of the 6 selected. Remaining numbers, i.e., 7, 8, 9 in 3! ways.
Hence, number of numbers are ${}^9C_6 \cdot 5 \cdot 3! = 84 \times 30 = 2520$

3. First arrange six '+' signs, which can be arranged in 1 way.
This creates 7 gaps.
For four '+' signs four places can be selected in ${}^7C_4$ ways, in which they can be arranged in only one way.
$\therefore$ Required number of arrangements are $= {}^7C_4 = 35$

4. Total number of things $= n$
We have to arrange r things out of n in which three particular things must occur together.
Therefore, combination of n things taken r at a time in which 3 things always occurs $= {}^{n-3}C_{r-3}$
If three things taken together, then it is considered as 1 group.
Arrangement of these three things = 3!
Now, we have to arrange $= (r-3) + 1 = (r-2)$ objects
$\therefore$ Arrangement of $(r-2)$ objects $= (r-2)!$
$\therefore$ Total number of arrangements $= {}^{n-3}C_{r-3} \times (r-2)! \times 3!$

5. Sum of digits is even if all the digits are even or one is digit even other two odd.
So, number of ways of selections are $({}^4C_3 + {}^4C_1 \times {}^3C_2) = 16$
Hence number of words $= 16 \times 3! = 96$

6. Let the particular side on which 3 particular sailors can work be named A; and another side B on which 2 particular sailors can work. Thus, we are left with 3 sailors only.
The number of ways of selection of one sailor for side A is ${}^3C_1 = 3$ and then we are left with 2 sailors for other sides. Now, on each side 4 sailors can be arranged in 4! ways. Hence, total number of arrangements is $3 \times 4! \times 4! = 3 \times 24 \times 24 = 1728$.

7. COMBINATORICS
Vowels in alphabetical order → A I I O O
Consonants in alphabetical order → B C C M N R S T
8 places for consonant's can be selected in ${}^{13}C_8$ ways in which consonants can be arranged in one way as order of letters is fixed.
Remaining 5 places can be filled by 5 vowels in one way can be filled by consonants in 1 way. Therefore,
Required numbers of arrangements $= {}^{13}C_5$

8. We have letters (C, C), (O, O), N, U, T
Required number of ways = Total number of ways without any restriction − when no C comes at odd place

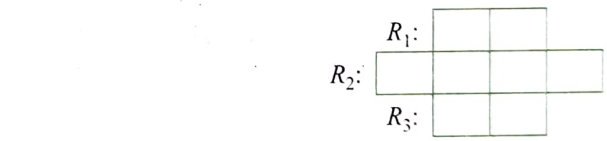
$$= \frac{7!}{2!2!} - {}^3C_2 \times \frac{5!}{2!} = 1080$$

9. We have letters M, D, C, L, E, I, A
Let us first arrange vowels, which has number of ways 4!.
Five places are created marked as '×'.

× M × D × C × L ×

In these five places, we have to arrange vowels A, E, I so that A and E are together but all are not together.
So, we have to arrange AE and I.
Number of ways are ${}^5C_2 \times 2! \times 2!$ (as AE can also be arranged in 2! ways)
So, total number of arrangements $= 4! \times {}^5C_2 \times 2! \times 2! = 960$

10. As all the X's are identical, the question is of selection of 6 squares from 8 squares, so that no row remains empty. Here R_1 has 2 squares, R_2 has 4 squares and R_3 2 squares. The selection scheme is as follows:



R_1	R_2	R_3
1	4	1
1	3	2
2	3	1
2	2	2

$\therefore$ Number of selections are $= {}^2C_1 \times {}^4C_4 \times {}^2C_1 + {}^2C_1 \times {}^4C_3 \times {}^2C_2 + {}^2C_2 \times {}^4C_3 \times {}^2C_1 + {}^2C_2 \times {}^4C_2 \times {}^2C_2$

$$= 4 + 8 + 8 + 6 = 26$$

Exercise 7.8

1. The number of selection of two ladies to sit together is ${}^3C_2$.
Let the two seats occupied by these two (selected) ladies be numbered as 1 and 2 and remaining seats by 3, 4, 5, and 6. The 3rd lady cannot occupy seat number 3 and 6, so there are only two choices (viz., numbers 4 and 5) left with the 3rd lady who can make selection of seats in ${}^2C_1$ ways.
Now, the two selected ladies (in seat numbers 1 and 2) can be permuted (i.e., change their seats) in 2! ways and 3 gentlemen can be permuted on remaining 3 seats in 3! ways.
Hence, by product rule, total number of ways is

$${}^3C_2 \times {}^2C_1 \times 2! \times 3! = 72.$$

2. Let the secretary A sit on one side of chairman B and deputy secretary C sit on other side of chairman. Then we have to arrange 12 persons and one group ABC in a circle.

So, number of ways of arrangement is 12!.

But in group ABC, members A and C can be arranged in 2! ways.

So, total number of arrangements = $2 \times 12!$

3. 6 men can dine at a round table in 5! ways.

Now, there are 6 gaps created.

These can be occupied by 5 women in ${}^6P_5$, i.e., 6! ways.

Therefore, the total number of ways is $6! \times 5!$.

4. Eight different beads can be arranged in a circular form in $(8 - 1)! = 7!$ ways. Since there is no distinction between the clockwise and anticlockwise arrangement, the required number of arrangements is $7!/2 = 2520$.

5. If four particular flowers are always together, flowers can be strung in $(4! \times 4!)/2 = 288$ ways.

Exercise 7.9

1. Let the number of candidates be n. Therefore, $(n - 2)$ are to be elected and so one can vote up to $n - 2$. Hence, the number of ways in which one can vote is

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-2} = 56 \text{ (given)}$$

or $\quad 2^n - ({}^nC_0 + {}^nC_{n-1} + {}^nC_n) = 56$

or $\quad 2^n - n = 58$

or $\quad 2^n = 58 + n$

which is satisfied by $n = 6$ only.

2. For each historical monument there are two possibilities, either he visits or does not visit.

The number of ways in which he can visit 5 historical monuments is 2^5. Similarly, he can visit 6 gardens in 2^6 ways.

To visit at least one of seven shopping malls, it could be in $2^7 - 1$ ways.

Therefore, the total number of ways he can visit the city is $2^5 \times 2^6 \times (2^7 - 1)$.

3. The required number of ways is: $(10 + 1)(9 + 1)(7 + 1) - 1$
$= 11 \times 10 \times 8 - 1 = 879$

4. We have

$$720 = 2^4 \times 3^2 \times 5^1$$

Now, number of divisors of 720 is equivalent to selection of zero or more integer from (2, 2, 2, 2), (3, 3), (5).

The number of ways of selection is $5 \times 3 \times 2 = 30$. This also includes 1 and 720.

Now, in even divisor factor 2 should be there. Then from set of 2, we must have nonempty selection. Hence, the number of even divisors is $4 \times 3 \times 2 = 24$. Also, the number of divisors of 720 can be seen as the number of different terms in the expansion of

$$(2^0 + 2^1 + 2^2 + 2^3 + 2^4) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1)$$

which also gives sum of divisors.

Hence, sum of divisors is

$$(2^0 + 2^1 + 2^2 + 2^3 + 2^4) \times (3^0 + 3^1 + 3^2) \times (5^0 + 5^1)$$

$$= \frac{2^5 - 1}{2 - 1} \times \frac{3^3 - 1}{3 - 1} \times \frac{5^2 - 1}{5 - 1} = 2418$$

5. $3^p 6^m 21^n = 2^m 3^{p+m+n} 7^n$

Therefore, the required number of proper divisors is equal to the number of selections of any number of 3's and 7's [$\because$ for odd number of selections of any number of 3's and 7's [$\because$ for odd divisors 2 must not be selected] and is given by $(p + m + n + 1)(n + 1) - 1$.

6. $7056 = 2^4 \times 3^2 \times 7^2$

Number of divisors $= (4 + 1)(2 + 1)(2 + 1) = 5 \times 3 \times 3 = 45$

Number of ways of resolving into 2 factors $= \frac{45 + 1}{2} = 23$

7. India must win at least 6 matches of 11 matches.

Then number of ways in which Indian can win the series

$$= {}^{11}C_6 + {}^{11}C_7 + \ldots + {}^{11}C_{11}$$

Now $\quad {}^{11}C_0 + {}^{11}C_1 + {}^{11}C_2 + \ldots + {}^{11}C_{10} + {}^{11}C_{11} = 2^{11}$

$\Rightarrow \quad 2({}^{11}C_6 + {}^{11}C_7 + \ldots + {}^{11}C_{11}) = 2^{11} \quad$ (Using ${}^nC_r = {}^nC_{n-r}$)

$\therefore \quad {}^{11}C_6 + {}^{11}C_7 + \ldots + {}^{11}C_{11} = 2^{10}$

8.

Number of objects from 21 different objects	Number of objects from 21 identical objects	Number of ways of selections
10	0	${}^{21}C_{10} \times 1$
9	1	${}^{21}C_9 \times 1$
...	...	...
...	...	...
0	10	${}^{21}C_0 \times 1$

Thus, total number ways of selection

$$= {}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{10} = 2^{20}$$

Exercise 7.10

1. 4 distinct balls can be divided into two nonempty groups as 1, 3 or 2, 2

Sine boxes are identical, number of ways of division and distribution are same

$\therefore \quad$ Number of ways $= \frac{4!}{1!\,3!} + \frac{4!}{2!\,2!\,2!} = 4 + 3 = 7.$

2. First divide 22 books into 5 groups of the size 5, 5, 4, 4, 4.

The number of ways is $\frac{22!}{3!2!(5!)^2(4!)^3}$

Now, number of ways of distribution of these groups among 5 students is 5! Then total number of ways of distribution is

$$\frac{22!}{3!2!(5!)^2(4!)^3} \times 5! = \frac{22!}{3!2!5!(4!)^3}$$

3. The number of ways 16 constables can be divided into 8 batches is $(16)!/8!(2!)^8$.

Now, the first batch may be assigned to patrol any of the 8 villages, the second one can then be assigned to any one of the remaining seven villages and so on.

Hence, the number of ways of assigning 8 batches to patrol 8 villages is

$$\frac{(16)!}{8!(2!)^8}(8!) = \frac{(16)!}{(2!)^8}$$

4. The number of selections of 4 prizes to the particular boy is ${}^{10}C_4$. From the remaining 6 prizes, each prize can be given to any of the four students. Therefore, the number of ways is 4^6.

Therefore, the total number of ways is

$${}^{10}C_4 \times 4^6 = 210 \times 4096 = 860160.$$

5. Any two months can be chosen in ${}^{12}C_2$ ways. The six birthdays can fall in these two months in 2^6 ways. Out of these 2^6 ways there are two ways when all the six birthdays fall in one month. So, the required number of ways is ${}^{12}C_2 \times (2^6 - 2)$.

6. Number of passengers to be allotted in the two decks bus
$= (u + l) - (r + s)$
Number of seats available in the upper deck
$= (u - s)$
Number of seats available in lower deck $= l - r$
$\therefore$ Total number of ways of distribution $= \dfrac{(u + l - r - s)!}{(l - r)!(u - s)!}$

7. This is equivalent to dividing $3n$ different objects into three equal size group. Hence, number of ways is $3n!/[(n!)^3 3!]$.

8. We have to select 4 professors for Roorkee and 6 professors for outside. Again, 2 professors prefer Roorkee and 3 outside, so we are left with 5 professors.
The number of ways in which two more professors for Roorkee can be selected is ${}^5C_2 = 10$.
And remaining 3 professors are left for outside. Now, 6 professors outside Roorkee can be allotted to 3 centres in $6!/(2!2!2!3!) \times 3!$ ways.
Now, 4 professors for 2 centers in Roorkee can be allotted in $4!/(2!2!2!) \times 2!$ ways. Hence, the total number of ways is

$$10 \times \frac{6!}{2!2!2!3!} \times 3! \times \frac{4!}{2!2!2!} \times 2! = 5400$$

Exercise 7.11

1. First, give ₹3 to each of four persons. Now for remaining ₹4 we have
$$x + y + z + w = 4 \quad (1)$$
where x, y, z, w are number of rupees gained by person 1, 2, 3, 4, respectively.
We have to find number of non-negative solutions of Eq. (1), which is ${}^{4+4-1}C_{4-1} = {}^7C_3 = 35$.

2. Let x_W, x_R, x_B be the number of white balls, red balls, and blue balls being selected. We must have
$$x_W + x_R + x_B = 10$$
The required number of ways is equal to the number of non-negative integral solutions of $x_W + x_R + x_B = 10$, which is ${}^{3+10-1}C_{10} = {}^{12}C_{10} = {}^{12}C_2$.

3. Let $a = 2p + 1$, $b = 2q + 1$, $c = 2r + 1$, $d = 2s + 1$ where p, q, r, s are non-negative integers. Therefore,
$$2p + 1 + 2q + 1 + 2r + 1 + 2s + 1 = 20$$
or $\quad p + q + r + s = 8$
Therefore, the required number is equal to the number of non-negative integral solutions of $p + q + r + s = 8$, which is given by ${}^{8+4-1}C_{4-1} = {}^{11}C_3 = 165$.

4. Number of ways in which two numbers can be selected from 100 integers is ${}^{100}C_2$.
Let us find the number of ways by selecting two numbers, if their difference is more than 10.
Let the two integers selected be P and Q.

$$\underset{x}{} \quad \underset{y}{P} \quad \underset{z}{Q}$$

Here, x, y, z are number of integers before P, between P and Q, and after Q respectively. Therefore, $x + y + z = 98$.
Now, we have to find number of integral solutions for $x, z \geq 0$ and $y \geq 10$.
$\therefore \quad x + y_1 + z = 88$, where $y_1 \geq 0$
Then, number of solutions of above equations is
${}^{88+3-1}C_{3-1} = {}^{90}C_2$.
Hence, the number of ways where two different natural numbers can be selected, which differ by almost 10 is
${}^{100}C_2 - {}^{90}C_2$.

5. $xyz = 21600 = 2^5 \times 3^3 \times 5^2$
Number of positive integral solutions
= (Number of ways (2, 2, 2, 2, 2) can be distributed into three boxes)
× (Number of ways (3, 3, 3) can be distributed into three boxes)
× (Number of ways (5, 5) can be distributed into three boxes)
$= {}^{5+3-1}C_{3-1} \times {}^{3+3-1}C_{3-1} \times {}^{2+3-1}C_{3-1}$
$= {}^7C_2 \times {}^5C_2 \times {}^4C_2$
$= 21 \times 10 \times 6 = 1260$

6. $(x_1 + x_2 + x_3)(y_1 + y_2) = 11 \times 7$ or 7×11
In the first case, $(x_1 + x_2 + x_3) = 11$ and $(y_1 + y_2) = 7$, which have ${}^{10}C_2 \cdot {}^6C_1$ solutions
In the second case, $(x_1 + x_2 + x_3) = 7$ and $(y_1 + y_2) = 11$, which have ${}^6C_2 \cdot {}^{10}C_1$ solutions
$\therefore$ Total number of solutions $= {}^{10}C_2 \cdot {}^6C_1 + {}^6C_2 \cdot {}^{10}C_1$
$= 270 + 150 = 420$

7. $\underset{x}{} \quad \underset{y}{B} \quad \underset{z}{B} \quad \underset{w}{B}$
First 3 boys can be arranged in 3! ways. After arranging the boys, 4 gaps are created. Let in these gaps x, y, z, and w girls sit as shown in the figure. Let us first find out the distribution ways of girls in the four gaps. As given in question, $y, z \geq 2$ and $x, w \geq 0$, we have to find the integral solutions of the equation $x + y + z + w = 15$ with the above condition. Let
$y = y_1 + 2$ and $z = z_1 + 2$ (where $y_1, z_1 \geq 0$)
$\Rightarrow \quad x + y_1 + z_1 + w = 11$
Number of solutions of above equation is ${}^{11+4-1}C_{4-1} = {}^{14}C_3$. After it is decided as in which gap how many girls will sit, they can be arranged in 15! ways.
Hence, the total number of ways is 3! 15! ${}^{14}C_3$.

8. Required number
$=$ coeff. of p^{30} in $(p^2 + p^3 + \ldots + p^{16})^8$
$=$ coeff. of p^{30} in $p^{16}(1 + p + \ldots + p^{14})^8$
$= \text{coeff. of } p^{14} \text{ in } \left(\dfrac{1 - p^{15}}{1 - p}\right)^8$
$=$ coeff. of p^{14} in $(1 - p)^{-8}$
$= {}^{21}C_{14}$

9. Number of integral solutions
$=$ coefficient of p^{24} in $(p + p^2 + p^3 + p^4 + p^5)$
$(p^{12} + p^{13} + \ldots + p^{18})(p^{-1} + p^0 + p^1 + \ldots + p^{11})$
$= \text{coefficient of } p^{24} \text{ in } \dfrac{p(1 - p^5)}{1 - p} \cdot \dfrac{p^{12}(1 - p^7)}{1 - p} \cdot \dfrac{p^{-1}(1 - p^{13})}{1 - p}$

$= \text{coefficient of } p^{12} \text{ in } (1-p^5)(1-p^7)(1-p^{13})(1-p)^{-3}$

$= \text{coefficient of } p^{12} \text{ in } (1-p^5-p^7+p^{12})(1-p)^{-3}$

$= {}^{14}C_{12} - {}^{9}C_7 - {}^{7}C_5 + 1$

$= 91 - 36 - 21 + 1 = 35$

10. Any number between 1 and 100000 must be of less than six digits.

Therefore, it must be of the form $a_1a_2a_3a_4a_5a_6$

where $a_1, a_2, a_3, a_4, a_5, a_6 \in \{0, 1, 2, \ldots, 9\}$

Now given that $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 18$ ...(1)

where $0 \le a_i \le 9,\ i = 1, 2, 3, \ldots, 9$

$\therefore$ Required number of integers

$= \text{coefficient of } p^{18} \text{ in } (1 + p + p^2 + \ldots + p^9)^6$

$= \text{coefficient of } p^{18} \text{ in } \left(\frac{1-p^{10}}{1-p}\right)^6$

$= \text{coefficient of } p^{18} \text{ in } [(1-p^{10})^6 (1-p)^{-6}]$

$= \text{coefficient of } p^{18} \text{ in } [(1 - {}^{6}C_1 p^{10})(1-p)^{-16}]$

$= {}^{33}C_{18} - 6 \times {}^{23}C_8$

Exercise 7.12

1. Number of n-digit numbers formed by using the digits 2, 3, 4, 5, 6, 7 is 6^n.

Number of n-digit numbers formed by using the digits 3, 4, 5, 6, 7 is $n(A) = 5^n$.

Number of n-digit numbers formed by using the digits 2, 3, 4, 5, 6 is $n(B) = 5^n$.

Number of n-digit numbers formed by using the digits 3, 4, 5, 6 is $n(A \cap B) = 4^n$.

So, total number of numbers which contains digits 2 and 7 is $6^n - 5^n - 5^n + 4^n$.

2. If exactly r element have self image then number of functions

= (Number of ways of selecting r elements)

$\times$ (Derangement of remaining elements)

$= {}^{n}C_r\, r! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots\ldots\right)$

$\therefore$ Required number of functions

$= {}^{6}C_3 \times 3! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) + {}^{6}C_4 \times 2! \left(1 - \frac{1}{1!} + \frac{1}{2!}\right)$

$+ {}^{6}C_5 \times 1! \times \left(1 - \frac{1}{1!}\right) + {}^{6}C_6$

$= 40 + 15 + 0 + 1$

$= 56$

3. Total number of ways such that at least 3 digits will not come in its position.

= Derangement of 3 digits + Derangement of 4 digits

+ Derangement of 5 digits

$= {}^{5}C_3 \times 3! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$

$+ {}^{5}C_4 \times 4! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$+ {}^{5}C_5 \times 5! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$= 10 \times 2 + 5 \times 9 + 44$

$= 20 + 45 + 44 = 109$

Exercises

Single Correct Answer Type

1. (3) We have 32 places for teeth. For each place, we have two choices, either there is a tooth or there is no tooth. Therefore, the number of ways to fill up these places is 2^{32}. As there is no person without a tooth, the maximum population is $2^{32} - 1$.

2. (3) For each bulb there are two possibilities. It will be switched either on or off. Hence, total number of ways in which the room can be illuminated is $2^{12} - 1$.

3. (3) The total number of ways is $6 \times 6 \times \cdots$ to n times $= 6^n$. The total number of ways to show only even numbers is $3 \times 3 \times \cdots$ to n times $= 3^n$. Therefore, the required number of ways is $6^n - 3^n$.

4. (1) If 7 is used at first place, the number of numbers is 9^4 and for any other four places it is 8×9^3

So, total number of numbers $= 9^4 + 4 \times 8 \times 9^3 = 41 \times 9^3$

5. (2) The number of one-digit numbers is 6.

The number of two-digit numbers is $5 \times 5 = 25$.

The number of thee-digit numbers is $5 \times 5 \times 4 = 100$.

Hence, the total number are is 131.

6. (2) Numbers greater than 1000 and less than or equal to 4000 will be of 4 digits and will have either 1 (except 1000) or 2 or 3 in the 1st place. After fixing 1st place, the 2nd place can be filled by any of the 5 digits. Similarly the 3rd place can be filled up in 5 ways and 4th place can be filled up in 5 ways. Thus, there will be $5 \times 5 \times 5 = 125$ ways in which 1 will be in first place but this also includes 1000. Hence, there will be 124 numbers having 1 in the first place. Similarly, 125 for each 2 or 3. One number will be there in which 4 will be in the first place, i.e., 4000. Hence, the required number of ways is $124 + 125 + 125 + 1 = 375$.

7. (1) Each position can be filled in 5 ways. Hence, the total number of numbers is 5^{20}.

8. (2) The smallest number of people

= Total number of possible forecasts

= Total number of possible results

$= 3 \times 3 \times 3 \times 3 \times 3$

9. (2) If n is odd,

$3^n = 4\lambda_1 - 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow$ $2^n + 3^n + 5^n$ is divisible by 4 if $n \ge 2$

Thus, $n = 3, 5, 7, 9, \ldots, 99$, i.e., n can take 49 different values.

If n is even, $3^n = 4\lambda_1 + 1,\ 5^n = 4\lambda_2 + 1$

$\Rightarrow$ $2^n + 3^n + 5^n$ is not divisible by 4

as $2^n + 3^n + 5^n$ will be in the form of $4\lambda + 2$.

Thus, the total number of ways of selecting 'n' is equal to 49.

10. (4) $N = 1! + 2! + \cdots + 2005!$

$= (1! + 2! + 3! + 4!) + (5! + \cdots + 2005!)$

= 33 + an integer having 0 in its unit's place

= an integer having 3 in its unit's place

Hence, N^{500} is an integer having 1 in its unit's place.

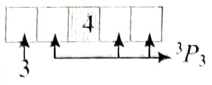
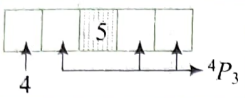
11. (2) Total number of triplets without restriction is $n \times n \times n$.

The number of triplets with all different coordinates is ${}^{n}P_3$.

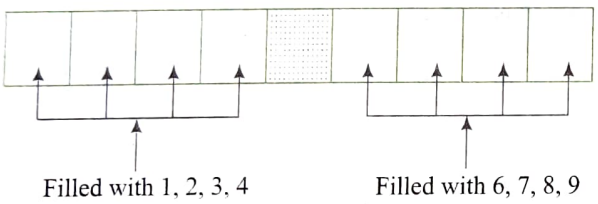
Therefore, the required number of triplets is $n^3 - n(n-1)(n-2)$.

12. (2) The number of numbers when repetition is allowed is 5^4.
The number of numbers when digits cannot be repeated is ${}^5P_5$.
Therefore, the required number of numbers is $5^4 - 5!$.

13. (4)

Middle digit	Digits available for remaining four places	Pattern	Number of ways filling remaining four places
4	0, 1, 2, 3		$3 \times {}^3P_3$
5	0, 1, ..., 4		$4 \times {}^4P_3$
6	0, 1, ..., 5	...	$5 \times {}^5P_3$
7	0, 1, ..., 6	...	$6 \times {}^6P_3$
8	0, 1, ..., 7	...	$7 \times {}^7P_3$
9	0, 1, ..., 8	...	$8 \times {}^8P_3$

14. (4) According to given conditions, numbers can be formed by the following format:



The required number of numbers is ${}^4P_4 \times {}^4P_4$.

15. (1) Total number of words without any restriction is 7!
Total number of words beginning with I is 6!
Total number of words ending with *B* is 6!
Total number of words beginning with I and ending with *B* is 5!
Thus the total number of required words is
$7! - 6! - 6! + 5! = 7! - 2(6!) + 5!$.

16. (3) Formed number can be utmost of nine digits. Total number of such numbers is

$$3 + 3^2 + 3^3 + \ldots + 3^8 + 2 \times 3^8$$

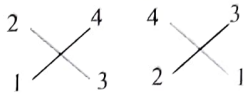
$$= \frac{3(3^8 - 1)}{3-1} + 2 \times 3^8 = \frac{3^9 - 3 + 4 \times 3^8}{2} = \frac{7 \times 3^8 - 3}{2}$$

17. (4) The order of letters of the word 'OBJECT' is B C E J O T
Words starting with B can be formed in 5! ways.
Words starting with C can be formed in 5! ways.
Words starting with E can be formed in 5! ways.
Words starting with J can be formed in 5! ways.
Words starting with O can be formed in 5! ways.
Words starting with TB can be formed in 4! ways.
Words starting with TC can be formed in 4! ways.
Words starting with TE can be formed in 4! ways.
Words starting with TJ can be formed in 4! ways.
Words starting with TOB can be formed in 3! ways.
Words starting with TOC can be formed in 3! ways.
Words starting with TOE can be formed in 3! ways.
Words starting with TOJB can be formed in 2! ways.
Words starting with TOJC can be formed in 2! ways.
Therefore, the total number of words is 718 words.
Hence, 717^{th} word is TOJCBE.

18. (2) 3 must be at thousand's place and since the number should be divisible by 5, or 5 must be at unit's place. Now, we have to fill two places (tens and hundreds), i.e., ${}^4P_2 = 12$.

19. (1) The number of ways of allotment without any restriction is ${}^8P_6$. Now, it is possible that none of the rooms of 2nd floor or 3rd floor are occupied. Thus, there are two ways in which one floor remains unoccupied. Hence, the number of ways of allotment in which a floor is unoccupied is $2 \times 6!$ Hence, number of ways in which none of the floor remains unoccupied is ${}^8P_6 - 2(6!)$.

20. (2) The natural numbers are 1, 2, 3, 4. Clearly, in one diagonal we have to place 1, 4 and in the other 2, 3.



(i) (ii)

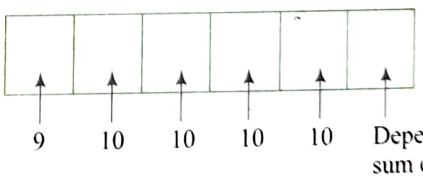
The number of ways in (i) is $2! \times 2! = 4$.
The number of ways in (ii) is $2! \times 2! = 4$.
Therefore, the total number of ways is 8.

21. (1) We can think of three packets. One consisting of three boys of class X, other consisting of 4 boys of class XI and last one consisting of 5 boys of class XII. These packets can be arranged in 3! ways and contents of these packets can be further arranged in 3! 4! and 5! ways, respectively. Hence, the total number of ways is $3! \times 3! \times 4! \times 5!$

22. (3) The number of numbers with 0 in the unit's place is $3! = 6$.
The number of numbers with 1 or 2 or 3 in the unit's place is $3! - 2! = 4$. Therefore, the sum of the digits in the unit's place is $6 \times 0 + 4 \times 1 + 4 \times 2 + 4 \times 3 = 24$.
Similarly, for the ten's and hundred's places, the sum of digits is 24. The sum of the digits in the thousand's place is $6 \times 1 + 6 \times 2 + 6 \times 3 = 36$.
Hence, the required sum is $36 \times 1000 + 24 \times 100 + 24 \times 10 + 24$.

23. (1) Total numbers ending with 2 is 3! as after fixing 2 in the unit's place, other three places can be filled by 3! ways. Thus, 2 appears in the unit's place 3! times.
Similarly, all other digits 4, 6, and 8 also appear 3! times. Then sum of the digits in the unit's place is $6(2 + 4 + 6 + 8) = 120$. Similarly, sum of digits in ten's place is 120 tens and that in hundred's place is 120 hundreds, etc. Hence, sum of all the 24 numbers is $120(1 + 10 + 10^2 + 10^3) = 120 \times 1111 = 133320$.

24. (1)



First place from left cannot be filled with 0. Next four places can be filled with any of the 10 digits. After filling the first five places, the last place can be filled in following ways:

Sum of digits in first five places	Digit in the unit's place
$5k$	0 or 5
$5k + 1$	4 or 9
$5k + 2$	3 or 8
$5k + 3$	2 or 7
$5k + 4$	1 or 6

Thus, in any case the last place can be filled in two ways.
Hence, the required number of numbers is $9 \times 10^4 \times 2$.

25. (3) Places for A, B, C can be chosen in ${}^{10}C_3$ ways. Remaining 7 persons can speak in 7! ways. Hence, the number of ways in which they can speak is ${}^{10}C_3 \times 7! = 10!/6$.

26. (3) R, R, R, O, O, H

Words starting with $R \rightarrow \dfrac{5!}{2!\,2!} = 30$

Words starting with $O \rightarrow \dfrac{5!}{3!} = 20$

Words starting with $HR \rightarrow \dfrac{4!}{2!\,2!} = 6$

The word following the above words will be "HORRRO"

The next word will be "HORROR"

$\therefore$ Rank of the word "HORROR" $= 30 + 20 + 6 + 1 + 1 = 58$

27. (1) $m+n$ counters on one side can be arranged in $\dfrac{(m+n)!}{m!n!}$ ways.

For each arrangement on one side, corresponding arrangement on the other side is just one as arrangements are symmetrical. Hence, the total number of arrangements is

$$\frac{(m+n)!}{m!n!} = {}^{m+n}C_m$$

28. (2) Let the arrangement be $x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8$. Clearly 5 should occupy the position x_4 or x_5. Thus, required number of arrangements is $2(7!)$.

29. (1)

Number of times 3 used	Pattern	Number of numbers	Number of times 3 appears
1	$--3$ $-3-$ $3--$	$3\times9\times9$	$1(3\times9\times9)$
2	-33 $33-$ $3-3$	3×9	$2(3\times9)$
3	333	1	3
		Total	300

Any place other than 3 is filled by 9 ways as 0 can appear anywhere which gives all types of numbers such as single digit, two digits, etc.

Alternate solution:

Three-digit numbers from 000 to 999 means 1000 numbers, each number constituting 3 digits. Hence, the total numbers of digits which we have to write is 3000.

Since the total number of digits is 10 (0 to 9) no digit is filled preferentially. Therefore, number of times we write 3 is $3000/10 = 300$.

30. (4) Since the balls are to be arranged in a row so that the adjacent balls are of different colours, we can therefore begin with a white ball or a black ball. If we begin with a white ball, we find that $n+1$ white balls numbered 1 to $n+1$ can be arranged in a row in $(n+1)!$ ways.

$$\times w \times w \times w \times w \times \ldots \times w$$

Since white and black balls are alternate, we have exactly $(n+1)$ place for black balls marked with '×'

So, the total number of arrangements in which adjacent balls are of different colours and first ball is a white ball is $(n+1)!$ $\times(n+1)! = [(n+1)!]^2$. But we can begin with a black ball also. Hence, the required number of arrangements is $2[(n+1)!]^2$.

31. (2) The number of trains a day (the digits 1, 2, 3) are three groups of like elements from which a sample must be formed. In the time-table for a week, the number 1 is repeated twice, the number 2 is repeated 3 times, and the number 3 is repeated twice.

The number of different time-tables is given by

$$p(2, 3, 2) = \frac{7!}{2!3!2!} = 210$$

32. (3) Sum of seven digits is a multiple of 9. Now, sum of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45. So, we have to remove two numbers whose sum is 9. So, pairs of removed numbers are (1, 8), (2, 7), (3, 6), (4, 5). With each pair left number of 7-digit number is 7!. So with all 4 pairs, total numbers are $4 \times 7!$.

33. (3)

Number of digits	Numbers ending with 0	Numbers ending with 5	Total
×	0	1	1
××	9	8	17
×××	$9\cdot8 = 72$	$8\cdot8 = 64$	136
××××	$9\cdot8\cdot7 = 504$	$8\cdot8\cdot7 = 448$	952
		Total	1106

34. (4) $\alpha = {}^mC_2 \Rightarrow \alpha = \dfrac{m(m-1)}{2}$

$$\therefore \quad {}^{\alpha}C_2 = \frac{\alpha(\alpha-1)}{2} = \frac{1}{2}\frac{m(m-1)}{2}\left\{\frac{m(m-1)}{2}-1\right\}$$

$$= \frac{1}{8}m(m-1)(m-2)(m+1)$$

$$= \frac{1}{8}(m+1)m(m-1)(m-2) = 3\,{}^{m+1}C_4$$

35. (1) ${}^nC_3 + {}^nC_4 > {}^{n+1}C_3$

$\Rightarrow \quad {}^{n+1}C_4 > {}^{n+1}C_3 \qquad ({}^{\because}\,{}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$

or $\dfrac{{}^{n+1}C_4}{{}^{n+1}C_3} > 1$

or $\dfrac{n-2}{4} > 1$

or $n > 6$

36. (4) Here, ${}^nP_3 - {}^nC_3 > 100$

or $\dfrac{n!}{(n-3)!} - \dfrac{n!}{3!(n-3)!} > 100$

or $\dfrac{5}{6}n(n-1)(n-2) > 100$

or $n(n-1)(n-2) > 120$

or $n(n-1)(n-2) > 6\times5\times4$

or $n = 7, 8, \ldots$

37. (2) Since 5 players are always to be excluded and 6 players always to be included, therefore 5 players are to be chosen from 14. Hence, required number of ways is ${}^{14}C_5 = 2002$.

38. (1) All strips are of different colors, then number of flags is $3! = 5$. When two strips are of same colour, then number of flags is ${}^3C_1 \times {}^2C_1 \times (3!/2!) = 18$. Total number of flags is $6 + 18 = 24 = 4!$

39. (1) $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, when the number is $x_1x_2x_3x_4x_5x_6$

Clearly no digit can be zero. Also, all the digits are distinct. So, let us first select six digits from the list of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 which can be done in ${}^9C_6$ ways.

After selecting these digits they can be put only in one order.

Thus, total number of such numbers is ${}^9C_6 \times 1 = {}^9C_3$.

40. (1) The number of ways of selecting four numbers from 1 to 30 without any restriction is ${}^{30}C_4$. The number of ways of selecting four consecutive [i.e. (1, 2, 3, 4), (2, 3, 4, 5), ..., (27, 28, 29, 30)] number is 27. Hence, the number of ways of selecting four integers which excludes consecutive four selections is

$${}^{30}C_4 - 27 = \frac{30\times29\times28\times27}{24} - 27 = 27378$$

41. (3) Let there be n men participants. Then the number of games that the men play between themselves is $2 \times {}^nC_2$ and the number of games that the men played with the women is $2 \times (2n)$.

$\therefore \quad 2 \times {}^nC_2 - 2 \times 2n = 66$ (by hypothesis)

or $\quad n^2 - 5n - 66 = 0$ or $n = 11$

Hence, the number of participants is 11 men + 2 women = 13.

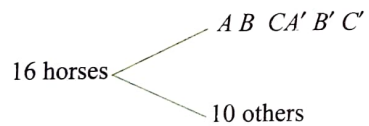
42. (3) The number of ways the candidate can choose questions under the given conditions is enumerated below.

Group 1	Group 2	Number of ways
4	2	$({}^5C_4)\,({}^5C_2) = 50$
3	3	$({}^5C_3)\,({}^5C_3) = 100$
2	4	$({}^5C_2)\,({}^5C_4) = 50$
	Total number of ways	200

43. (1) The selection can be made in ${}^5C_3 \times {}^{22}C_9$ ways.

(Since 3 vacancies are filled from 5 candidates in ${}^5C_3$ ways and now remaining candidates are 22 and remaining seats are 9.)

44. (3)



The number of ways is ${}^{10}C_3 \times$ number of ways of choosing out of $ABC\,A'B'C'$, so that AA', BB' or CC' are not together

$= {}^{10}C_3$ (one from each of pairs AA', BB', CC')

$= {}^{10}C_3 \times 8$

$$= \frac{10\times9\times8}{1\times2\times3} \times 8 = 960$$

45. (3) Let S_1 and S_2 refuse to be together and S_3 and S_4 want to be together only. The total number of ways when S_3 and S_4 are selected is $({}^8C_2 + {}^2C_1 \times {}^8C_1) = 44$. The total ways when S_3 and S_4 are not selected is $({}^8C_4 + {}^2C_1 \times {}^8C_3) = 182$. Thus, the total number of ways is 44 + 182 = 226.

46. (3) The number of ways can be given as follows:

2 bowlers and 9 other players: ${}^4C_2 \times {}^9C_9$

3 bowlers and 8 other players: ${}^4C_3 \times {}^9C_8$

4 bowlers and 7 other players: ${}^4C_4 \times {}^9C_7$

Hence, required number of ways is $6 \times 1 + 4 \times 9 + 1 \times 36 = 78$.

47. (1) Matches whose predictions are correct can be selected in ${}^{20}C_{10}$ ways. Now each wrong prediction can be made in 2 ways. Thus, the total number of ways is ${}^{20}C_{10} \times 2^{10}$.

48. (3) **(i)** Ms. C is taken

(A) B included $\Rightarrow$ A excluded $\Rightarrow {}^4C_1 \times {}^4C_2 = 24$

(B) B excluded $\Rightarrow {}^4C_1 \times {}^5C_3 = 40$

(ii) Ms. C is not taken

$\Rightarrow$ B does not come $\Rightarrow {}^4C_2 \times {}^5C_3 = 60$

$\Rightarrow$ Total = 124

Alternate method:

Case I:

Mr. B is present

$\Rightarrow$ A is excluded and C included

Hence, the number of ways is ${}^4C_2\,{}^4C_1 = 24$.

Case II:

Mr. B is absent

$\Rightarrow$ No constraint

Hence, the number of ways is ${}^5C_3\,{}^5C_2 = 100$.

$\therefore$ Total = 124.

49. (3) Number of points required for the fixed circle is 3. So, first select any three points from the 10 points in ${}^{10}C_3$ ways.

In these ways, circle with four concyclic points is selected in ${}^4C_3$ ways. But it should be taken once then total number of circles is $({}^{10}C_3 - {}^4C_3) + 1$.

50. (3) Select any three points from total $3p$ points, which can be done ${}^{3p}C_3$ ways. But this also includes selection of three collinear points. Now three collinear points from each straight line can be selected in ${}^pC_3$ ways. Then the number of triangles is ${}^{3p}C_3 - 3{}^pC_3 = p^2(4p - 3)$.

51. (3) Two circles intersect at two distinct points. Two straight lines intersect at one point. One circle and one straight line intersect at two distinct points. Then the total numbers of points of intersections are as follows:

Number of ways of selection	Points of intersection
Two straight lines: ${}^5C_2$	${}^5C_2 \times 1 = 10$
Two circles: ${}^4C_2$	${}^4C_2 \times 2 = 12$
One line and one circle: ${}^5C_1 \times {}^4C_1$	${}^5C_1 \times {}^4C_1 \times 2 = 40$
Total	62

52. (1) The number of selection of two parallel lines from m lines is ${}^mC_2$.

The number of selection of two parallel lines from n lines is ${}^nC_2$.

Hence, the number of parallelograms is

$${}^mC_2 \times {}^nC_2 = \frac{1}{4}mn(m-1)(n-1)$$

53. (4) No group of four numbers from the first 12 natural numbers can have the common difference 4.

If one group including 1 is selected with the common difference 1, then the other two group can have the common difference 1 or 2.

If one group including 1 is selected with the common difference 2, then one of the other two groups can have the common difference 2 and the remaining group will have common difference 1.

If one group including 1 is selected with the common difference 3, then the other two groups can have the common difference 3.

Therefore, the required number of ways is 2 + 1 + 1 = 4.

54. (2) Let the numbers selected be x_1, x_2, x_3. We must have

$2x_2 = x_1 + x_3$

$\Rightarrow$ $x_1 + x_3$ = even.

Therefore, x_1, x_3 both are odd or both are even.

If x_1 and x_3 both are even, we can select them in ${}^{12}C_2$ ways.

Similarly, if x_1 and x_3 both are odd, we can again select them in ${}^{12}C_2$ ways. Thus, the total number of ways is $2 \times {}^{12}C_2 = 132$.

55. (2) Given number can be grouped as

$1, 4, 7, \ldots, 3n-2 \rightarrow 3\lambda - 2$

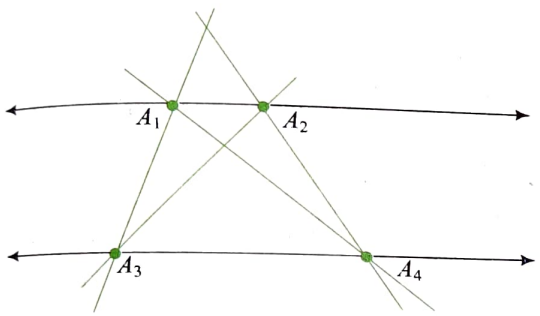
$2, 5, 8, \ldots, 3n-1 \rightarrow 3\lambda - 1$

$3, 6, 9, \ldots, 3n \rightarrow 3\lambda$

That means, we must take two numbers from last row or one number each from first and second rows. Therefore, the total number of ways is

$$ {}^nC_2 + {}^nC_1 \times {}^nC_1 = \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2} $$

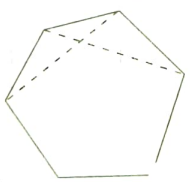
56. (2)



For intersection point, we must have two straight lines, for which two points from each straight line must be selected. Now selection of these points can be done in ${}^mC_2 \times {}^nC_2$ ways. Now as shown in diagram these four points can give two different sets of straight lines, which generate two distinct points of intersection.

Then total number of points of intersection is ${}^mC_2 \times {}^nC_2 \times 2$.

57. (1) Let the number of sides be n. A selection of four vertices of the polygon gives an interior point of intersection of diagonals.



$\therefore$ ${}^nC_4 = 70$

$\Rightarrow$ $n(n-1)(n-2)(n-3) = 24 \times 70$

$= 8 \times 7 \times 6 \times 5$

$\Rightarrow$ $n = 8$

58. (1) 26 cards can be chosen out of 52 cards in ${}^{52}C_{26}$ ways. There are two ways in which each card can be dealt, because a card can be either from the first pack or from the second. Hence the total number of ways is ${}^{52}C_{26} \times 2^{26}$.

59. (2) Two elements for set $P \cap Q$ can be selected in ${}^nC_2$ ways.

Each of the remaining $(n-2)$ elements can be put in any of the three sets $(P \cap Q')$, $(P' \cap Q)$ or $(P \cup Q)'$.

So, total number of subsets $= {}^nC_2 \times 3^{n-2}$

60. (1) There can be two types of numbers.

(i) Any one of the digits 1, 2, 3, 4 appears thrice and the remaining digits only once, i.e., of the type 1, 2, 3, 4, 4, 4, etc. Number of ways of selection of digit which appears thrice is ${}^4C_1$.

Then the number of numbers of this type is $(6!/3!) \times {}^4C_1 = 480$.

(ii) Any two of the digits 1, 2, 3, 4 appears twice and the remaining two only once, i.e., of the type 1, 2, 3, 3, 4, 4, etc. The number of ways of selection of two digits which appear twice is ${}^4C_2$. Then the number of numbers of this type is $[6!/(2!2!) \times {}^4C_2]$. Therefore, the required number of numbers is $480 + 1080 = 1560$.

61. (4) Let us first select two places for vowels, which can be selected from 4 places in ${}^4C_2$ ways. Now this places can be filled by vowels in $5 \times 5 = 25$ ways as repetition is allowed. The remaining two places can be filled by consonants in 21×21 ways. Then the total number of words is ${}^4C_2 \times 25 \times 21^2 = 150 \times 21^2$.

62. (2) Other than two, remaining five places can be filled by 1 and 3 for each place. The number of ways for five places is $2 \times 2 \times 2 \times 2 \times 2 = 2^5$. For 2, selecting two places out of 7 is ${}^7C_2$. Hence, the required number of ways is ${}^7C_2 \times 2^5$.

63. (3) 1 S, 3A, 1 H, 2 R, 1 N, 1 P, 1 U

When all letters are different, corresponding ways is ${}^7C_3 \times 3! = {}^7P_3 = 210$. When two letters are of one kind and other is different, corresponding number of ways is ${}^2C_1 \times {}^6C_1 \times (3!/2!) = 36$. When all letters are alike, corresponding number of ways is 1. Thus, total number of words that can be formed is $210 + 36 + 1 = 247$.

64. (2) There are 6 different letters. We have to select 6 squares, taking at least one from each row and then arranging in each selection. Let us first select places in each row such that no row remains empty.

R_1	R_2	R_3	Number of selections
1	1	4	${}^2C_1 {}^2C_1 {}^4C_4 = 4$
1	2	3	${}^2C_1 {}^2C_2 {}^4C_3 = 8$
2	1	3	${}^2C_2 {}^2C_1 {}^4C_3 = 8$
2	2	2	${}^2C_2 {}^2C_2 {}^4C_2 = 6$

Therefore, the total number of selections of 6 squares is $4 + 8 + 8 + 6 = 26$. For each selection of 6 squares, the number of arrangements of 6 letters is $6! = 720$. Hence, the required number of ways is $26 \times 720 = 18720$.

65. (2) There are 11 letters A, A; I, I; N, N; E, X, M, T, O. For the selection of 4 letters we have the following possibilities:

(i) 2 alike, 2 alike

(ii) 2 alike, 2 different

(iii) All four different

(i) There are 3 pairs of 2 letters. So, the number of ways of selection of 2 pairs is ${}^3C_2$ and permutation of these 4 letters is 4! / 2! 2! Therefore, the number of words in this case is ${}^3C_2 \times 4! / 2!\,2! = 18$.

(ii) We have to select one pair from 3 pairs and 2 distinct letters from remaining 7 distinct letters. For illustration, let us select both A, A; then we have I, N, E, X, M, T, O, i.e., 7 as remaining distinct letters. Hence, the number of selections is ${}^3C_1 \times {}^7C_2$ and these 4 (2 same, 2 distinct) can be permuted in 4!/2! ways. Therefore, number of words is ${}^3C_1 \times {}^7C_2 \times 4! / 2! = 3 \times 21 \times 12 = 756$.

(iii) There are 8 distinct letters so number of words of 4 letters is ${}^8C_4 \times 4! = 1680$. By sum rule, the total number of words is $18 + 756 + 1680 = 2454$.

66. (2)

Number of girls	Number of boys	Number of groups going to picnic	Total number of dolls
1	4	${}^3C_1\,{}^4C_4$	$1({}^3C_1\,{}^4C_4)=3$
2	3	${}^3C_2\,{}^4C_3$	$2({}^3C_2\,{}^4C_3)=24$
3	2	${}^3C_3\,{}^4C_2$	$3({}^3C_3\,{}^4C_2)=18$
		Total	45

67. (2) The number of times the teacher goes to the zoo is ${}^nC_3$. The number of times a particular child goes to the zoo is equal to number of ways two other children can be selected who accompany a particular child, i.e., ${}^{n-1}C_2$. From the question,

$${}^nC_3 - {}^{n-1}C_2 = 84$$

or $(n-1)(n-2)(n-3) = 6\times 84 = 9\times 8\times 7$

$\Rightarrow \quad n-1=9$

68. (2) We first select 2 men out of 7 in ${}^7C_2$ ways. Now we exclude the wives of these two selected men and so select 2 ladies from remaining 5 ladies in ${}^5C_2$ ways. Let A, B be two men and X, Y be the ladies playing in one set. Then we can have

(i) A and X plying against B and Y.

(ii) A and Y playing against B and X.

Then the total number of ways is ${}^7C_2\times {}^5C_2\times 2 = 21\times 10\times 2 = 420$.

69. (2) Suppose there are n players in the beginning. The total number of games to be played was equal to ${}^nC_2$ and each player would have played $n-1$ games.

Let us assume that A and B fell ill. Now the total number of games of A and B is $(n-1)+(n-1)-1 = 2n-3$. But they have played 3 games each. Then their remaining number of games is $2n-3-6 = 2n-9$. Given,

$${}^nC_2 - (2n-9) = 84$$

$\Rightarrow \quad n^2 - 5n - 150 = 0$

or $\quad n = 15$

Alternate solution:

The number of games excluding A and B is ${}^{n-2}C_2$. But before leaving A and B played 3 games each. Then,

$${}^{n-2}C_2 + 6 = 84$$

Solving this equation, we get $n = 15$.

70. (1) Three IIT students who will be between the IIT students can be selected in ${}^{10}C_3$ ways. Now, two DCE students having three IIT students between them can be arranged in $2!\times 3!$ ways. Finally, a group of above five students and the remaining seven students together can be arranged in 8! ways. Hence, total number of ways is ${}^{10}C_3\times 2!\times 3!\times 8!$

71. (2)

Dashes	Dots	Arrangements
5	2	${}^7C_2$
4	3	${}^7C_3$
3	4	${}^7C_4$
2	5	${}^7C_5$
1	6	${}^7C_6$
0	7	${}^7C_7$

The total number of ways is ${}^7C_2+{}^7C_3+{}^7C_4+{}^7C_5+{}^7C_6+{}^7C_7 = 2^7-8 = 120$.

72. (1) If zero is included it will be at $z \Rightarrow {}^9C_2$

If zero is excluded $\begin{cases} x,y,z \text{ all diff.} & \Rightarrow & {}^9C_3\times 2! \\ x=z<y & \Rightarrow & {}^9C_2 \\ x<y=z & \Rightarrow & {}^9C_2 \end{cases}$

The total number of ways is 276.

Alternate method:

y can be from 2 to 9; so total number of ways is

$$\sum_{r=2}^{9}(r^2-1) = 276$$

73. (4) We will consider the following cases:

Case	Flags	No. of signals
4 alike and 2 others alike	4 white and 2 red	$\frac{6!}{4!\,2!}=15$
4 alike and 2 others different	4 white, 1 red and 1 blue	$\frac{6!}{4!}=30$
3 alike and 3 others alike	3 white, 3 red	$\frac{6!}{3!\,3!}=20$
3 alike and 2 other alike and 1 different	3 white, 1 blue, 2 red or 3 red, 1 blue, 2 white	${}^2C_1\times\frac{6!}{3!2!}=120$
	Total	185

74. (3) The total number of ways of selection without restriction is ${}^{20}C_3$. The number of ways of selection when two are adjacent is $20\times {}^{16}C_1$. The number of ways of selection when all the three are adjacent is 20. The required number of ways is

$${}^{20}C_3 - 20\times 16 - 20 = \frac{20\times 19\times 18}{6} - 20\times 16 - 20$$

$$= 20\,[57-16-1]$$

$$= 20\times 40$$

$$= 800$$

75. (1) The number of ways he can select at least one parantha is $2^2-1=7$. The number of ways he can select at least one vegetable dish is $2^4-1=15$. The number of ways he can select zero or more items from salads and sauces is 2^5.

Hence, the total number of ways is $7\times 15\times 32 = 3360$.

76. (3) Let there be n candidates. Then

$${}^nC_1 + {}^nC_2 + \cdots + {}^nC_{n-1} = 254$$

$\Rightarrow \quad 2^n - 2 = 254$

or $\quad 2^n = 2^8$ or $n = 8$

77. (2) The candidate is unsuccessful if he fails in 9 or 8 or 7 or 6 or 5 papers. Therefore, the number of ways to be unsuccessful is

$${}^9C_9+{}^9C_8+{}^9C_7+{}^9C_6+{}^9C_5 = {}^9C_0+{}^9C_1+{}^9C_2+{}^9C_3+{}^9C_4$$

(recall the concept of half series)

$$= \frac{1}{2}\,({}^9C_0+{}^9C_1+\cdots+{}^9C_9)$$

$$= \frac{1}{2}\times 2^9 = 2^8$$

78. (2) $N = 12600 = 2^3\times 3^2\times 5^2\times 7^1$

The number of even divisors is equal to number of ways in which one or more '2', zero or more '3', zero or more '5' and zero or more '7' can be selected, and is given by $(3)(2+1)(2+1)(1+1) = 54$.

80. (1) P_1 must win at least $n+1$ games. Let P_1 win $n+r$ games ($r = 1, 2, \ldots, n$). Therefore, corresponding number of ways is ${}^{2n}C_{n+r}$. The total number of ways is

$$\sum_{r=1}^{n} {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + {}^{2n}C_{n+2} + \cdots + {}^{2n}C_{2n}$$

$$= \frac{2^{2n} - {}^{2n}C_n}{2}$$

80. (3) $480 = 2^5 \times 3 \times 5$

Now, $4n + 2 = 2(2n + 1)$ = odd multiple of 2. Thus, the total number of such divisors is $1 \times 2 \times 2 = 4$.

81. (3) When at least one one-rupee coin is selected, we can select any number of twenty-five coins and ten paisa coins. Then number of ways of such selection is $4\ (2 + 1)(5 + 1) = 72$ as we can select zero or more twenty-five paisa and ten paisa coins to ensure that amount selected is ₹1 or more.

But when none of one-rupee coins is selected we have to select all twenty-five paisa coins and ten paisa coins to ensure sum of ₹1, which can be done only in one way. Then the total number of ways is 73.

82. (1) Obviously, A, B, and C get 4, 5, and 7 objects, respectively. Then, number of distribution ways is equal to number of ways ways of division which is given by 16!/(4!5!7!).

83. (1) Here, we have to divide 12 books into 4 sets of 3 books each. Therefore, the required number of ways is

$$\frac{12!}{(3!)^4 4!} \times 4!$$

84. (1) Required number of ways

= (Number of ways in which 16 players can be divided into 8 pairs) – (Number of pairs in which S_1 and S_2 are together)

$$= \frac{16!}{2^8 \cdot 8!} - \frac{(14)!}{2^7 \cdot 7!} = \frac{16 \cdot 15 \cdot 14!}{2 \cdot 2^7 \cdot 8 \cdot 7!} - \frac{(14)!}{2^7 \cdot 7!}$$

$$= \frac{(14)!}{2^7 \cdot 7!}\left[\frac{16 \cdot 15}{16} - 1\right] = \frac{(14) \cdot (14)!}{14 \cdot (6!) \cdot 2^6} = \frac{(14)!}{2^6 \cdot 6!}$$

85. (1) Three elements from set A can be selected in ${}^7C_3$ ways. Their image has to be y_2. Remaining 2 images can be assigned to remaining 4 pre-images in 2^4 ways. But the function is onto, hence the number of ways is $2^4 - 2$. Then the total number of functions is ${}^7C_3 \times 14 = 490$.

86. (4) Make 1 group of 2 persons, 1 group of 4 persons, and 3 groups of 3 persons among 15 persons (except 2 particular persons). Hence, the number of ways by grouping method is

$$\frac{15!}{2!4!(3!)^3 3!}.$$

Now we add those two persons in the group of 2 persons and thus the number of arrangements of these groups into cars and autos is

$$\frac{15!}{2!4!(3!)^3 3!} \times 2! \times 3! = \frac{15!}{4!(3!)^3}$$

87. (4) Let x, y, z be the friends and a, b, c denote the case when x is invited a times, y is invited b times, and z is invited c times.

Now, we have the following possibilities:

$(a, b, c) = (1, 2, 3)$ or (3 3 0) or (2 2 2)

[grouping of 6 days of week]

Hence, the total number of ways is

$$\frac{6!}{1!2!3!}3! + \frac{6!}{3!3!2!}3! + \frac{6!}{(2!2!2!)3!}3!$$

$$= 360 + 60 + 90 = 510$$

88. (2) Let $x = p - 5$, $y = q - 5$ and $z = r - 5$, where $p, q, r \geq 0$. Then the given equation reduces to

$$p + q + r = 15 \quad (1)$$

Now, we have to find non-negative integral solution of Eq. (1). The total number of such solutions is ${}^{15+3-1}C_{3-1} = {}^{17}C_2 = 136$.

89. (1) Let the balls put in the box are x_1, x_2, x_3, x_4 and x_5. We have

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15,\ x_i \geq 2$$

or $(x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 5$

$\Rightarrow$ $y_1 + y_2 + y_3 + y_4 + y_5 = 5,\ y_i = x_i - 2 \geq 0$

The total number of ways is equal to the number of non-negative integral solutions of the last equation, which is equal to ${}^{5+5-1}C_5 = {}^9C_5$.

90. (1) Let x be the number of objects to the left of the first object chosen, y the number of objects between the first and the second, z the number of objects between the second and the third, and u the number of objects to the right of the third objects. Then, $x, u \geq 0$; $y, z \geq 1$ and $x + y + z + u = n - 3$. Let $y_1 = y - 1$ and $z_1 = z - 1$. Then, $y_1 \geq 0$, $z_1 \geq 0$ such that $x + y_1 + z_1 + u = n - 5$.

The total number of non-negative integral solutions of this equation is ${}^{n-5+4-1}C_{4-1} = {}^{n-2}C_3$.

91. (1) Let x_1, x_2, x_3, x_4 be the number of times T, I, D, E appears on the coupon. Then we must have $x_1 + x_2 + x_3 + x_4 = 8$, where $1 \leq x_1, x_2, x_3, x_4 \leq 8$ (as each letter must appear once). Then the required number of combinations of coupons is equivalent to the number of positive integral solutions of the above equation, which is further equivalent to number of ways of 8 identical objects distributed among 4 persons when each gets at least one objects, and is given by ${}^{8-1}C_{4-1} = {}^7C_3$.

92. (3) If we put minimum number of balls required in each box, balls left are $n(n-1)/2$ which can be put in ${}^{(n^2+n-2)/2}C_{n-1}$ ways without restriction.

93. (1) Let the blankets received by the persons are x_1, x_2, x_3 and x_4. We have

$$x_1 + x_2 + x_3 + x_4 = 15 \text{ and } x_i \geq 2$$

$\Rightarrow$ $(x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) = 7$

$\Rightarrow$ $y_1 + y_2 + y_3 + y_4 = 7$ (where $y_i = x_i - 2 \geq 0$)

The required number is equal to the number of non-negative integral solutions of this equation which is equal to ${}^{4-7-1}C_7$, i.e., ${}^{10}C_7 = {}^{10}C_3$.

94. (3) Let person P_i gets x_i number of things such that

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25$$

Let $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$. Then

$$2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + 5 = 25$$

or $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 10$

We have to simply obtain the number of non-negative integral solutions of this equation, which is equal to ${}^{14}C_4$.

95. (4) Suppose i^{th} person receives ₹ x_i; $i = 1, 2, 3, 4$

Then, $x_1 + x_2 + x_3 + x_4 = 18$, where $x_i \geq 3$

Let $y_i = x_i - 3$, $i = 1, 2, 3, 4$. Then,

$$y_1 + y_2 + y_3 + y_4 = 6$$

The total number of ways is equal to number of solutions of the above equation, which is given by ${}^{6+4-1}C_{4-1} = {}^9C_3 = 84$.

96. (1) Since the total number of selections of r things from n things where each thing can be repeated as many times as one can is ${}^{n+r-1}C_r$. Therefore, the required number is ${}^{6+3-1}C_6 = 28$.

97. (1) $15 < x_1 + x_2 + x_3 \le 20$

$\Rightarrow \quad x_1 + x_2 + x_3 = 16 + r,\ r = 0, 1, 2, 3, 4$

Now, the number of positive integral solution of

$x_1 + x_2 + x_3 = 16 + r$ is ${}^{13+r+3-1}C_{13+r}$,

i.e., ${}^{15+r}C_{13+r} = {}^{15+r}C_2$

Thus, the total number of solutions is

$$\sum_{r=0}^{4} {}^{15+r}C_2 = {}^{15}C_2 + {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2$$

$$= \frac{1}{2}(15\times14+16\times15+17\times16+18\times17+19\times18)$$

$$= 685$$

Multiple Correct Answers Type

1. (2), (3)

If a, b, c are in A.P., then a and c both are odd or both are even.

Case I: n is even.

The number of ways of selection of two even numbers a and c is ${}^{n/2}C_2$. Number of ways of selection of two odd numbers is ${}^{n/2}C_2$. Hence the number of A.P.'s is

$$2\,{}^{n/2}C_2 = 2\,\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{2} = \frac{n(n-2)}{4}$$

Case II: n is odd.

The number of ways of selection of two odd numbers a and c is ${}^{(n+1)/2}C_2$. The number of ways of selection of two even numbers a and c is ${}^{(n-1)/2}C_2$. Hence, the number of A.P.'s is

$${}^{(n+1)/2}C_2 + {}^{(n-1)/2}C_2$$

$$= \frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}-1\right)}{2} + \frac{\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}-1\right)}{2}$$

$$= \frac{1}{8}(n-1)\big((n+1)+(n-3)\big)$$

$$= \frac{(n-1)^2}{4}$$

2. (2), (3)

The number of ways of inviting, with the couple not included, is ${}^8C_5$. The number of ways of inviting with the couple included, is ${}^8C_3$. Therefore the required number of ways is

$${}^8C_5 + {}^8C_3 = {}^8C_3 + {}^8C_3 \quad (\because {}^8C_5 = {}^8C_3)$$

Also,

$${}^{10}C_5 - 2\times{}^8C_4 = \frac{10!}{5!5!} - 2\times\frac{8!}{4!4!}$$

$$= \frac{10\times9\times8\times7\times6}{120} - 2\times\frac{8\times7\times6\times5}{24}$$

$$= 9\times4\times7 - 140$$

$$= 112 = 2\times\frac{8!}{3!5!}$$

3. (1), (2), (4)

$p = {}^5C_4 \times {}^2C_1 = 10$

$q = {}^5C_2({}^2C_1)^3 = 80$

$r = {}^5C_0({}^2C_1)^5 = 32$

$\Rightarrow \quad 2q = 5r,\ 8p = q$ and $2(p + r) > q$

4. (1), (2), (4)

Clearly, each player will play 9 games. And total number of games is ${}^{10}C_2 = 45$. Clearly,

$w_i + l_i = 9$ and $\Sigma w_i = \Sigma l_i = 45$

$\Rightarrow \quad w_i = 9 - l_i \Rightarrow w_i^2 = 81 + l_i^2 - 18l_i$

$\Rightarrow \quad \Sigma w_i^2 = 81 \times 10 + \Sigma l_i^2 - 18 \times 45 = \Sigma l_i^2$

5. (2), (3), (4)

When $z = n + 1$, we can choose x, y from $\{1, 2, \ldots, n\}$.

When $z = n + 1$, x, y can be chosen in n^2 ways and when $z = n$, x, y can be chosen in $(n-1)^2$ ways and so on. Therefore, the number of ways of choosing triplets is

$$n^2 + (n-1)^2 + \cdots + 1^2 = \frac{1}{6}n(n+1)(2n+1)$$

Alternatively triplets with $x = y < z,\ x < y < z,\ y < x < z$ can be chosen in ${}^{n+1}C_2,\ {}^{n+1}C_3,\ {}^{n+1}C_3$ ways. Therefore,

$${}^{n+1}C_2 + 2({}^{n+1}C_3) = {}^{n+2}C_2 + {}^{n+1}C_3 = 2({}^{n+2}C_3) - {}^{n+1}C_2$$

6. (1), (2), (3)

$$\frac{(200)!}{\underbrace{2!\,2!\ldots2!}_{100\text{ times}}(100)!} = \frac{(200)!}{100!\,2^{100}} = 1\times3\times5\cdots199$$

Also, $\dfrac{(200)!}{100!\,2^{100}} = \left(\dfrac{101}{2}\right)\left(\dfrac{102}{2}\right)\cdots\left(\dfrac{200}{2}\right)$

7. (1), (2), (3), (4)

8, 7, 6, 4, 2, x and y

Any number is divisible by 3 if sum of digits is divisible by 3, i.e., $x + y + 27$ is divisible by 3. x and y can take values from 0, 1, 3, 5, 9. Possible pairs are (5, 1) (3, 0) (9, 0) (9, 3) and (1, 5), (0, 3) (0, 9) (3, 9).

8. (1), (4)

Problem is same as dividing 17 identical things in two groups.

$\therefore \quad n = \dfrac{17+1}{2} = 9$

There is no effect if two diamonds are different as necklace can be flipped over. Hence, $n = m = 9$.

9. (1), (2), (3)

Let the number of regions for n circles be $f(n)$. Clearly, $f(1) = 2$.

Now,

$f(n) = f(n-1) + 2(n-1),\ \forall\ n \ge 2$

$\Rightarrow \quad f(n) - f(n-1) = 2(n-1)$

Putting $n = 2, 3, \ldots, n$, we get

$f(n) - f(1) = 2(1 + 2 + 3 + \cdots (n-1)) = (n-1)n$

$\Rightarrow \quad f(n) = n(n-1) + 2 = (n^2 - n + 2)$ (which is always even)

$\Rightarrow \quad f(20) = 20^2 - 20 + 2 = 382$

Also, $n^2 - n + 2 = 92$

or $\quad n^2 - n - 90 = 0$ or $n = 10$

10. (1), (3)

$3^p = (4-1)^p = 4\lambda_1 + (-1)^p$

$5^q = (4+1)^q = 4\lambda_2 + 1$

$7^r = (8-1)^r = 4\lambda_3 + (-1)^r$

Hence, any positive integer power of 5 will be in the form of $4\lambda_2 + 1$. Even power of 3 and 7 will be in the form of $4\lambda + 1$ and odd power of 3 and 7 will be in the form of $4\lambda - 1$. Hence, both p and r must be odd or both must be even. Thus $p + r$ is always even. Also, $p + q + r$ can be odd or even.

11. **(1), (2), (3)**

When $n = 3k$, there are exactly $n/3$ integers of each type $3p$, $3p+1$, $3p+2$.

Now, sum of three selected integers is divisible by 3. Then either all the integers are of the same type $3p$, $3p+1$, or $3p+2$ or one-one integer from each type. Then number of selection ways is ${}^{n/3}C_3 + {}^{n/3}C_3 + {}^{n/3}C_3 + ({}^{n/3}C_1)({}^{n/3}C_1)({}^{n/3}C_1) = 3({}^{n/3}C_3) + (n/3)^3$.

If $n = 3k+1$, then there are $(n-1)/3$ integers of the type $3p$, $3p+2$, and $(n+2)/3$ integers of the type $3p+1$. Then number of selection ways is

$$2({}^{(n-1)/3}C_3) + ({}^{(n+2)/3}C_3) + ((n-1)/3)^2((n+2)/3).$$

When $n = 3k+2$, the number of selection ways are same as in the case of $n = 3k+1$.

12. **(1), (3)**

$${}^{n+5}P_{n+1} = \frac{11(n-1)}{2} \times {}^{n+3}P_n$$

or $${}^{n+5}P_{n+1} = \frac{(n+5)!}{4!} = \frac{11(n-1)}{2}\frac{(n+3)!}{3!}$$

or $$(n+5)(n+4) = 22(n-1)$$

After solving, we get $n = 6$ or $n = 7$.

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C_2 \equiv 15$ or 21.

13. **(1), (2), (4)**

Total number of units to be covered is $3 + 7 + 11 = 21$. A person can choose 3 units in ${}^{21}C_3$ ways. A person can choose 7 units in ${}^{18}C_7$ ways. The rest 11 units can be chosen in 1 way. Therefore, total number of ways is ${}^{21}C_3 \times {}^{18}C_7 \times 1 = 21!/(3!7!11!)$.

For correct answers (1) and (4), see the respective theory.

For correct answer (2) see the theory of multinomial expansion in the binomial theorem.

14. **(1), (3)**

Let person P_i get x_i number of things such that

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 30$$

If x_i is odd or $x_i = 2\lambda_i + 1$, where $\lambda_i \geq 0$, then

$$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) + 6 = 30$$

or $$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 12$$

Then number of solutions is ${}^{12+6-1}C_{6-1} = {}^{17}C_5$. If x_i is even

or $x_i = 2\lambda_i$, where $\lambda_i \geq 1$, then

$$2(\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6) = 30$$

or $$\lambda_i + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 15$$

Therefore, required number of ways is ${}^{15-1}C_{6-1} = {}^{14}C_5$.

15. **(1), (3)**

Let x_i $(1 \leq i \leq n)$ be the number of objects selected of the ith type. Since each object is to be selected at least once, we must have $x_i \geq 1$ and $x_1 + x_2 + \cdots + x_n = r$. We have to find the number of positive integral solutions of the above equation. Total number of such solutions is ${}^{r-1}C_{n-1} = {}^{r-1}C_{r-n}$.

16. **(1), (4)**

Let $A = \{a_1, a_2, \ldots, a_n\}$. For each $a_i (1 \leq i \leq n)$, we have either $a_i \in P_j$ or $a_i \notin P_j$ $(1 \leq j \leq m)$. That is, there are 2^m choices in which a_i $(1 \leq i \leq n)$ may belong to the P_j's. One of these, there is only one choice, in which $a_i \in P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cap P_2 \cap \cdots \cap P_m$ to be ϕ. Thus, $a_i \notin P_1 \cap P_2 \cap \cdots \cap P_m$ in $2^m - 1$ ways.

Since there are n elements in set A, the total number of choices is $(2^m - 1)^n$.

Also, there is exactly one choice, in which, $a_i \notin P_j$ for all $j = 1, 2, \ldots, m$ which is not favourable for $P_1 \cup P_2 \cup \cdots \cup P_m$ to be equal to A.

Thus, a_i can belong to $P_1 \cup P_2 \cup \cdots \cup P_m$ in $(2^m - 1)$ ways.

Since there are n elements in set A, the number of ways in which $P_1 \cup P_2 \cup \cdots \cup P_m$ can be equal to A is $(2^m - 1)^n$.

17. **(1), (2), (4)**

Exponent of 2 is

$$\left[\frac{10}{2}\right] + \left[\frac{10}{2^2}\right] + \left[\frac{10}{2^3}\right] = 5 + 2 + 1 = 8$$

Exponent of 3 is

$$\left[\frac{10}{3}\right] + \left[\frac{10}{3^2}\right] = 3 + 1 = 4$$

Exponent of 5 is

$$\left[\frac{10}{5}\right] = 2$$

Exponent of 7 is

$$\left[\frac{10}{7}\right] = 1$$

The number of divisors of 10! is $(8+1)(4+1)(2+1)(1+1) = 270$. The number of ways of putting N as a product of two natural numbers is $270/2 = 135$.

18. **(2), (3), (4)**

$$P = 21(21+1)(21-1)(21+2)(21-2)\cdots(21+10)(21-10)$$
$$= (21-10)(21-9)\cdots(21-1)\,21\,(21+1)(21+10)$$
$$\cdots(21+10)$$
$$= 11 \times 12 \times \cdots \times 30 \times 31$$

which is divisible by 21!, and hence by 20! and 19!

19. **(1), (4)**

When n is odd:

8 ways as 0 does not appear | 8 | 7 | 5 ways, i.e., with 1, 3, 5, 7, 9

The number of such numbers is $8 \times 8 \times 7 \times 5 = 2240$.

When n is even:

If unit's place is filled with 0, then the total number is $9 \times 8 \times 7 = 504$.

8 ways as 0 does not appear | 8 | 7 | 4 ways i.e., with 2, 4, 6, 8

If unit's place is not filled with 0, then the total number is $8 \times 8 \times 7 \times 4 = 1792$.

Hence, the total number of even numbers is $y = 504 + 1792 = 2296$.

Linked Comprehension Type

1. (1) When one all-rounder and 10 players from bowlers and batsmen play, number of ways is ${}^4C_1\ {}^{14}C_{10}$.

When one wicketkeeper and 10 players from bowlers and batsmen play, number of ways is ${}^2C_1\ {}^{14}C_{10}$.

When one all-rounder, one wicketkeeper, and nine from batsmen and bowlers play, number of ways is ${}^4C_1\ {}^2C_1\ {}^{14}C_9$.

When all 11 players play from bowlers and batsmen, then number of ways is ${}^{14}C_{11}$.

Total number of selections is ${}^4C_1\ {}^{14}C_{10} + {}^2C_1\ {}^{14}C_{10} + {}^4C_1\ {}^2C_1\ {}^{14}C_9 + {}^{14}C_{11}$.

2. (2) If a particular bowler plays, then two batsmen will not play. So, rest of 10 players can be selected from 17 other players. Number of such selections is ${}^{17}C_{10}$.

If a particular bowler does not play, then number of selections is ${}^{19}C_{11}$.

If all the three players do not play, then number of selections is ${}^{17}C_{11}$.

Total number of selections is ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$.

3. (2) If the particular batsman is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If particular wicketkeeper is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If both are not selected, then number of ways is ${}^{18}C_{11}$.

Hence, total number of ways is $2\ {}^{18}C_{10} + {}^{18}C_{11} = {}^{19}C_{11} + {}^{18}C_{10}$.

4. (3) Seven persons can be selected for the first table in ${}^{12}C_7$ ways. Now these seven persons can be arranged in 6! ways. The remaining five persons can be arranged on the second table in 4! ways. Hence, total number of ways is ${}^{12}C_5 6!4!$.

5. (2) Here, A can sit on first table and B on the second or A on the second table and B on the first table.

If A is on the first table, then remaining six for first table can be selected in ${}^{10}C_6$ ways. Now these seven persons can be arranged in 6! ways. Remaining five can be arranged on the other table in 4! ways.

Hence, total number of ways is $2\ {}^{10}C_6\ 6!4!$.

6. (1) If A, B are on the first table, then remaining five can be selected in ${}^{10}C_5$ ways.

Now seven persons including A and B can be arranged on the first table in which A and B are together in 2!5! ways. Remaining five can be arranged on the second table in 4! ways. Total number of ways is ${}^{10}C_5\ 4!5!2!$

If A, B are on the second table, then remaining three can be selected in ${}^{10}C_3$ ways.

Now five persons including A and B can be arranged on the second table in which A and B are together in 2!3! ways. Remaining seven can be arranged on the first table in 6! ways. Hence, total number of ways ${}^{10}C_7\ 6!3!2!$.

7. (2) 8. (3), 9. (4)

If no box remains empty, then we can have (1, 1, 3) or (1, 2, 2) distribution pattern.

7. (2) When balls are different and boxes are identical, number of distributions is equal to number of divisions in (1, 1, 3) or (1, 2, 2) ways. Hence, total number of ways is

$$\frac{5!}{(1!)^2\,3!\,2!} + \frac{5!}{1!\,(2!)^2\,2!} = 25$$

8. (3) When balls as well as boxes are identical, we have only two ways (1, 1, 3) and (1, 2, 2). Hence, number of ways is 2.

9. (4) When boxes are kept in a row, they will be treated as different. In this case, the number of ways will be ${}^{5-1}C_{3-1} = {}^4C_2 = 6$.

10. (2) 11. (3), 12.(2)

$6 = 0(2) + 6(1) = 1(2) + 4(1) = 2(2) + 2(1) = 3(2) + 0(1)$

Number of 2s	Number of 1s	Number of permutations
0	6	1
1	4	$\frac{5!}{4!}=5$
2	2	$\frac{4!}{2!\,2!}=6$
3	0	$\frac{3!}{3!}=1$
		Total = 13

$\therefore \quad f(6) = 13$

Now, $f(f(6)) = f(13)$

Number of 1s	Number of 2s	Number of permutations
13	0	1
11	1	$\frac{12!}{11!}=12$
9	2	$\frac{11!}{9!2!}=55$
7	3	$\frac{10!}{7!\,3!}=120$
5	4	$\frac{9!}{5!4!}=126$
3	5	$\frac{8!}{3!5!}=56$
1	6	$\frac{7!}{6!}=7$
		Total = 377

$\therefore \quad f(f(6)) = f(13) = 377$

$f(1) = 1$ (1)

$f(2) = 2$ (1, 1 or 2)

$f(3) = 3$ (1, 1, 1 or 2, 1 or 1, 2)

$f(4) = 5$ (explained in the paragraph)

By taking higher value of n in $f(n)$, we always get more value of $f(n)$. Hence, $f(x)$ is one-one. Clearly, $f(x)$ is into.

13. (3) Let x_0 denote the number of empty seats to the left of the first person, x_i $(1 \le i \le n-1)$ the number of empty seats between the ith and $(i+1)$th persons of the nth person. Let x_n be the number of seats after the last person. Then $x_0, x_n \ge 0$ and $x_i \ge 1$ for $1 \le i \le n-1$.

$$x_0 + x_1 + \cdots + x_n = (m-n) \qquad (1)$$

Putting $x_i = y_i + 1$, where $y_i \ge 0$, we have

$$x_0 + y_1 + \cdots + y_{n-1} + x_n + (1 + 1 + 1 + \cdots + (n-1) \text{ times}) = (m-n)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - n - (n-1)$$

$$\Rightarrow \quad x_0 + y_1 + \cdots + y_{n-1} + x_n = m - 2n + 1$$

Now number of non-negative integral solutions is
$${}^{n+1+(m-2n+1)-1}C_{n+1-1} = {}^{m-n+1}C_n.$$

Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-n+1}C_n\right)(n!) = \frac{(m-n+1)!}{n!(m-2n+1)!}\,n! = \frac{(m-n+1)!}{(m-2n+1)!}$$

14. (1) Let $n = 2k$, where k is some positive integer. Let x_0 denote the number of empty seats to the left of the first pair, $x_i\,(1 \le i \le k-1)$ the number of empty seats between ith and $(i+1)$th pair and x_k the number of empty seats to the right of the kth pair. Note that $x_0, x_k \ge 0$, $x_i \ge 1\ (1 \le i \le k-1)$ and

$$x_0 + x_1 + \cdots + x_k = (m - 2k) \qquad (2)$$

The number of integral solutions of Eq. (2) is ${}^{m-2k+1}C_k$ (see the solution to problem 13).

Since we can permute n persons in $n!$ ways, the required number of ways is

$$\left({}^{m-2k+1}C_k\right)(2k)! = \frac{(m-2k+1)!}{k!(m-3k+1)!}(2k)!$$

$$= \frac{(2k)!}{(k)!}\frac{(m-2k+1)!}{(m-3k+1)!}$$

$$= ({}^{2k}P_k)\,({}^{m-2k+1}P_k)$$

$$= ({}^{n}P_{n/2})\,({}^{m-n+1}P_{n/2})$$

15. (4) m is even. Let $m = 2k$, where k is some positive integer. We can choose n seats out of the k seats to the left of the middle seat in ${}^kC_n$ ways. Each chosen seat can be either empty or occupied. Thus, the number of ways of choosing seats for n persons is equal to $({}^kC_n)\,(2^n)$. We can arrange n persons at these seats in ${}^nP_n$ ways. Hence, the required number of arrangements is given by

$$(n!)\,({}^kC_n)\,(2^n) = ({}^kP_n)\,(2^n) = ({}^{m/2}P_n)(2^n)$$

16. (4) Since there are 5 even places and 3 pairs of repeated letters, therefore at least one of these must be at an odd place. Therefore, the number of ways is 11!/(2!2!2!).

17. (2) Make a group of both M's and another group of T's. Then except A's we have 5 letters remaining. So M's, T's, and the letters except A's can be arranged in 7! ways. Therefore, total number of arrangements is $7! \times {}^8C_2$.

18. (3) Consonants can be placed in 7!/(2!2!) ways. Then there are 8 places and 4 vowels. Therefore, number of ways is

$$\frac{7!}{2!2!}\,{}^8C_4\,\frac{4!}{2!}$$

19. (1) 13 places for spades can be selected in ${}^{52}C_{13}$ ways.

In these places, spades can be arranged in only one ways as there are in increasing order of denomination.

The remaining 39 cards can be arranged in 39!

So, total number of arrangements $= {}^{52}C_{13} \times 39! = \dfrac{52!}{13!}$.

20. (3) There are 13 red cards of diamond and 13 red cards of heart.

26 places for red cards can be selected in ${}^{52}C_{26}$ ways.

Since red cards appear in non–decreasing order of denominations in first two positions of above selected, put two red cards of denomination '1' in 2! ways. In next two positions put two red cards of denomination '2' in 2! ways and so on.

So, total number of arrangements

$= {}^{52}C_{26} \times 2 \times 2 \times \ldots \times 2$ (13 times) $= {}^{52}C_{26} \times 2^{13}$.

21. (4) Since cards appear in non–decreasing order of denomination, cards of same denomination are together.

i.e., first four places, put four cards of denomination '1' in 4! ways. In next four places, put four cards of denomination '2' in 4! ways and so on.

So, total number of arrangements $= (4!)^{13}$

22. (2) 13 places for spade cards can be selected in ${}^{52}C_{13}$ ways in which spade cards can be arranged in only one way in increasing order.

Similarly, we select 13 places for each suit.

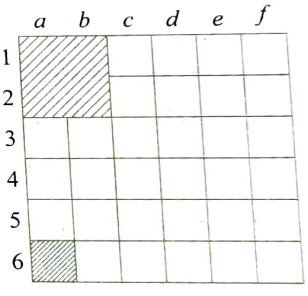
So, number of arrangements $= {}^{52}C_{13} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$

$$= \frac{52!}{13!13!13!13!}$$

Matrix Match Type

1. **a → r; b → s; c → p; d → q.**
   - **a.** ${}^{10}C_2 - {}^4C_2 + 1 = 45 - 6 + 1 = 40$
   - **b.** $1 \times {}^{10}C_2 = 45$
   - **c.** $2 \times {}^6C_2 = 30$
   - **d.** ${}^6C_2 \times 4 = 60$

2. **a → q; b → r; c → s; d → p.**
   - **a.** Number of rectangles is equal to number of ways we can select two vertical lines and two horizontal lines. Total number of ways is ${}^7C_2 \times {}^7C_2 = 441$.
   - **b.** If the square is of 1 sq. units such as $a6$, then we have such $6 \times 6 = 36$ squares.



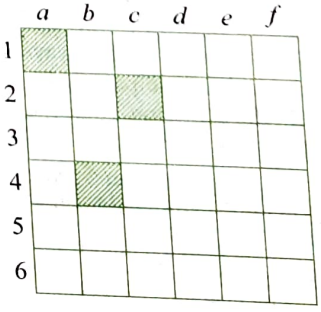
If the square is of 4 sq. units like the shaded region of the squares $a1$, $a2$, $b1$, $b2$, then we have such 5 squares in the belt formed by rows 1 and 2. Similarly, we have 4 more belts 23, 34, 45, and 56. Hence, there are $5 \times 5 = 25$ such squares.

Similarly, we have 4×4, 3×3, 2×2, 1×1 squares of increasing sizes.

Hence, total number of squares is

$$1 + 4 + 9 + 16 + 25 + 36 = 91.$$

   - **c.** The first square can be selected in 36 ways. If one such square $a1$ is selected, we are left with 25 squares; second square cannot be selected from row 1 and column a. If second square is $c2$, we are left with 16 squares, from which third square can be selected, e.g., $b4$.



Hence, number of ways of selections is $36 \times 25 \times 16$. But in this one-by-one type of selection order of selection is also considered. Hence, actual number of ways is $(36 \times 25 \times 16)/3! = 2400$.

d. Given number of ways is equivalent to selecting 11 squares from 36 squares if no row remains empty.

Suppose $x_1, x_2, x_3, x_4, x_5, x_6$ be the number of squares selected from the 1st, 2nd, 3rd, 4th, 5th and 6th row.

Then we must have $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 11$ (where $1 \le x_i \le 6$).

The number of positive integral solutions of the above equation is ${}^{11-1}C_{6-1} = {}^{10}C_5$.

3. a → p; b → r; c → q; d → s.

a. Number of surjective functions is

$$3^6 - {}^3C_1(3-1)^6 + {}^3C_2(3-2)^6 = 729 - 192 + 3 = 540$$

b. If $f(a_i) \neq b_i$, then pre-image a_1, a_2, a_3 cannot be assigned images b_1, b_2, b_3, respectively.

Hence, each of a_1, a_2, a_3 can be assigned images in 2 ways. a_4, a_5, a_6 can be assigned images in 3 ways each.

Hence, number of functions is $2^3 3^3 = 216$.

c. One-one functions are not possible as pre-images are more than images.

d. Number of many-one functions is

Total number of functions – Number of one-one functions $= 3^6 - 0 = 729$

4. a → p, s; b → q, r; c → p, s; d → r.

a. Total number of required functions is equal to number of derangement of 5 objects, which is given by

$$5!\left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44$$

b. $x_1 x_2 x_3 = 2 \times 35 \times 7 = 2 \times 49 \times 5 = 10 \times 7 \times 7 = 14 \times 7 \times 5$

So total number of solution set is $3 \times 3! + 3!/2! = 21$.

c. $3780 = 2^2 \times 3^3 \times 5 \times 7$

Number of divisors which are divisible by 2 but not by 3 is $2 \times 2 \times 2 = 8$.

Number of divisors which are divisible by 3 but not by 2 is $3 \times 2 \times 2 = 12$.

Number of divisors which are divisible by 2 as well as 3 is $2 \times 3 \times 2 \times 2 = 24$.

Hence, total number of divisors is 44.

d. $4\lambda + 2 = 2(2\lambda + 1)$ = odd multiple of 2

Thus, total number of divisors is $1 \times 5 \times 11 - 1 = 54$. (1 is subtracted and powers of three and five are zero each and this will make $\lambda = 0$.)

5. a → q, s; b → p, r; c → p, r; d → p, r.

a. There are two cases.

(i) 5, 4, 1, 1, 1

Number of ways = 5!/3! = 20.

(ii) 5, 2, 2, 1, 1

Number of ways = 5!/2!2! = 30.

Hence, total number is 20 + 30 = 50.

b. Select 4 pairs in ${}^5C_4 = 5$ ways. Now select exactly one shoe from each of the pairs selected in $({}^2C_1)^4$ ways. This will fulfill the condition. Hence, required answer is $5 \times 16 = 80$.

c. The first child C_1 can be chosen in 3 ways; his/her mother can be interviewed in 5 ways; the second child C_2 can be chosen in 2 ways, and his/her mother can be interviewed in 3 ways.

Hence, total number of ways is $3 \times 5 \times 2 \times 3 = 90$.

d. Required number of ways is

$$5! - 4! - 3! = 120 - 64 - 6 = 90.$$

(Number will be less than 56000 only if either 4 occurs on the first place or 5, 4 occurs on the first two places.)

6. a → r, s; b → p, r; c → p, q; d → r, s

a. We have

$$a = {}^{x+2}P_{x+2} = (x+2)!,$$

$$b = {}^xP_{11} = \frac{x!}{(x-11)!}$$

$$c = {}^{x-11}P_{x-11} = (x-11)!$$

Now, $a = 182\,bc$

$$\Rightarrow \quad (x+2)! = 182 \times \frac{x!}{(x-11)!}(x-11)!$$

$$\Rightarrow \quad (x+2)! = 182x!$$

$$\Rightarrow \quad (x+2)(x+1) = 182$$

$$\Rightarrow \quad x = 12$$

b. ×|×|×|

Even digits occupy odd places shown by crosses. Crosses can be filled in $2 \times 2 \times 1$ ways (since 0 cannot go in the first place from the left). The remaining places can be filled in 3! ways.

Therefore, the required number of numbers is

$$2 \times 2 \times 1 \times 3! = 24.$$

c. Total number of numbers without restriction is 2^5. Two numbers have all the digits equal. So, the required number of numbers is $2^5 - 2$.

d. Let number of sides of polygon be n. Number of sides of polygon is equal to number of vertices of polygon. Now number of diagonals of polygon is

$${}^nC_2 - n = 54$$

or $\dfrac{n(n-1)}{2} - n = 54$

or $n^2 - 3n - 162 = 0$

or $(n-12)(n+9) = 0$

or $n = 12$

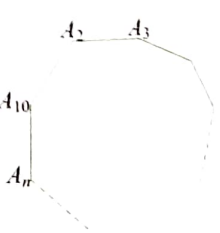
7. (1)

a. If polygon has n sides, then number of diagonals is ${}^nC_2 - n = 35$ (given). Solving we get $n = 10$. Thus, there are 10 vertices, from which ${}^{10}C_3$ (= 120) triangles can be formed.

b. Four vertices can be selected in ${}^{10}C_4$ (= 210) ways. Using these four vertices two diagonals can be formed, which has exactly one point of intersection lying inside the polygon.

Hence, number of points of intersections of diagonal which lies inside the polygon is ${}^{10}C_4 \times 1 = 210$.

c. Suppose one of the sides of the triangle is A_1A_2. Then the third vertex cannot be A_3 or A_{10}. Thus, for the third vertex, six vertices are left. There are six triangles in which side A_1A_2 is common with that of polygon. Similarly, for each of the sides A_2A_3, A_3A_4, ⋯, A_9A_{10}, there are six triangles. Then total number of triangles is 60.



d. Triangles $A_1A_2A_3$, $A_2A_3A_4$, …, $A_8A_9A_{10}$ have two sides common with that of polygon. Hence, there are 10 such triangles.

8. (3)

a. The number of possible outcomes with 2 on at least one dice
= Total number of outcomes with 2 on at least one dice
= (Total number of outcomes) − (Number of outcomes in which 2 does not appear on any dice)
$= 6^4 - 5^4$
$= 1296 - 625 = 671$

b. Any selection of four digits from 10 digits 0, 1, 2, 3, …, 9 gives one number. So, the required number of numbers is ${}^{10}C_4$.

c. Let the number be $n = pqr$. Since $p + q + r$ is even, p can be filled in 9 ways and q can be filled in 10 ways.
r can be filled in number of ways depending upon what is the sum of p and q.
If $p + q$ is odd, then r can be filled with any one of five odd digits.
If $p + q$ is even, then r can be filled with any one of five even digits.
In any case, r can be filled in five ways.
Hence, total number of numbers is $9 \times 10 \times 5 = 450$.

d. After fixing 1 at one position out of 4 places, 3 places can be filled by ${}^7P_3$ ways. But for some numbers whose fourth digit is zero, such type of ways is ${}^6P_2$. Therefore, total number of ways is ${}^7P_3 - {}^6P_2 = 480$.

Numerical Value Type

1. (5) ${}^nP_r = {}^nP_{r+1}$

$$\text{or} \quad \frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \text{ or } n - r = 1 \qquad (1)$$

$$\text{Again } {}^nC_r = {}^nC_{r-1} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r+1)!(r-1)!}$$

$$\text{or} \quad \frac{1}{r} = \frac{1}{n-r+1} \text{ or } n - 2r = -1 \qquad (2)$$

Solving (1) and (2), $n = 3$, $r = 2$

2. (8) We have ${}^nC_2 = 28$
$\Rightarrow \quad n = 8$ (as there are 7 days in week)

3. (4) Number of arrangements are $2\,n!\,n!$
Given that $2\,n!\,n! = 1152$
or $\quad (n!)^2 = 576$
or $\quad n! = 24$
or $\quad n = 4$

4. (8) Let $n(A)$ = Number divisible by 60 = (60, 120, …, 960) = 16
$n(B)$ = Number divisible by 24 = (24, 48, …, 984) = 41
$n(A \cap B)$ = Number divisible by both
$= 120 + 240 + \cdots + 960 = 8$
Hence, $n(A \cap B) = n(A) - n(A \cap B) = 16 - 8 = 8$

(361) We have A's = 2; B's = 4; C's = 2

$$\text{Total words formed} = \frac{8!}{4!\,2!\,2!} = 420 \qquad (1)$$

Let $ABBC$ = '×'

Number of ways in which × $ABBC$ can be arranged = $\frac{5!}{2!} = 60$
but this includes × $ABBC$ and $ABBC$ ×.

But the word $ABBCABBC$ is counted twice in 60 hence it should be 59
Hence, required number of ways = 420 − 59 = 361

6. (18) Let T and S denote teacher and student, respectively.
Then we have following possible patterns according to question:
(i) T S S T S S T S S
(ii) S T S S T S S T S
(iii) S S T S S T S S T
Hence, total number of arrangements are $3 \cdot (3!)6! = 18 \times 6!$

7. (8) To form a triangle, 3 points out of 5 can be chosen in ${}^5C_3 = 10$ ways.
But of these, the three points lying on the 2 diagonals will be collinear.
So 10 − 2 = 8 triangles can be formed

8. (64) Here A is common letter in words SUMAN and DIVYA
Now for selecting six different letters we must select A either from word SUMAN or from word DIVYA.
Hence, for possible selections, we have
A excluded from SUMAN + A included in SUMAN
$= {}^4C_3 \cdot {}^5C_3 + {}^4C_2 \cdot {}^4C_3 = 40 + 24 = 64$

9. (5) Let r no. of books of algebra and $20 - r$ of calc. no. of selections
$= {}^rC_5 \times {}^{20-r}C_5$
Which has maximum value when $r = 10$

10. (9) Number of digits are 9
Select 2 places for the digit 1 and 2 in ${}^9C_2$ ways
From the remaining 7 places, select any two places for 3 and 4 in ${}^7C_2$ ways and from the remaining 5 places, select any two for 5 and 6 in ${}^5C_2$ ways.
Now, the remaining 3 digits can be filled in 3! ways
$\therefore$ Total ways $= {}^9C_2 \cdot {}^7C_2 \cdot {}^5C_2 \cdot 3!$

$$= \frac{9!}{2!\cdot 7!} \cdot \frac{7!}{2!\cdot 5!} \cdot \frac{5!}{2!\cdot 3!} \cdot 3!$$

$$= \frac{9!}{8} = \frac{9 \cdot 8 \cdot 7!}{8} = 9 \cdot 7!$$

11. (960) Including the two specified people, 4 others can be selected in ${}^5C_4$ ways.
The two adjacent seats can be taken in 4 ways and the two specified people can be arranged in 2!ways, remaining 4 people can be arranged in 4! ways. Therefore,
${}^5C_4 \cdot 4 \cdot 2!4! = 960$

12. (162) | x | x | | when two consecutive digits are 11, 22, etc = 9 · 9 = 81

| | 0 | 0 | when two consecutive digits are 0 0 = 9

| | x | x | when two consecutive digits are 11, 22, 33, ... = 9 . 8 = 72

Total number of numbers are N = 162.

13. (24) We have N = | a | b | c | d |
First place a can be filled in 2 ways, i.e., 4, 5 ($4000 \le N < 6000$)
For b and c, total possibilities are 6 ($3 \le b < c \le 6$)
i.e., 34, 35, 36, 45, 46, 56
Last place d can be filled in 2 ways, i.e., 0, 5 (N is a multiple of 5)
Hence, total numbers = 2 × 6 × 2 = 24

14. (10) A AAAA | B BBBBB
Since word reads the same backwards and forwards, the middle digit must be A.

M

× × × × × ↓ × × × × ×

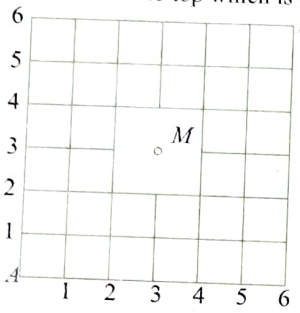
so that even number of A's and B's are available for arrangement about middle position M in the above figure.

Take AABBB on one side of M (6th place) and then their image about M in a unique way

$\therefore$ Number of ways, $N = \dfrac{5!}{2!\cdot 3!} = 10$

15. (524) Here the path which leads from A to B is of length 12 units.

Now without considering the constrain of passing through the point M, number of ways in which we can reach B from A is equal to number of ways we can select 6 steps from left to right and 6 from bottom to top which is equal to ${}^{12}C_6$.



Now we can reach from A to M in 6 steps in ${}^6C_3$ ways and can reach from M to B in ${}^6C_3$ ways.

Hence, we can reach from A to B through M in ${}^6C_3 \times {}^6C_3$ ways.

Hence, required number of ways $= {}^{12}C_6 - [{}^6C_3 \times {}^6C_3] = 924 - 400 = 524$

16. (6) Number of numbers beginning with 1 = 120

Number of numbers beginning with 2 = 120

Starting with 31 ..... = 24

Starting with 3214 ..... = 2

Finally = 1

1					
2					
3	1				
3	2	1	4		
3	2	1	5	4	6

Hence, unit place digit of 267th number is 6.

17. (21) If x denotes the number of times he can take unit step and y denotes the number of times he can take 2 steps, then $x + 2y = 7$.

We must have $x = 1, 3, 5,$

If $x = 1$, the steps will be 1 2 2 2

$\Rightarrow$ Number of ways $= \dfrac{4!}{3!} = 4$

If $x = 3$, the steps will be 1 1 1 2 2

$\Rightarrow$ Number of ways $= \dfrac{5!}{2!.3!} = 10$

If $x = 5$, the steps will be 1 1 1 1 1 2

$\Rightarrow$ Number of ways $= {}^6C_1 = 6$

If $x = 7$, the steps will be 1 1 1 1 1 1 1

$\Rightarrow$ ${}^7C_0 = 1$

Hence total number of ways is 21

18. (630) 3 women can be selected in ${}^7C_3$ ways and can be paired with 3 men in 3! ways.

Remaining 4 women can be grouped into two couples in $\dfrac{4!}{2!\cdot 2!\cdot 2!} = 3$

$\therefore$ Total $= {}^7C_3 \cdot 3! \cdot 3 = 630$

Then the value of $N/90$ is 7

19. (8) Here $P_n = {}^{n-2}C_3$ and $P_{n+1} = {}^{n-1}C_3$

Hence, ${}^{n-1}C_3 - {}^{n-2}C_3 = 15$

or ${}^{n-2}C_3 + {}^{n-2}C_2 - {}^{n-2}C_3 = 15$

or ${}^{n-2}C_2 = 15$ or $n = 8$

20. (510)

(i) He can invite 2 friends three times each

Let us select first those 2 friends in ${}^3C_2$ ways

Now these two friends each three times can be invited on 6 days in $\dfrac{6!}{3!3!}$

Thus total number of ways 2 friends can be invited three times $= {}^3C_2 \dfrac{6!}{3!3!}$

(ii) Another possibility is that he invites all three friends two times each. Then

Number of ways $= \dfrac{6!}{2!2!2!}$

(iii) One more possibility is that he invites one friend three times, one two times, and one three times. Then

Number of ways $= \dfrac{6! \times 6}{3!2!}$

Hence, total number of ways

$$= {}^3C_2 \times \frac{6!}{3!3!} + \frac{6! \times 6}{3!2!} + \frac{6!}{2!2!2!} = 510$$

21. (8) $\displaystyle\sum_{k=r}^{n} {}^kC_r = {}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r$

$= 1 + {}^{r+1}C_1 + {}^{r+2}C_2 + {}^{r+3}C_3 + \cdots + {}^nC_{n-r}$

$= {}^{r+1}C_0 + {}^{r+1}C_1 + {}^{r+2}C_2 + \cdots + {}^nC_{n-r}$

$= {}^{r+2}C_1 + {}^{r+2}C_2 + \cdots + {}^nC_{n-r}$

$= {}^{r+3}C_2 + \cdots + {}^nC_{n-r}$ and so on finally ${}^{n-1}C_{n-r}$

Now, ${}^{n+1}C_{n-r} = {}^{n+1}C_{r+1}$

$\therefore \quad f(n) = \displaystyle\sum_{r=0}^{n} {}^{n+1}C_{r+1} = {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \ldots + {}^{n+1}C_{n+1}$

$= {}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_{n+1} - 1$

$f(n) = (2^{n+1}) - 1$

$f(9) = 2^{10} - 1 = 1023 = 3 \cdot 11 \cdot 31$

Hence, number of divisors are $(1 + 1)(1 + 1)(1 + 1) = 8$

22. (191)

The whole family has 24 children. Children of Geeta and Sohan are $24 - x - (x + 1) = 23 - 2x$ in number. When Sohan's child fights with a Geeta's child, there are $x(x + 1)$ fights. When Sohan's child fights with Sohan and Geeta's children, there are $x(23 - 2x)$ fights. Again, when Geeta's child fights with Sohan and Geeta's children, there are $(x + 1)(23 - 2x)$ fights. Therefore, total number of fights is

$$F(x) = x(x + 1) + x(23 - 2x) + (x + 1)(23 - 2x)$$

$$= 23 + 45x - 3x^2$$

$$= 3\left[\frac{23}{3} + \frac{225}{4} - \left(x - \frac{15}{2}\right)^2\right]$$

For $F(x)$ to be maximum,

$$x - \frac{15}{2} = 0,$$

i.e., $x = 7.5$

Since x is an integer, therefore, $x = 7$ or 8. For both $x = 7$ and $x = 8$. Total number of fights is $F(x) = 191$.

23. (9) $5400 = 2^3 \times 3^3 \times 5^2$

Since divisor is divisible by 6 but not by 9, divisor must be even, i.e., 2 must occur at least once. Also 3 must occur exactly once.

So, number of required divisors $= 3 \times 1 \times 3 = 9$

14. (30) First we will consider a face to be fixed with a specific colour and find out the relative arrangements of the rest.

When we have fixed a colour , the opposite face can be painted with the rest 5 colours , thus there are 5 choices.

Now the rest four face we can calculate using the circular permutations.

That is 4 colours can be painted on the remaining 4 faces in $(4-1)! = 3!$

Thus total number ways $= 5 \times 3! = 30$

15. (180) Consider the cuboid has dimensions $x \times y \times z$, that is there are three pairs of different faces, the x × y faces, the $x \times z$ faces and the $y \times z$ faces.

Suppose that one of the colours we are using is red. There are 3 choices as to which type of face will be painted red. Once we have made that choice, rotate the cuboid until the red face is underneath. There are 5 choices for the colour of the topmost face (the other face of the same type as the one painted red).

Consider one of the four remaining colours. There are two choices as to which type of face will be painted with that colour. Once we have made that choice, rotate the cuboid so that the red face is still underneath, and our newly painted face is facing the front. We cannot rotate the cuboid any more, so there are 3! ways of painting the remaining three faces.

So, total of $3 \times 5 \times 2 \times 3! = 180$

Alternative method:

There are ${}^6C_2$ choices for the pair of colours to paint the $x \times y$ faces, then ${}^4C_2$ further choices for the colours with which to paint the $x \times z$ faces, with the remaining two colours going with the last pair of faces. There are just two ways of painting the cuboid with these colour pairs, so the number of colourings is

$${}^6C_4 \times {}^4C_2 \times 2 = 180$$

Archives

JEE MAIN

Single Correct Answer Type

1. (4) Four novels can be selected from six novels in ${}^6C_4$ ways. One dictionary can be selected from three dictionaries in ${}^3C_1$ ways. As the dictionary selected is fixed in the middle, the remaining four novels can be arranged in 4 ! ways.

Hence, the required number of ways of arrangement is

$${}^6C_4 \times {}^3C_1 \times 4! = 1080.$$

2. (1) Number of ways

= number of ways in which we select two balls from each box

$= {}^3C_2 \times {}^9C_2 = 3 \times 36 = 108$

3. (3) Number of ways to distribute 10 identical balls in four distinct boxes such that no box remains empty $= {}^{10-1}C_{4-1} = {}^9C_3$

Number of ways to select 3 different places from 9 places $= {}^9C_3$

Clearly, statement 2 is not a correct explanation of statement 1.

4. (2) Every element has 3 options. Either set Y or set Z or none.

So, number of ordered pairs $= 3^5$.

5. (4) Number of ways of selecting zero or more balls from 10 identical white balls $= 1 + 1 + 1 \ldots$ (11 times)

$= 11$

Similarly, numbers of ways of selecting zero or more balls from 9 identical green balls and 7 identical black balls will be $(9 + 1)$ and $(7 + 1)$ ways respectively.

So total number of ways of selecting one or more balls from all the balls $= 11 \times 10 \times 8 - 1 = 879$.

6. (3) $A = \{x, y\}$

$B = \{a, b, c, d\}$

$A \times B$ having $2 \times 4 = 8$ elements

Total subsets of $A \times B = 2^8 = 256$

$\therefore$ Total no. of subsets of $A \times B$ having 3 or more elements

$= 256 - (1$ null set $+ 8$ singleton set $+ {}^8C_2$ subset having 2 elements$)$

$= 256 - 1 - 8 - 28 = 219$

7. (2) T_n = number of all possible triangles formed by joining vertices of n-sided regular polygon

= selecting three vertices from n vertices $= {}^nC_3$

Given $T_{n+1} - T_n = 10$

$\therefore \quad {}^{n+1}C_3 - {}^nC_3 = 10$

$\Rightarrow \quad (n+1)n(n-1) - n(n-1)(n-2) = 60$

$\Rightarrow \quad n(n-1)(n+1-n+2) = 60$

$\Rightarrow \quad n(n-1) = 20$

$\therefore \quad n = 5$

8. (2) Number of integers greater than 6000 may be 4 digit or 5 digit.

Number of 4 digit numbers $= 3 \times 4 \times 3 \times 2 \times 1$

$= 72$ ($\because$ 1st digit can be 6, 7 or 8)

Number of 5 digit numbers $= 5! = 120$

$\therefore$ Total numbers $= 72 + 120 = 192$

9. (3) SMALL

Words starting with A $= \frac{4!}{2!} = 12$

Words starting with L $= 4! = 24$

Words starting with M $= \frac{4!}{2!} = 12$

Words starting with SA $= \frac{3!}{2!} = 3$

Words starting with SL $= 3! = 6$

Next word is SMALL

$\therefore$ Its position is 58^{th}

10. (2) According to the question, we have following possibilities:

X		Y	
Ladies (4)	**Men (3)**	**Ladies (3)**	**Men (4)**
0	3	3	0
3	0	0	3
1	2	2	1
2	1	1	2

So, total number of ways

$= {}^4C_0 \cdot {}^3C_3 \cdot {}^3C_3 \cdot {}^4C_0 + {}^4C_3 \cdot {}^3C_0 \cdot {}^3C_0 \cdot {}^4C_3 + {}^4C_1 \cdot {}^3C_2 \cdot {}^3C_2 \cdot {}^4C_1 + {}^4C_2 \cdot {}^3C_1 \cdot {}^3C_1 \cdot {}^4C_2$

$= 1 + 16 + 144 + 324 = 485$

11. (2) We have 6 novels and 3 dictionaries.

One dictionary can be selected in ${}^3C_1$ ways.

This dictionary is to be put in the middle.

Now, four novels can be selected in ${}^6C_4$ ways.

These four novels, two on each side of the dictionary, can be arranged in 4! ways.

$\therefore$ Total number of ways of arrangements

$= {}^3C_1 \times {}^6C_4 \times 4! = 3 \times 15 \times 24 = 1080$

JEE ADVANCED

Single Correct Answer Type

1. (4) Total number of unordered pairs of disjoint subsets

$$= \frac{3^4+1}{2} = 41.$$

2. (3) If '2' goes in '1' then it is derangement of 4 things which can be done in $4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)=9$ ways

If '2' doesn't go in '1', it is derangement of 5 things which can be done in $5!\left(\frac{1}{2!}-\frac{1}{3!}-\frac{1}{4!}+\frac{1}{5!}\right) = 44$ ways.

Hence, total 53 ways are there.

3. (1) Number of ways

= (selecting 4 girls and a captain) or (selecting 3 girls and 1 boy and a captain)

$= {}^6C_4 \times 4 + {}^6C_3 \times {}^4C_1 \times 4$

$= 380$

4. (4) $N_1 + N_2 + N_3 + N_4 + N_5$

= Total number of ways – Number of ways when no odd number is selected

Since only available even numbers are 2, 4, 6, 8; number of ways when no odd number is selected, is zero.

$\therefore \quad N_1 + N_2 + N_3 + N_4 + N_5 = {}^9C_5 - 0 = 126$

Linked Comprehension Type

1. (2) $a_n = b_n + c_n$

$b_n = a_{n-1}$

$c_n = a_{n-2} \Rightarrow a_n = a_{n-1} + a_{n-2}$

As $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8 \Rightarrow b_6 = 8.$

2. (1) As $a_n = a_{n-1} + a_{n-2}$

for $n = 17$

$\Rightarrow \quad a_{17} = a_{16} + a_{15}.$

Matrix Match Type

1. (3) 6 Boys and 5 girls

α_1 = Number of ways of selecting 3 boys and 2 girls

$= {}^6C_3 \times {}^5C_2 = 200$

α_2 = Boys and girls are equal and members are greater than or equal to 2

$= {}^6C_1 \cdot {}^5C_1 + {}^6C_2 \cdot {}^5C_2 + {}^6C_3 \cdot {}^5C_3 + {}^6C_4 \cdot {}^5C_4 + {}^6C_5 \cdot {}^5C_5$

$= 30 + 150 + 200 + 75 + 6 = 461$

α_3 = Number of ways of selecting 5 members having at least 2 girls

= Total cases without any restriction

– cases when no girl is selected

– cases when exactly one girl is selected

$= {}^{11}C_5 - {}^6C_5 - {}^6C_4 \cdot {}^5C_1 = {}^{11}C_5 - 81 = 381$

For α_4, Number of ways in which G_1 is included and M_1 is excluded

$= {}^4C_1 \cdot {}^5C_2 + {}^4C_2 \cdot {}^5C_1 + {}^4C_3 = 40 + 30 + 4 = 74$

Number of ways in which M_1 is included and G_1 is excluded

$= {}^4C_2 \cdot {}^5C_1 + {}^4C_3 = 34$

Number of ways in which both G_1 and M_1 are excluded

$= {}^4C_4 + {}^4C_3 \cdot {}^5C_1 + {}^4C_2 \cdot {}^5C_2 = 81$

Hence, $\alpha_4 = 74 + 34 + 81 = 189$

Numerical Value Type

1. (5) Let (1, 1, 1), (–1, 1, 1), (1, –1, 1), (–1, –1, 1) be vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$. Rest of the vectors are $-\vec{a}, -\vec{b}, -\vec{c}, -\vec{d}$ and let us find the number of ways of selecting coplanar vectors.

Observe that out of any 3 coplanar vectors, two will be collinear (antiparallel)

Number of ways of selecting the antiparallel pair = 4

Number of ways of selecting the third vector = 6

Total = 24

Number of non-coplanar selections $= {}^8C_3 - 24 = 32 = 2^5$

$\therefore p = 5$

2. (5) Number of red lines $= {}^nC_2 - n$

Number of blue lines $= n$

Hence, ${}^nC_2 - n = n$

$\Rightarrow \quad {}^nC_2 = 2n$

$\Rightarrow \quad \frac{n(n-1)}{2} = 2n$

$\Rightarrow \quad n - 1 = 4 \Rightarrow n = 5$

3. (7) Possible solutions are

1, 2, 3, 4, 10

1, 2, 3, 5, 9

1, 2, 3, 6, 8

1, 2, 4, 5, 8

1, 2, 4, 6, 7

1, 3, 4, 5, 7

2, 3, 4, 5, 6

Hence, 7 solutions are there.

4. (5) For n

$B_1, B_2, B_3, B_4, B_5, (G_1, G_2, G_3, G_4, G_5)$

Number of arrangements $= n = 5! \times 6!$

For m

First arrange 5 boys in 5! ways.

$\uparrow B_1 \uparrow B_2 \uparrow B_3 \uparrow B_4 \uparrow B_5 \uparrow$

Now, we have to arrange 5 girls in such a way that group of four girls and the fifth girl are arranged in any two of the six positions shown as arrows.

Two positions can be selected in ${}^6C_2$ ways.

Four girls can be selected in ${}^5C_4$ ways.

Now, this group and the fifth girl can be arranged in selected two positions is 2! ways.

Also, four girls arrange among themselves in 4! ways.

Hence, number of arrangements $= m = 5! \times {}^6C_2 \times {}^5C_4 \times 2! \times 4!$

$= 5! \times 15 \times 2 \times 5!$

$\therefore \quad \frac{m}{n} = \frac{5! \times 15 \times 2 \times 5!}{5! \times 6!} = 5$

(15) According to question, $x = 10!$

Now for y, we can select one letter which will be repeated twice in ${}^{10}C_1$ ways.

Remaining eight letters can be selected in ${}^{9}C_8$ ways.

So, $y = {}^{10}C_1 \times {}^{9}C_8 \times \frac{10!}{2!}$

$$\Rightarrow \quad \frac{y}{9x} = \frac{{}^{10}C_1 \times {}^{9}C_8}{9 \times 2!} = \frac{10 \times 9}{9 \times 2} = 5$$

(15) Number is divisible by 4 if number formed by last two digits is divisible by 4.

So, last two digits are 12, 24, 32, 44, 52.

For each of the above cases, first three places can be filled in $5 \times 5 \times 5 = 125$ ways.

So, total number of numbers $= 5 \times 125 = 625$

(19) $n(X) = 5$

$n(Y) = 7$

α_1 = Number of ways of selecting 3 boys and 2 girls

α = Number of one-one function from set X to Y

= Number of ways of selecting five images from seven images and then assigning to five preimages $= {}^{7}C_5 \times 5!$

β = Number of onto function Y to X

= Number of ways in which all five images are associated with seven preimages.

= Number of ways in which seven different objects can be distributed among five persons if each gets at least one object

So, possible group sizes for distribution are (1, 1, 1, 1, 3) and (1, 1, 1, 2, 2).

Hence, $\beta = \frac{7!}{(1!)^4 3!\,4!} \times 5! + \frac{7!}{(1!)^3 (2!)^2 3! 2!} \times 5!$

$$\therefore \quad \frac{\beta - \alpha}{5!} = \frac{7!}{3!4!} + \frac{7!}{(2!)^3 3!} - {}^{7}C_5$$

$$= {}^{7}C_3 + 3{}^{7}C_3 - {}^{7}C_2$$

$$= 4 \times 35 - 21 = 119$$

Chapter 8

Concept Application Exercises

Exercise 8.1

1. $T_1 = {}^nC_0 = 1$

$T_2 = {}^nC_1\, ax = 6x$ (1)

$T_3 = {}^nC_2\,(ax)^2 = 16x^2$ (2)

From (2),

$na = 6$ (3)

From (3),

$$\frac{n(n-1)}{2}a^2 = 16 \quad (4)$$

Eliminating a from (3) and (4),

$$\frac{n-1}{2n} = \frac{4}{9} \text{ or } n = 9$$

From (3),

$a = 2/3$

2. $T_4 = T_{3+1} = {}^nC_3\, a^{n-3}b^3$

$\Rightarrow \quad {}^nC_3 = 56 \Rightarrow \dfrac{n!}{3!(n-3)!} = 56$

or $\quad n(n-1)(n-2) = 56 \times 6$

or $\quad n(n-1)(n-2) = 8 \times 7 \times 6$

or $\quad n = 8$

3. Let the two successive terms in the expansion of $(1+x)^{24}$ be $(r+1)$th and $(r+2)$th terms.

So, $\quad T_{r+1} = {}^{24}C_r\, x^r$

and $\quad T_{r+2} = {}^{24}C_{r+1}\, x^{r+1}$

Given that, $\dfrac{{}^{24}C_r}{{}^{24}C_{r+1}} = \dfrac{1}{4}$

$$\Rightarrow \quad \frac{r+1}{24-(r+1)+1} = \frac{1}{4}$$

$\Rightarrow \quad 4r + 4 = 24 - r$

$\Rightarrow \quad r = 4$

$\therefore \quad T_{4+1} = T_5$ and $T_{4+2} = T_6$

4. Number of terms in the expansion of $(x+y+z)^n$ is

$(n+1)(n+2)/2 = 36$

or $\quad (n+1)(n+2) = 72$

or $\quad n = 7$

5. We have

$$\frac{1}{81^n} - \frac{10}{81^n}{}^{2n}C_1 + \frac{10^2}{81^n}{}^{2n}C_2 - \frac{10^3}{81^n}{}^{2n}C_3 + \cdots + \frac{10^{2n}}{81^n}$$

$$= \frac{1}{81^n}[{}^{2n}C_0 - {}^{2n}C_1 10^1 + {}^{2n}C_2 10^2 - {}^{2n}C_3 10^3 + \cdots + {}^{2n}C_{2n}10^{2n}]$$

$$= \frac{1}{81^n}[1-10]^{2n}$$

$$= \frac{(-9)^{2n}}{81^n} = \frac{81^n}{81^n} = 1$$

6. $\displaystyle\sum_{r=0}^{n}(-1)^r \times {}^nC_r\left(\frac{1}{2^r} + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \cdots m \text{ terms}\right)$

$$= \sum_{r=0}^{n}{}^nC_r\left(\frac{-1}{2}\right)^r + \sum_{r=0}^{n}{}^nC_r\left(\frac{-3}{9}\right)^n + \sum_{r=0}^{n}{}^nC_r\left(\frac{-7}{8}\right)^r + \cdots, m \text{ terms.}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \cdots m \text{ terms}$$

$$= \frac{1}{2^n} + \frac{1}{2^{2n}} + \frac{1}{2^{3n}} + \frac{1}{2^{4n}} + \cdots m \text{ terms}$$

$$= \frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{1-\frac{1}{2^n}} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

7. $$\frac{1}{6} = \frac{{}^nC_6\left(2^{1/3}\right)^{n-6}\left(3^{-1/3}\right)^6}{{}^nC_{n-6}\left(2^{1/3}\right)^6\left(3^{-1/3}\right)^{n-6}}$$

or $\quad 6^{-1} = 6^{-4} \times 6^{n/3} = 6^{n/3-4} \quad \Rightarrow \dfrac{n}{3} - 4 = -1 \quad \Rightarrow n = 9$

8. The coefficients of $(r-5)$th and $(2r-1)$th terms of the expansion $(1+x)^{34}$ are ${}^{34}C_{r-6}$ and ${}^{34}C_{2r-2}$, respectively.

Since they are equal so ${}^{34}C_{r-6} = {}^{34}C_{2r-2}$

Therefore, either $r-6 = 2r-2$ or $r-6 = 34-(2r-2)$

[Using the fact that if ${}^nC_r = {}^nC_p$, then either $r = p$ or $r = n-p$]

So, we get $r = -4$ or $r = 14$. r being a natural number, $r = -4$ is not possible.

So, $r = 14$.

9. T_{r+1}, the $(r+1)$th term in the expansion of $(5^{1/6} + 2^{1/8})^{100}$, is

$T_{r+1} = {}^{100}C_r\,(5^{1/6})^{100-r}(2^{1/8})^r$

As 5 and 2 are relatively prime, T_{r+1} will be rational if $(100-r)/6$ and $r/8$ are both integers, i.e., $100-r$ is a multiple of 6 and r is a multiple of 8.

As $0 \le r \le 100$, multiples of 8 up to 100 and corresponding values of $100-r$ are given by

$r = 0, 8, 16, 24, \ldots, 88, 96$

$100 - r = 100, 92, 84, 76, \ldots, 12, 4$

In $100-r$ values, multiples of 6 are 84, 60, 36, and 12.

Hence, there are just four rational terms. Therefore, the number of irrational terms is $101 - 4 = 97$.

10. $\cos 6\theta$ = Real part of $(\cos\theta + i\sin\theta)^6$

$$= {}^6C_0\cos^6\theta - {}^6C_2\cos^4\theta\sin^2\theta + {}^6C_4\cos^2\theta\sin^4\theta - {}^6C_6\sin^6\theta$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-2\cos^2\theta+\cos^4\theta) - (1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$$

$$= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$$

11. Given expression is

$$2\left[1 + {}^9C_2\,(3\sqrt{2}x)^2 + {}^9C_4\,(3\sqrt{2}x)^4 + {}^9C_6\,(3\sqrt{2}x)^6 + {}^9C_8\,(3\sqrt{2}x)^8\right]$$

Therefore, the number of nonzero terms is 5.

$(x+a)^n - (x-a)^n = 2[{}^nC_1 x^{n-1}a + {}^nC_3 x^{n-3}a^3 + {}^nC_5 x^{n-5}a^5 + \cdots]$

13. $(\sqrt{2}+1)^6 - (\sqrt{2}-1)^6$

$\therefore \quad = 2[{}^6C_1(\sqrt{2})^5(1)^1 + {}^6C_3(\sqrt{2})^3(1)^3 + {}^6C_5(\sqrt{2})^1(1)^5]$

$= 2[6 \times 4\sqrt{2} + 20 \times 2\sqrt{2} + 6\sqrt{2}]$

$= 2[24\sqrt{2} + 40\sqrt{2} + 6\sqrt{2}] = 140\sqrt{2}$

14. We have

$$\frac{1}{\sqrt{4x+1}}\left\{\left(\frac{1+\sqrt{4x+1}}{2}\right)^n - \left(\frac{1-\sqrt{4x+1}}{2}\right)^n\right\}$$

$$= \frac{1}{2^n\sqrt{4x+1}}\{2[{}^nC_1\sqrt{4x+1} + {}^nC_3(\sqrt{4x+1})^3 + \cdots]\}$$

$$= \frac{1}{2^{n-1}}[{}^nC_1 + {}^nC_3(\sqrt{4x+1})^2 + {}^nC_5(\sqrt{4x+1})^4 + \cdots]$$

Since R.H.S. of the given expression contains x^5, so this expression should go up to $(\sqrt{4x+1})^{10}$ and hence $n = 11$ or 12.

14. Here $f = R - [R]$ is the fractional part of R. Thus, if I is the integral part of R, then

$R = I + f = (5\sqrt{5} + 11)^{2n+1}$, and $0 < f < 1$

Let $f' = (5\sqrt{5} - 11)^{2n+1}$.

Then $0 < f' < 1$ (as $5\sqrt{5} - 11 < 1$)

Now, $I + f - f' = (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$

$= 2[{}^{2n+1}C_1(5\sqrt{5})^{2n} \times 11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2} \times 11^3 + \cdots]$

$=$ an even integer (1)

$\Rightarrow$ $f - f'$ must also be an integer

$\Rightarrow$ $f - f' = 0$, ($\because \quad 0 < f < 1, 0 < f' < 1$)

$\Rightarrow$ $f = f'$

$\therefore \quad Rf = Rf' = (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$

$= (125 - 121)^{2n+1} = 4^{2n+1}$

15. Here middle term is T_6

$\therefore \quad T_6 = {}^{10}C_5 \frac{1}{x^5} \cdot x^5 \sin^5 x = \frac{63}{8}$

or $\sin^5 x = \frac{1}{2^5}$

or $\sin x = \frac{1}{2}$

$\Rightarrow \quad x = n\pi + (-1)^n \frac{\pi}{6}, n \in Z$

16. $\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left(x + \frac{1}{x}\right)^{2n}$

Hence, the middle term is

$$T_{\frac{2n}{2}+1} = T_{n+1} = {}^{2n}C_n(x)^n\left(\frac{1}{x}\right)^n = {}^{2n}C_n = \frac{(2n)!}{(n!)^2}$$

17. Number of terms in the expansion

$= {}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+3)(n+2)(n+1)}{3!}$

According to the question,

$$\frac{(n+3)(n+2)(n+1)}{3!} = 286$$

$\therefore \quad n = 10$

So, the term containing (xyz) is $\frac{10!}{7!1!1!1!}(1)^7(2x)^1(-3y)^1(4z)^1$

$= -17280xyz$.

Therefore, the required coefficient is -17280.

Exercise 8.2

1. $1 + 99^n = 1 + (100 - 1)^n$

$= 1 + [{}^nC_0 100^n - {}^nC_1 100^{n-1} + {}^nC_2 100^{n-2} + \cdots - {}^nC_n]$ [$\because$ n is odd]

$= 100[{}^nC_0 100^{n-1} - {}^nC_1 100^{n-2} + \cdots + {}^nC_{n-1}]$

$= 100 \times$ integer

Hence, the number of zeros at the end of the sum $99^n + 1$ is 2.

2. $2^{3n} - 1 = (2^3)^n - 1 = (1+7)^n - 1$

$= [1 + {}^nC_1(7) + {}^nC_2(7)^2 + \cdots + {}^nC_n(7)^n] - 1$

$= 7[{}^nC_1 + {}^nC_2(7) + \cdots + {}^nC_n(7)^{n-1}]$

Thus, $2^{3n} - 1$ is divisible by 7 for all $n \in N$

3. $27^{27} = 3(10 - 1)^{40}$

$= 3(10^{40} - {}^{40}C_1 \cdot 10^{39} + \ldots + {}^{40}C_{38} \cdot 10^2 - {}^{40}C_{39} \cdot 10 + 1)$

$= 3(1000\lambda - 400 + 1)$

$= 3(1000\lambda - 399)$

Therefore, last 3 digits of this number is 803.

4. $(1 + 100)^{100} = 1 + 100 \times 100 + \frac{100 \times 99}{1 \times 2} \times (100^2)$

$+ \frac{100 \times 99 \times 98}{1 \times 2 \times 3}(100)^3 + \ldots$

$\Rightarrow \quad (101)^{100} - 1$

$= 100 \times 100\left[1 + \frac{100 \times 99}{1 \times 2} + \frac{100 \times 9 \times 98}{1 \times 2 \times 3} \times 100 + \ldots\right]$

From above, it is clear that $(101)^{100} - 1$ is divisible by $(100)^2 = 10000$. So, the greatest value of m is 4.

5. In order to show that $9^{n+1} - 8n - 9$ is divisible by 64, it has to be proved that $9^{n+1} - 8n - 9 = 64k$, where k is some natural number

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1(8) + {}^{n+1}C_2(8)^2 + \ldots + {}^{n+1}C_{n+1}(8)^{n+1}$

or $9^{n+1} = 1 + (n+1)(8) + 8^2[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}]$

or $9^{n+1} = 9 + 8n + 64[{}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}]$

or $9^{n+1} - 8n - 9 = 64k$, where $k = {}^{n+1}C_2 + {}^{n+1}C_3 \times 8 + \ldots + {}^{n+1}C_{n+1}(8)^{n-1}$ is a natural number

Thus, $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.

6. We have

$2^{4n+4} - 15n - 16$

$= 2^{4(n+1)} - 15n - 16$

$= 16^{n+1} - 15n - 16$

$= (1 + 15)^{n+1} - 15n - 16$

$= {}^{n+1}C_0 15^0 + {}^{n+1}C_1 15^1 + {}^{n+1}C_2 15^2 + {}^{n+1}C_3 15 + \ldots + {}^{n+1}C_{n+1}(15)^{n+1} - 15n - 16$

$= 1 + (n+1)15 + {}^{n+1}C_2 15^2 + {}^{n+1}C_3 15^3 + \ldots + {}^{n+1}C_{n+1}(15)^{n+1} - 15n - 16$

$= 15^2[{}^{n+1}C_2 + {}^{n+1}C_3 15 + \cdots]$

Thus, $2^{4n+4} - 15n - 16$ is divisible by 225.

7. $\frac{7^{103}}{25} = \frac{7(49)^{51}}{25}$

$= \frac{7(50-1)^{51}}{25}$

$$= \frac{7(25k-1)}{25}$$
$$= \frac{175(k)-25+25-7}{25}$$
$$= \frac{25(7k-1)+18}{25}$$

Therefore, the remainder is 18.

8. $E = 3^{2003} = 3^{2001} \times 3^2 = 9(27)^{667} = 9(28-1)^{667}$

$\Rightarrow \quad E = 9[{}^{667}C_0\, 28^{667} - {}^{667}C_1\, (28)^{666} + \cdots - {}^{667}C_{667}]$

$= 9 \times 28k - 9$

$\Rightarrow \quad \frac{E}{28} = 9k - \frac{9}{28} = 9k - 1 + \frac{19}{28}$

That means if we divide 3^{2003} by 28, the remainder is 19. Thus,

$$\left\{\frac{3^{2003}}{28}\right\} = \frac{19}{28}$$

9. $3456^{2222} = (7 \times 493 + 5)^{2222}$

$= (7k+5)^{2222}$

$= 7m + 5^{2222}$

Now, $5^{2222} = 5^2(5^3)^{740}$

$= 25(125)^{740}$

$= 25(126-1)^{740}$

$= 25[7n+1]$

$= 175n + 25$

Remainder when $175n + 25$ is divided by 7 is 4.

10. Let $(\sqrt{3}+1)^{2m} = I + f$, where I is the integral and f the fractional part.

$\therefore \quad 0 < f < 1$

Also, let $f' = (\sqrt{3}-1)^{2m}$. Now,

$I + f + f' = (\sqrt{3}+1)^{2m} + (\sqrt{3}-1)^{2m}$

$= (4+2\sqrt{3})^m + (4-2\sqrt{3})^m$

$= 2^m[(2+\sqrt{3})^m + (2-\sqrt{3})^m]$

$= 2^{m+1}\,[2^m + {}^mC_2\, 2^{m-2}\,(\sqrt{3})^2 + \cdots]$

or $\quad I + f + f' =$ even integer $\qquad (1)$

$\Rightarrow \quad f + f'$ is an integer

$\because \quad 0 < f < 1$ and $0 < f' < 1$

$\therefore \quad 0 < f + f' < 2$

Thus, $f + f'$ is an integer between 0 and 2.

$\therefore \quad f + f' = 1$

Hence, from (1), we conclude that $I + f + f'$ is an integer next above $(\sqrt{3}+1)^{2m}$ and it contains 2^{m+1} as a factor.

Exercise 8.3

1. $T_{r+1} = {}^{15}C_r\,(x^4)^{15-r}\left(\frac{1}{x^3}\right)^r = {}^{15}C_r\, x^{60-7r}$

$\Rightarrow \quad 60 - 7r = 4$

$\Rightarrow \quad r = 8$

Then, the required term is 9^{th}.

2. The $(r+1)$th term is ${}^{2n}C_r(x^2)^{2n-r}(1/x)^r = {}^{2n}C_r\, x^{4n-3r}$.

For the coefficient of x^p,

$4n - 3r = p$

or $\quad r = \frac{1}{3}(4n-p)$

Therefore, the coefficient of x^p is

$${}^{2n}C_{(4n-p)/3} = \frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[2n - \frac{1}{3}(4n-p)\right]!}$$
$$= \frac{(2n)!}{\left[\frac{1}{3}(4n-p)\right]!\left[\frac{1}{3}(2n+p)\right]!}$$

3. $E = (1 + 2t^2 - t^3)^9$

$= {}^9C_0(1+2t^2)^9 - {}^9C_1\,(1+2t^2)^8 \cdot t^3 + {}^9C_2(1+2t^2)^7 \cdot t^6$
$- {}^9C_3\,(1+2t^2)^6 \cdot t^9 + \ldots - {}^9C_9(t^3)^9$

$\therefore \quad$ Coefficient of t^8 in $E = {}^9C_0$ (Coefficient of t^8 in $(1+2t^2)^9$)
$- {}^9C_1$(Coefficient of t^5 in $(1+2t^2)^9$)
$+ {}^9C_2$(Coefficient of t^2 in $(1+2t^2)^7$)

$= {}^9C_0 \cdot {}^9C_4 \cdot 2^4 - 0 + {}^9C_2 \cdot {}^7C_1 \cdot 2$

$= 2520$

4. $(1+x)^{11}\,(1+x^2)^{11} = (1 + {}^{11}C_1\, x + {}^{11}C_2\, x^2 + {}^{11}C_3 x^3 + {}^{11}C_4 x^4 + \ldots)$
$\times (1 + {}^{11}C_1\,(x^2) + {}^{11}C_2 (x^2)^2 + \ldots)$

Coefficient of x^4 is ${}^{11}C_2 \cdot 1 + {}^{11}C_1 \cdot {}^{11}C_2 + {}^{11}C_4 = 990$

5. $\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}$

$= \frac{(x^{1/3}+1)(x^{2/3} - x^{1/3} + 1)}{x^{2/3} - x^{1/3} + 1} - \frac{x^{1/2}+1}{x^{1/2}}$

$= x^{1/3} + 1 - 1 - x^{-1/2}$

$= x^{1/3} - x^{-1/2}$

$\therefore \quad \left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}}\right)^{10} = (x^{1/3} - x^{-1/2})^{10}$

Let $\quad T_{r+1} = {}^{10}C_r\,(x^{1/3})^{10-r}\,(-1)^r\,(x^{-1/2})^r$

For the term independent of x,

$$\frac{10-r}{3} - \frac{r}{2} = 0$$

$\Rightarrow \quad 20 - 2r - 3r = 0$

$\Rightarrow \quad r = 4$

So, required coefficient $= {}^{10}C_4\,(-1)^4 = 210$

6. $(1 + 3x + 2x^2)^6 = [1 + x(3+2x)]^6$

$= 1 + {}^6C_1 x(3+2x) + {}^6C_2 x^2(3+2x)^2 + {}^6C_3 x^3(3+2x)^3 +$
${}^6C_4 x^4\,(3+2x)^4 + {}^6C_5 x^5\,(3+2x)^5 + {}^6C_6 x^6\,(3+2x)^6$

We get x^{11} only from ${}^6C_6\, x^6\,(3+2x)^6$

$\therefore \quad$ Coefficient of $x^{11} = {}^6C_5 \times 3 \times 2^5 = 576$

7. $((1-x^2) + x^4)^3(1-x)^7$

$= ((1-x^2)^3 + 3x^4(1-x^2)^2 + 3x^8(1-x^2) + x^{12})$
$\times ({}^7C_0 - {}^7C_1 x + {}^7C_2 x^2 - \cdots - {}^7C_7 x^7)$

$\therefore \quad$ Coefficient of $x^{12} = {}^7C_0 - 3 \times {}^7C_2 + 6 \times {}^7C_4 - 7 \times {}^7C_6$

$= 99$

Exercise 8.4

1. We have

$$(1+x)^{50} = \sum_{r=0}^{50} {}^{50}C_r x^r$$

Therefore, sum of the coefficients of odd powers of x *is*

$${}^{50}C_1 + {}^{50}C_3 + \cdots + {}^{50}C_{49} = \frac{1}{2}(2^{50}) = 2^{49}$$

3. $\frac{1}{n!}+\frac{1}{2!(n-2)!}+\frac{1}{4!(n-4)!}+\ldots$

$=\frac{1}{n!}\left[{}^nC_0+{}^nC_2+{}^nC_4+\ldots\right]=\frac{2^{n-1}}{n!}$

4. There are 60 terms in the expansion of $(1+x)^{59}$. Sum of the last 30 coefficients is

$S={}^{59}C_{30}+{}^{59}C_{31}+\ldots+{}^{59}C_{58}+{}^{59}C_{59}$

$\therefore\quad S={}^{59}C_{29}+{}^{59}C_{28}+\ldots+{}^{59}C_1+{}^{59}C_0$ [Using ${}^nC_r={}^nC_{n-r}$]

Adding the above two expansions, we get

$2S={}^{59}C_0+{}^{59}C_1+\ldots+{}^{59}C_{59}=2^{59}$

or $S=2^{58}$

5. $\sum_{j=0}^{n}\left({}^{4n+1}C_j+{}^{4n+1}C_{2n-j}\right)=({}^{4n+1}C_0+{}^{4n+1}C_1+\ldots+{}^{4n+1}C_n)$

$+({}^{4n+1}C_{2n}+{}^{4n+1}C_{2n-1}+\ldots+{}^{4n+1}C_n)$

$=({}^{4n+1}C_0+{}^{4n+1}C_1+\ldots+{}^{4n+1}C_{2n})+{}^{4n+1}C_n$

$=2^{4n}+{}^{4n+1}C_n$

6. We have $E=\left(\sum_{r=1}^{5}{}^{20}C_{2r-1}\right)^6$

We know that ${}^{20}C_1+{}^{20}C_3+{}^{20}C_5+{}^{20}C_7+\ldots+{}^{20}C_{19}=2^{19}$

Now, ${}^{20}C_1={}^{20}C_{19}$, ${}^{20}C_3={}^{20}C_{17}$... etc.

$\therefore\quad 2({}^{20}C_1+{}^{20}C_3+\ldots+{}^{20}C_9)=2^{19}$

$\Rightarrow\quad {}^{20}C_1+{}^{20}C_3+\ldots+{}^{20}C_9=2^{18}$

$\Rightarrow\quad E=(2^{18})^6=2^{108}$

$=8(2^5)^{21}$

$=8(33-1)^{21}$

$=8(33k-1)$

$=8\times 33k-8$

$=11(8\times 3k-1)+3$

Therefore, the remainder when $\left(\sum_{r=1}^{5}{}^{20}C_{2r-1}\right)^6$ is divided by 11 is 3.

7. $S={}^nC_0+5\times{}^nC_1+9\times{}^nC_2+\ldots+(4n-3)\times{}^nC_{n-1}+(4n+1)\times{}^nC_n\quad\ldots(1)$

$\therefore\quad S=(4n+1)\,{}^nC_n+(4n-3)\,{}^nC_{n-1}+\ldots+5\,{}^nC_1+{}^nC_0$

$\Rightarrow\quad S=(4n+1)\,{}^nC_0+(4n-3)\,{}^nC_1+\ldots+5\,{}^nC_{n-1}+{}^nC_n\ldots(2)$

Adding (1) and (2), we get

$2S=(4n+2)\,({}^nC_0+{}^nC_1+\ldots+{}^nC_{n-1}+{}^nC_n)$

$=(4n+2)2^n$

$\Rightarrow\quad S=(2n+1)2^n$

8. Consider $(1+x)^n={}^nC_0+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots+{}^nC_nx^n$

Now, put $x=1$, ω and ω^2, where ω is complex cube root of unity.

$\therefore\quad 2^n={}^nC_0+{}^nC_1+{}^nC_2+{}^nC_3+{}^nC_4+{}^nC_5+{}^nC_6+\ldots$

$(1+\omega)^n={}^nC_0+{}^nC_1\omega+{}^nC_2\omega^2+{}^nC_3\omega^3+{}^nC_4\omega^4+{}^nC_5\omega^5+{}^nC_6\omega^6+\ldots$

$={}^nC_0+{}^nC_1\omega+{}^nC_2\omega^2+{}^nC_3+{}^nC_4\omega+{}^nC_5\omega^2+{}^nC_6+\ldots$

$(1+\omega^2)^n={}^nC_0+{}^nC_1\omega^2+{}^nC_2\omega^4+{}^nC_3\omega^6+{}^nC_4\omega^8+{}^nC_5\omega^{10}+{}^nC_6\omega^{12}+\ldots$

$={}^nC_0+{}^nC_1\omega^2+{}^nC_2\omega+{}^nC_3+{}^nC_4\omega^2+{}^nC_5\omega+{}^nC_6+\ldots$

$\therefore\quad 2^n+(1+\omega)^n+(1+\omega^2)^n=3\,({}^nC_0+{}^nC_3+{}^nC_6+\ldots)$

Now, $2^n+(1+\omega)^n+(1+\omega^2)^n=2^n+2\mathrm{Re}((1+\omega)^n)$

$=2^n+2\mathrm{Re}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)^n$

$=2^n+2\mathrm{Re}\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n$

$=2^n+2\cos\frac{n\pi}{3}$

Hence ${}^nC_0+{}^nC_3+{}^nC_6+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$

8. We have,

${}^{4n}C_0+{}^{4n}C_2x^2+{}^{4n}C_4x^4+\ldots+{}^{4n}C_{4n}x^{4n}$

$=\frac{1}{2}\left[(1+x)^{4n}+(1-x)^{4n}\right]$

Putting $x=1$ and $x=i$, we get

${}^{4n}C_0+{}^{4n}C_2+{}^{4n}C_4+\ldots+{}^{4n}C_{4n}=\frac{1}{2}[2^{4n}]$

and ${}^{4n}C_0-{}^{4n}C_2+{}^{4n}C_4-\ldots+{}^{4n}C_{4n}$

$=\frac{1}{2}[(1+i)^{4n}+(1-i)^{4n}]$

Thus,

$2[{}^{4n}C_0+{}^{4n}C_4+\ldots+{}^{4n}C_{4n}]=2^{4n-1}+\frac{1}{2}\left[(1+i)^{4n}+(1-i)^{4n}\right]$

Now, $(1+i)^{4n}+(1-i)^{4n}$

$=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{4n}+\left[\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)\right]^{4n}$

$=2^{2n}(\cos n\pi+i\sin n\pi)+2^{2n}(\cos n\pi-i\sin n\pi)$

$=2^{2n+1}\cos n\,\pi=2^{2n+1}(-1)^n$

$\therefore\quad 2[{}^{4n}C_0+{}^{4n}C_4+\ldots+{}^{4n}C_{4n}]=2^{4n-1}+\frac{1}{2}2^{2n+1}(-1)^n$

$\Rightarrow\quad {}^{4n}C_0+{}^{4n}C_4+\ldots+{}^{4n}C_{4n}=2^{4n-2}+(-1)^n\,2^{2n-1}$

9. $\sum_{r=0}^{s}\sum_{\substack{s=1\\ r\le s}}^{n}{}^nC_s\,{}^sC_r=\sum_{s=1}^{n}{}^nC_s({}^sC_0+{}^sC_1+{}^sC_2\ldots+{}^sC_s)$

$=\sum_{s=1}^{n}{}^nC_s2^s$

$=\sum_{s=0}^{n}{}^nC_s2^s-{}^nC_02^0$

$=(1+2)^n-1$

$=3^n-1$

10. Let

$(1+x-3x^2)^{4163}=a_0+a_1x+a_2x^2+\ldots$

Putting $x=1$ on both sides, we get

$(1+1-3)^{4163}=a_0+a_1+a_2+\ldots$

Hence, sum of the coefficients is $(-1)^{4163}=-1$.

11. Sum of the coefficients in the expansion of $(x-2y+3z)^n$ is $(1-2+3)^2=2^n$ (putting $x=y=z=1$)

$\therefore\quad 2^n=128$ or $n=7$

Therefore, the greatest coefficient in the expansion of $(1+x)^7$ is ${}^7C_3$ or ${}^7C_4$ because both are equal to 35.

12. For sum of coefficients, put $x=1$. Therefore,

$(1+2+3+\ldots n)^2=\left(\frac{n(n+1)}{2}\right)^2=\Sigma n^3$

13. $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + \cdots + a_{12}x^{12}$

Putting $x = 1$ and $x = -1$ and adding the results, we get

$64 = 2(1 + a_2 + a_4 + \ldots)$

$\therefore \quad a_2 + a_4 + a_6 + \cdots + a_{12} = 31$

Exercise 8.5

1. Since the 7^{th} and 8^{th} terms are equal, we have

$${}^nC_6\, x^6 = {}^nC_7\, x^7$$

or $$x = \frac{{}^nC_6}{{}^nC_7} = \frac{7}{n-6}$$

or $$\left(\frac{7}{x}+6\right)^2 = n^2$$

2. $t_r = r^2 \dfrac{{}^nC_r}{{}^nC_{r-1}}$

$= r^2 \dfrac{n-r+1}{r}$

$= r(n+1-r)$

$= (n+1)r - r^2$

$\therefore \quad \text{Sum} = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$

$= (n+1)\dfrac{n(n+1)}{2} - \dfrac{n(n+1)(2n+1)}{6}$

$= \dfrac{n(n+1)}{6}\{3(n+1) - (2n+1)\}$

$= \dfrac{n(n+1)(n+2)}{6}$

3. **(a)** Let ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ be in G.P.

$\therefore \quad \dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{{}^nC_{r+1}}{{}^nC_r}$

or $$\frac{n-r+1}{r} = \frac{n-(r+1)+1}{r+1}$$

or $(r+1)(n-r+1) = r(n-r)$

or $nr - r^2 + r + n - r + 1 = nr - r^2$

or $n + 1 = 0$

or $n = -1$

This is not possible.

Hence not in G.P.

(b) Let ${}^nC_{r-1}$, ${}^nC_r$, ${}^nC_{r+1}$ be in H.P.

$\therefore \quad \dfrac{1}{{}^nC_{r-1}}, \dfrac{1}{{}^nC_r}, \dfrac{1}{{}^nC_{r+1}}$ are in A.P.

$\Rightarrow \quad \dfrac{2}{{}^nC_r} = \dfrac{1}{{}^nC_{r-1}} + \dfrac{1}{{}^nC_{r+1}}$

or $$2 = \frac{{}^nC_r}{{}^nC_{r-1}} + \frac{{}^nC_r}{{}^nC_{r+1}}$$

or $$2 = \frac{n-r+1}{r} + \frac{r+1}{n-r}$$

or $2r(n-r) = (n-r)^2 + (n-r) + r^2 + r$

or $2rn - 2r^2 = n^2 - 2nr + r^2 + n - r + r^2 + r$

or $n^2 + 4r^2 - 4nr + n = 0$ (1)

or $(n-2r)^2 + n = 0$

This is not possible as both $(n-2r)^2$ and n are positive.

Hence not in H.P.

4. We know that

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\frac{\alpha}{x}$$

$T_3 = a,\ T_4 = b,\ T_5 = c,\ T_6 = d$

Putting $r = 3, 4, 5$ in the above, we get

$$\frac{T_4}{T_3} = \frac{n-2}{3}\frac{\alpha}{x} = \frac{b}{a} \quad (1)$$

$$\frac{T_5}{T_4} = \frac{n-3}{4}\frac{\alpha}{x} = \frac{c}{b} \quad (2)$$

$$\frac{T_6}{T_5} = \frac{n-4}{5}\frac{\alpha}{x} = \frac{d}{c} \quad (3)$$

We have to prove

$$\frac{b^2-ac}{c^2-bd} = \frac{5a}{3c}$$

or $$\frac{\frac{b}{c}-\frac{a}{b}}{\frac{c}{b}-\frac{d}{c}} = \frac{5a}{3c}$$

Now, $$\frac{\frac{b}{c}-\frac{a}{b}}{\frac{c}{b}-\frac{d}{c}} = \frac{\frac{4x}{(n-3)\alpha} - \frac{3x}{(n-2)\alpha}}{\frac{(n-3)\alpha}{4x} - \frac{(n-4)\alpha}{5x}}$$

$$= \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-2)(n-3)}\frac{4n-8-3n+9}{5n-15-4n+16}$$

$$= \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-2)(n-3)}$$

Also, $$\frac{5a}{3c} = \frac{x^2}{\alpha^2}\frac{4\times 5}{(n-2)(n-3)}$$

$\therefore$ L.H.S. = R.H.S.

5. We have $(3+2x)^{50}$

Now $$\frac{T_{r+1}}{T_r} = \frac{50-r+1}{r}\cdot\frac{2x}{3}$$

$\Rightarrow \quad \dfrac{T_{r+1}}{T_r} = \dfrac{51-r}{r}\cdot\dfrac{2}{15}$ (Putting $x = 1/5$)

Let $$\frac{T_{r+1}}{T_r} \geq 1$$

$\Rightarrow \quad 102 - 2r \geq 15r$

$\Rightarrow \quad 102 \geq 17r$

$\Rightarrow \quad r \leq 6$

Hence, 7^{th} and 6^{th} terms are the greatest.

Now, $T_7 = T_{6+1} = 3^{50} \times {}^{50}C_6\left(\dfrac{2}{15}\right)^6$

$$T_6 = T_{5+1} = 3^{50} \times {}^{50}C_5\left(\frac{2}{15}\right)^5$$

$\therefore \quad \dfrac{T_7}{T_6} = \dfrac{{}^{50}C_6}{{}^{50}C_5}\dfrac{2}{15}$

$= \dfrac{50-6+1}{6}\dfrac{2}{15}$

$= \dfrac{45}{6}\dfrac{2}{15} = 1$

6. We have $(1+4x)^8$.

Now $\dfrac{T_{r+1}}{T_r} = \dfrac{n-r+1}{r}(4x) = \dfrac{9-r}{r}\cdot\dfrac{4}{3}$ (Putting $n=8, x=1/3$)

$\therefore \quad \dfrac{T_{r+1}}{T_r} \geq 1$

$\Rightarrow \quad 36 - 4r \geq 3r$

$\Rightarrow \quad 36 \geq 7r$

$\Rightarrow \quad r \leq 5\dfrac{1}{7}$

Hence, $r = 5$, and therefore, T_{5+1} i.e., 6^{th} term is greatest.

$\Rightarrow \quad T_6 = {}^8C_5(4x)^5 = \dfrac{8!}{5!\times 3!}\cdot\left(\dfrac{4}{3}\right)^5$

$= \dfrac{8\times 7\times 6}{6}\left(\dfrac{4}{3}\right)^5$

$= 56\left(\dfrac{4}{3}\right)^5$

7. If n is even, the greatest coefficient is ${}^nC_{n/2}$. Therefore, the greatest term is ${}^nC_{n/2}\,x^{n/2}$.

$\therefore \quad {}^nC_{n/2}\,x^{n/2} > {}^nC_{(n/2)-1}x^{(n-2)/2}$ and ${}^nC_{n/2}x^{n/2} > {}^nC_{(n/2)+1}x^{(n/2)+1}$

$\Rightarrow \quad \dfrac{n-\frac{n}{2}+1}{\frac{n}{2}}x > 1$ and $\dfrac{\frac{n}{2}}{\frac{n}{2}+1}x < 1$

$\Rightarrow \quad x > \dfrac{\frac{n}{2}}{\frac{n}{2}+1}$ and $x < \dfrac{\frac{n}{2}+1}{\frac{n}{2}}$

$\Rightarrow \quad x > \dfrac{n}{n+2}$ and $x < \dfrac{n+2}{n}$

8. In the expansion of $(2x+5)^{10}$, the middle term is T_6.

Consider the expansion of $(1+2x/5)^{10}$. Now,

$\left|\dfrac{T_6}{T_5}\right| > 1$ and $\left|\dfrac{T_7}{T_6}\right| < 1$

$\Rightarrow \quad \left|\dfrac{10-5+1}{5}\dfrac{2x}{5}\right| > 1$ and $\left|\dfrac{10-6+1}{6}\dfrac{2x}{5}\right| < 1$

$\Rightarrow \quad \left|\dfrac{12}{25}x\right| > 1$ and $\left|\dfrac{x}{3}\right| < 1$

$\Rightarrow \quad \dfrac{25}{12} < |x| < 3$

$\Rightarrow \quad x \in \left(-3, -\dfrac{25}{12}\right) \cup \left(\dfrac{25}{12}, 3\right)$

Exercise 8.6

1. $({}^{10}C_0) + ({}^{10}C_0 + {}^{10}C_1) + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2) + \cdots + ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \ldots + {}^{10}C_9)$

$= 10\,{}^{10}C_0 + 9\,{}^{10}C_1 + 8\,{}^{10}C_2 + \ldots + {}^{10}C_9$

$= {}^{10}C_1 + 2\,{}^{10}C_2 + 3\,{}^{10}C_3 + \ldots + 10\,{}^{10}C_{10}$

$= \sum_{r=1}^{10} r\cdot{}^{10}C_r = 10\sum_{r=1}^{10} {}^9C_{r-1} = 10\times 2^9$

2. $S = \sum_{r=1}^{n} \dfrac{1^2+2^2+\cdots+r^2}{2r+1}\,{}^nC_r$

$= \sum_{r=1}^{n} \dfrac{r(r+1)(2r+1)}{6(2r+1)}\,{}^nC_r$

$= \dfrac{1}{6}\sum_{r=1}^{n} r(r+1)\,{}^nC_r$

$= \dfrac{1}{6}\sum_{r=1}^{n} (r+1)\cdot n\cdot{}^{n-1}C_{r-1}$

$= \dfrac{1}{6}n\sum_{r=1}^{n}((r-1)+2)^{n-1}C_{r-1}$

$= \dfrac{1}{6}n\cdot\sum_{r=1}^{n}\left((r-1)\cdot{}^{n-1}C_{r-1} + 2\cdot{}^{n-1}C_{r-1}\right)$

$= \dfrac{1}{6}n\cdot\sum_{r=1}^{n}\left((n-1)\cdot{}^{n-2}C_{r-2} + 2\cdot{}^{n-1}C_{r-1}\right)$

$= \dfrac{1}{6}n\cdot(n-1)\cdot 2^{n-2} + \dfrac{n}{3}\cdot 2^{n-1} = \dfrac{1}{6}n(n+3)\,2^{n-2}$

3. $\sum_{r=0}^{n} r^2\,{}^nC_r p^r q^{n-r}$

$= \sum_{r=0}^{n} nr\,{}^{n-1}C_{r-1}p^r q^{n-r}$

$= n\sum_{r=0}^{n}[(r-1)+1]\,{}^{n-1}C_{r-1}p^r q^{n-r}$

$= n\sum_{r=0}^{n}[(r-1)^{n-1}C_{r-1} + {}^{n-1}C_{r-1}]p^r q^{n-r}$

$= n\sum_{r=0}^{n}[(n-1)^{n-2}C_{r-2} + {}^{n-1}C_{r-1}]p^r q^{n-r}$

$= p^2n(n-1)\sum_{r=0}^{n} {}^{n-2}C_{r-2}p^{r-2}q^{n-r} + np\sum_{r=0}^{n} {}^{n-1}C_{r-1}\,p^{r-1}q^{n-r}$

$= p^2n(n-1)(p+q)^{n-2} + np(p+q)^{n-1}$

$= p^2n(n-1) + np$

$= p^2n^2 + np(1-p)$

$= p^2n^2 + npq$

4. $S = 1 - {}^nC_1\left(\dfrac{1+x}{1+nx}\right) + {}^nC_2\dfrac{1+2x}{(1+nx)^2} + \cdots$

$= \sum_{r=0}^{n}(-1)^r\,{}^nC_r\dfrac{(1+rx)}{(1+nx)^r}$

$= \sum_{r=0}^{n}(-1)^r\left[\dfrac{{}^nC_r}{(1+nx)^r} + \dfrac{{}^nC_r\,rx}{(1+nx)^r}\right]$

$= \sum_{r=0}^{n} {}^nC_r\left(-\dfrac{1}{1+nx}\right)^r + x\sum_{r=0}^{n}\dfrac{n\cdot{}^{n-1}C_{r-1}}{(1+nx)^r}(-1)^r$

$= \left[1-\dfrac{1}{1+nx}\right]^n - \left(\dfrac{nx}{1+nx}\right)\sum_{r=0}^{n} {}^{n-1}C_{r-1}\left(-\dfrac{1}{1+nx}\right)^{r-1}$

$= \left[1-\dfrac{1}{1+nx}\right]^n - \left(\dfrac{nx}{1+nx}\right)\left[1-\dfrac{1}{1+nx}\right]^{n-1}$

$= \left[1-\dfrac{1}{1+nx}\right]^{n-1}\left[1-\dfrac{1}{1+nx}-\dfrac{nx}{1+nx}\right] = 0$

5. $S = \frac{{}^nC_0}{1} + \frac{{}^nC_2}{3} + \frac{{}^nC_4}{5} + \frac{{}^nC_6}{7} + \cdots$

The general term of the series is

$$\frac{{}^nC_{2r}}{2r+1} = \frac{{}^{n+1}C_{2r+1}}{n+1}, \text{ where } r = 0, 1, 2, \ldots$$

$$\therefore \quad S = \frac{1}{n+1}[{}^{n+1}C_1 + {}^{n+1}C_3 + {}^{n+1}C_5 + \cdots] = \frac{2^n}{n+1}$$

6. $C_2 + 2C_3 + 3C_4 + \cdots + 14C_{15}$

$$= \sum_{r=1}^{14} r\ {}^{15}C_{r+1}$$

$$= \sum_{r=1}^{14} [(r+1)-1]\ {}^{15}C_{r+1}$$

$$= \sum_{r=1}^{14} [(r+1)\ {}^{15}C_{r+1} - {}^{15}C_{r+1}]$$

$$= \sum_{r=1}^{14} (15\ {}^{14}C_r - {}^{15}C_{r+1})$$

$$= 15(2^{14} - 1) - (2^{15} - {}^{15}C_0 - {}^{15}C_1)$$

$$= 13 \times 2^{14} + 1$$

7. Given polynomial is $n + 1$ degree polynomial.

$$(x + {}^nC_0)\ (x + 3\ {}^nC_1)\ (x + 5\ {}^nC_2) \cdots [x + (2n + 1)\ {}^nC_n]$$
$$= x^{n+1} + a_1 x^n + \cdots + a_{n+1}$$

Then,

$$-\frac{a_1}{a_0} = -{}^nC_0 - 3\ {}^nC_1 - 5\ {}^nC_2 - \cdots - (2n + 1)\ {}^nC_n$$

$$\Rightarrow \quad a_1 = {}^nC_0 + 3\ {}^nC_1 + 5\ {}^nC_2 - \cdots + (2n + 1)\ {}^nC_n$$

$$= \sum_{r=0}^{n} {}^nC_r\ (2r+1) = 2\sum_{r=0}^{n} r\ {}^nC_r + \sum_{r=0}^{n} {}^nC_r$$

$$= 2\sum_{r=0}^{n} n \cdot {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^nC_r$$

$$= 2n\ 2^{n-1} + 2^n = (n + 1)2^n$$

8. ${}^{20}C_0 - \frac{{}^{20}C_1}{2} + \frac{{}^{20}C_2}{3} - \frac{{}^{20}C_3}{4} + \cdots$

$$= \sum_{r=0}^{20} \frac{{}^{20}C_r}{r+1}(-1)^r$$

$$= \sum_{r=0}^{20} \frac{{}^{21}C_{r+1}}{20+1}(-1)^r$$

$$= -\frac{1}{21}\sum_{r=0}^{20} {}^{21}C_{r+1}(-1)^{r+1}$$

$$= -\frac{1}{21}\left[-{}^{21}C_1 + {}^{21}C_2 - {}^{21}C_3 + \cdots\right]$$

$$= -\frac{1}{21}\left[({}^{21}C_0 - {}^{21}C_1 + {}^{21}C_2 - {}^{21}C_3 + \cdots) - {}^{21}C_0\right]$$

$$= -\frac{1}{21}[(1 - 1)^{21} - 1]$$

$$= \frac{1}{21}$$

9. $S = {}^{10}C_1(x-1)^2 - {}^{10}C_2(x-2)^2 + {}^{10}C_3(x-3)^2 + \ldots - {}^{10}C_{10}(x-10)^{10}$

$$= \sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r\ (x - r)^2$$

$$= \sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r\ (x^2 - 2x\,r + r^2)$$

$$= \sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r\ (x^2) - 2x\sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r\ r + \sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r\ r^2$$

$$= x^2 \sum_{r=1}^{10} (-1)^{r+1}\ {}^{10}C_r - 2x\ (0) + 0$$

$$= x^2\ ({}^{10}C_1 - {}^{10}C_2 + {}^{10}C_3 - {}^{10}C_4 + \ldots - {}^{10}C_{10})$$
$$= x^2\ (1) = x^2$$

10. $S = \frac{{}^nC_1}{2} - \frac{2({}^nC_2)}{3} + \frac{3({}^nC_3)}{4} \cdots + (-1)^{n+1}\frac{n({}^nC_n)}{n+1}$

$$= \sum_{r=1}^{n} \frac{r\ {}^nC_r}{(r+1)}(-1)^{r+1}$$

$$= \sum_{r=1}^{n} r\frac{{}^{n+1}C_{r+1}}{n+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n} [(r+1)-1]\ {}^{n+1}C_{r+1}(-1)^{r+1}$$

$$= \frac{1}{n+1}\sum_{r=1}^{n} [(r+1)\ {}^{n+1}C_{r+1}(-1)^{r+1} - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= \frac{1}{n+1}\sum_{r=1}^{n} [-(n+1)\ {}^nC_r(-1)^r - {}^{n+1}C_{r+1}(-1)^{r+1}]$$

$$= -[-{}^nC_1 + {}^nC_2 - {}^nC_3 + \ldots + (-1)^n\ {}^nC_n]$$

$$-\frac{1}{n+1}[{}^{n+1}C_2 - {}^{n+1}C_3 + \cdots (-1)^{n+1}\ {}^{n+1}C_{n+1}]$$

$$= -[(1 - 1)^n - 1] - \frac{1}{n+1}[(1-1)^{n+1} - {}^{n+1}C_0 + {}^{n+1}C_1]$$

$$= 1 - \frac{1}{n+1}[(n+1)-1]$$

$$= 1 - \frac{n}{n+1} = \frac{1}{n+1}$$

Exercise 8.7

1. $\sum_{r=0}^{n} r(n-r)({}^nC_r)^2 = \sum_{r=0}^{n} r\ {}^nC_r(n-r)\ {}^nC_{n-r}$

$$= \sum_{r=0}^{n} n\ {}^{n-1}C_{r-1}\ n\ {}^{n-1}C_{n-r-1}$$

$$= n^2\sum_{r=0}^{n} {}^{n-1}C_{r-1}\ {}^{n-1}C_{n-r-1}$$

$$= n^2 \times \text{coefficient of } x^{n-2} \text{ in } (1+x)^{n-1}\ (1+x)^{n-1}$$
$$= n^2 \times {}^{2n-2}C_{n-2} = n^2\ {}^{2n-2}C_n$$

$$\sum_{r=0}^{2n}(-1)^r\,({}^{2n}C_r)^2 = \sum_{r=0}^{2n}(-1)^r\,{}^{2n}C_r\,{}^{2n}C_r$$

$$= \sum_{r=0}^{2n}(-1)^r\,{}^{2n}C_r\,{}^{2n}C_{2n-r}$$

$$= \text{Coefficient of } x^{2n} \text{ in } (1-x)^{2n}(1+x)^{2n}$$

$$= \text{Coefficient of } x^{2n} \text{ is } (1-x^2)^{2n}$$

$$= (-1)^n\,{}^{2n}C_n$$

3. ${}^{84}C_4 + 6\times{}^{84}C_5 + 15\times{}^{84}C_6 + 20\times{}^{84}C_7 + 15\times{}^{84}C_8 + 6\times{}^{84}C_9 + {}^{84}C_{10}$

$$= {}^6C_6\times{}^{84}C_4 + {}^6C_5\times{}^{84}C_5 + {}^6C_4\times{}^{84}C_6 + {}^6C_3\times{}^{84}C_7 + {}^6C_2\times{}^{84}C_8 + {}^6C_1\times{}^{84}C_9 + {}^6C_0\times{}^{84}C_{10}$$

$$= \text{Coefficient of } x^{10} \text{ in } (1+x)^6(1+x)^{84}$$

$$= \text{Coefficient of } x^{10} \text{ in } (1+x)^{90}$$

$$= {}^{90}C_{10}$$

4. $S = {}^nC_0\,{}^nC_2 + 2\,{}^nC_1\,{}^nC_3 + 3\,{}^nC_2\,{}^nC_4 + \cdots + (n-1)\,{}^nC_{n-2}\,{}^nC_n$

$$= {}^nC_0\,{}^nC_{n-2} + 2\,{}^nC_1\,{}^nC_{n-3} + 3\,{}^nC_2\,{}^nC_{n-4} + \cdots + (n-1)\,{}^nC_{n-2}\,{}^nC_0$$

$$= \sum_{r=1}^{n} r\,{}^nC_{r-1}\,{}^nC_{n-r-1}$$

$$= \sum_{r=1}^{n}((r-1)+1)\,{}^nC_{r-1}\,{}^nC_{n-r-1}$$

$$= \sum_{r=1}^{n}\left[(r-1)\,{}^nC_{r-1}\,{}^nC_{n-r-1} + {}^nC_{r-1}\,{}^nC_{n-r-1}\right]$$

$$= \sum_{r=1}^{n}\left[n\,{}^{n-1}C_{r-2}\,{}^nC_{n-r-1} + {}^nC_{r-1}\,{}^nC_{n-r-1}\right]$$

$$= n\,{}^{2n-1}C_{n-3} + {}^{2n}C_{n-2}$$

5. $S = C_0 - 2^2C_1 + 3^2C_2 - \cdots + (-1)^n(n+1)^2C_n$

$$T_r = (-1)^r r^2\,{}^nC_r$$

$$= (-1)^r r(r\,{}^nC_r)$$

$$= (-1)^r r(n\,{}^{n-1}C_{r-1})$$

$$= n(-1)^r((r-1)+1)({}^{n-1}C_{r-1})$$

$$= n(-1)^r((r-1)\,{}^{n-1}C_{r-1} + {}^{n-1}C_{r-1})$$

$$= n(-1)^r((n-1)\,{}^{n-2}C_{r-2} + {}^{n-1}C_{r-1})$$

$$= n(n-1)\,{}^{n-2}C_{r-2}(-1)^{r-2} - n\,{}^{n-1}C_{r-1}(-1)^{r-1}$$

$$\Rightarrow \quad S = \sum_{r=0}^{n}T_r$$

$$= n(n-1)(1-1)^{n-2} - n(1-1)^{n-1}$$

$$= 0$$

6. $S = \sum\sum_{1\le i<j\le n-1}(i\,{}^nC_i)(j\,{}^nC_j)$

$$= n^2\sum\sum_{1\le i<j\le n-1}{}^{n-1}C_{i-1}\,{}^{n-1}C_{j-1}$$

$$= n^2\left(\frac{2^{2(n-1)} - {}^{2(n-1)}C_{n-1}}{2}\right)$$

7. ${}^nC_0\,{}^nC_0 - {}^nC_1\,{}^{n+1}C_1 + {}^nC_2\,{}^{n+2}C_2 - \cdots$

$$= {}^nC_0\,{}^nC_n - {}^nC_1\,{}^{n+1}C_n + {}^nC_2\,{}^{n+2}C_n - \cdots$$

$$= \text{Coefficient of } x^n \text{ in } [{}^nC_0(1+x)^n - {}^nC_1(1+x)^{n+1} + {}^nC_2(1+x)^{n+2} + \cdots + (-1)^n\,{}^nC_n(1+x)^{2n}]$$

$$= \text{Coefficient of } x^n \text{ in } [{}^nC_0(1+x)^n - {}^nC_1(1+x)^{n+1} + {}^nC_2(1+x)^{n+2} - \cdots]$$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^n[{}^nC_0 - {}^nC_1(1+x) + {}^nC_2(1+x)^2 - \cdots + (-1)^n\,{}^nC_n(1+x)^n]$$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^n[1-(1+x)]^n$$

$$= \text{Coefficient of } x^n \text{ in } (1+x)^n(-x)^n$$

$$= (-1)^n$$

8. L.H.S. $= {}^nC_0\,{}^{2n}C_n - {}^nC_1\,{}^{2n-2}C_n + {}^nC_2\,{}^{2n-4}C_n - \cdots$

$$= \text{Coefficient of } x^n \text{ in } [{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2} + {}^nC_2(1+x)^{2n-4} - \cdots]$$

$$= \text{Coefficient of } x^n \text{ in } [(1+x)^2 - 1]^n$$

$$= \text{Coefficient of } x^n \text{ in } (2x+x^2)^n$$

$$= \text{Coefficient of } x^n \text{ in } x^n(2+x)^n$$

$$= 2^n$$

$$= \text{R.H.S.}$$

9. $S = \sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\,{}^mC_p\right)$

$$= \sum_{p=1}^{n}({}^nC_p\,{}^pC_p + {}^nC_{p+1}\,{}^{p+1}C_p + \cdots + {}^nC_n\,{}^nC_p)$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{{}^nC_p(1+x)^p + {}^nC_{p+1}(1+x)^{p+1} + \cdots + {}^nC_n(1+x)^n\}]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{{}^nC_0 + {}^nC_1(1+x) + {}^nC_2(1+x)^2 + \cdots + {}^nC_{p-1}(1+x)^{p-1} + {}^nC_p(1+x)^p + \cdots + {}^nC_n(1+x)^n\}]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } \{1+(1+x)\}^n]$$

$$= \sum_{p=1}^{n}[\text{coefficient of } x^p \text{ in } (2+x)^n]$$

$$= \sum_{p=1}^{n}[{}^nC_p\,2^{n-p}]$$

$$= {}^nC_1\,2^{n-1} + {}^nC_2\,2^{n-2} + \cdots + {}^nC_n$$

$$= {}^nC_0\,2^n + {}^nC_1\,2^{n-1} + {}^nC_2\,2^{n-2} + \cdots + {}^nC_n - {}^nC_0\,2^n$$

$$= (1+2)^n - 2^n$$

$$= 3^n - 2^n$$

Alternate solution:

$$\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\,{}^mC_p\right) = \sum_{p=1}^{n}\left(\sum_{m=p}^{n}\frac{n!}{(n-m)!m!}\,\frac{m!}{(m-p)!p!}\right)$$

$$= \sum_{p=1}^{n}\left(\sum_{m=p}^{n}\frac{n!}{(n-m)!}\,\frac{1}{(m-p)!p!}\right)$$

$$= \sum_{p=1}^{n}\frac{n!}{(n-p)!p!}\left(\sum_{m=p}^{n}\frac{(n-p)!}{(n-m)!(m-p)!}\right)$$

$$= \sum_{p=1}^{n}{}^nC_p\left(\sum_{m=p}^{n}{}^{n-p}C_{m-p}\right)$$

$$= \sum_{p=1}^{n}{}^nC_p\,2^{n-p}$$

$$= {}^nC_1 2^{n-1} + {}^nC_2 2^{n-2} + \cdots + {}^nC_n 2^0$$
$$= (2+1)^n - {}^nC_0 2^n$$
$$= 3^n - 2^n$$

Also,

$$\lim_{n\to\infty} \frac{1}{3^n} \sum_{p=1}^{n} \left(\sum_{m=p}^{n} {}^nC_m \, {}^mC_p \right) = \lim_{n\to\infty} \frac{3^n - 2^n}{3^n} = 1$$

Exercise 8.8

1. We have

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \cdots$$

Given that the third term is $-(1/8)x^2$, hence

$$\frac{m(m-1)}{2}x^2 = -\frac{1}{8}x^2$$

or $\quad 4m^2 - 4m = -1$

or $\quad (2m-1)^2 = 0$ or $m = \frac{1}{2}$

2. $(217)^{1/3} = (6^3+1)^{1/3} = 6\left(1+\frac{1}{6^3}\right)^{1/3}$

$$= 6\left(1 + \frac{1}{3\times 216} - \frac{1\times 2}{3\times 3\times 2}\left(\frac{1}{216}\right)^2 + \cdots\right) = 6.01$$

3. $\sqrt{3} = \left(\frac{1}{3}\right)^{-1/2}$

$$= \left(1-\frac{2}{3}\right)^{-1/2}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-\frac{2}{3}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{2}{3}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(-\frac{2}{3}\right)^3 + \cdots$$

$$= 1 + \frac{1}{2}\cdot\frac{2}{3} + \frac{1}{2}\cdot\left(\frac{3}{2}\right)\cdot\left(\frac{2}{3}\right)^2\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\left(\frac{2}{3}\right)^3\cdot\frac{1}{3!} + \cdots$$

$$= 1 + \frac{1}{3} + \frac{1}{3}\left(\frac{3}{6}\right) + \frac{1}{3}\left(\frac{3}{6}\right)\left(\frac{5}{9}\right) + \cdots$$

4. $\left(\frac{a+x}{a}\right)^{-1/2} + \left(\frac{a-x}{a}\right)^{-1/2} = \left(1+\frac{x}{a}\right)^{-1/2} + \left(1-\frac{x}{a}\right)^{-1/2}$

$$= \left[1 + \left(-\frac{1}{2}\right)\left(\frac{x}{a}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2.1}\left(\frac{x}{a}\right)^2 + \cdots\right]$$
$$+ \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{a}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2.1}\left(-\frac{x}{a}\right)^2 + \cdots\right]$$

$$= 2 + \frac{3x^2}{4a^2} + \cdots$$

Hence, the coefficient of x^2 is $(3/4)a^2$.

5. Given expression is

$$(1+x+x^2+\cdots)^2 = [(1-x)^{-1}]^2$$
$$= (1-x)^{-2}$$
$$= 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + (n+1)x^n + \ldots$$

Therefore, the coefficient of x^n is $n+1$.

6. Given that $|x| > 1$.

So, the given expression can be written as

$$x^{-2}\left(1+\frac{1}{x}\right)^{-2} = x^{-2}\left[1 - \frac{2}{x} + \frac{3}{x^2} - \frac{4}{x^3} + \cdots\right]$$
$$= \left[\frac{1}{x^2} - \frac{2}{x^3} + \frac{3}{x^4} - \frac{4}{x^5} + \cdots\right]$$

7. Since $1 + 2x + 3x^2 + 4x^3 + \cdots \infty = (1-x)^{-2}$, we have

$$(1 + 2x + 3x^2 + 4x^3 + \cdots \infty)^{1/2} = [(1-x)^{-2}]^{1/2}$$
$$= (1-x)^{-1}$$
$$= 1 + x + x^2 + \cdots + x^n + \cdots \infty$$

Therefore, the coefficient of x^n is 1.

8. We have

$$T_{r+1} = \frac{\frac{7}{2}\left(\frac{7}{2}-1\right)\left(\frac{7}{2}-2\right)\cdots\left(\frac{7}{2}-r+1\right)x^r}{r!}$$

This will be the first negative term when

$$\frac{7}{2} - r + 1 < 0, \text{ i.e., } r > \frac{9}{2}$$

Hence, $r = 5$.

9. We have $\dfrac{1}{(1-x)(1-2x)(1-3x)}$

$$= \frac{1}{2(1-x)} - \frac{4}{1-2x} + \frac{9}{2(1-3x)} \quad \text{[By resolving into partial fractions]}$$

$$= \frac{1}{2}(1-x)^{-1} - 4(1-2x)^{-1} + \frac{9}{2}(1-3x)^{-1}$$

$$= \frac{1}{2}(1 + x + x^2 + \ldots + x^n + \ldots) - 4(1 + 2x + (2x)^2 + \ldots + (2x)^n + \ldots) + \frac{9}{2}(1 + (3x) + (3x)^2 + \ldots + (3x)^n + \ldots)$$

$\therefore$ Coefficient of $x^n = \frac{1}{2}[1 - 8.2^n + 9.3^n]$

$$= \frac{1}{2}[1 - 2^{n+3} + 3^{n+2}].$$

10. $(r+1)\times {}^nC_0 - r\times {}^nC_1 + (r-1)\times {}^nC_2 - (r-2)\,{}^nC_3 + \ldots + (-1)^r \times {}^nC_r + \ldots$

= Coefficient of x^r in

$$({}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \ldots + (-1)^r \times {}^nC_r + \ldots) \times (1 + 2x + 3x^2 + 4x^3 + \ldots (r+1)x^r + \ldots)$$

= Coefficient of x^r in $(1-x)^n(1-x)^{-2}$

= Coefficient of x^r in $(1-x)^{n-2}$

$= (-1)^r \times {}^{n-2}C_r$

Exercises

Single Correct Answer Type

1. (3) Coefficient of T_5 is ${}^nC_4$, that of T_6 is ${}^nC_5$ and that of T_7 is ${}^nC_6$.
According to the condition, $2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$$\therefore \quad 2\left[\frac{n!}{(n-5)!5!}\right] = \left[\frac{n!}{(n-4)!4!} + \frac{n!}{(n-6)!6!}\right]$$

$$\text{or} \quad 2\left[\frac{1}{(n-5)5}\right] = \left[\frac{1}{(n-4)(n-5)} + \frac{1}{6\times 5}\right]$$

After solving, we get $n = 7$ or 14.

2. (3) Middle term of $(1+\alpha x)^4$ is T_3.
Its coefficient is ${}^4C_2\,(\alpha)^2 = 6\alpha^2$.
Middle term of $(1-\alpha x)^6$ is T_4.
Its coefficient is ${}^6C_3\,(-\alpha)^3 = -20\alpha^3$.
According to question,

$$6\alpha^2 = -20\alpha^3$$

$$\text{or} \quad 3\alpha^2 + 10\alpha^3 = 0$$

$$\text{or} \quad \alpha^2(3 + 10\alpha) = 0$$

$$\text{or} \quad \alpha = -\frac{3}{10}$$

3. (2) Put $x = i$

$$(1+i)^5 = (a_0 - a_2 + a_4) + i(a_1 - a_3 + a_5)$$

$$\Rightarrow \quad |1+i|^5 = |(a_0 - a_2 + a_4) + i(a_1 - a_3 + a_5)|$$

$$\Rightarrow \quad (a_0 - a_2 + a_4)^2 + (a_1 - a_3 + a_5)^2 = 2^5 = 32$$

4. (1) We have,

$$\frac{2}{\sqrt{2x^2+1} + \sqrt{2x^2-1}} = \frac{2\left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)}{\left(2x^2+1\right) - \left(2x^2-1\right)}$$

$$= \sqrt{2x^2+1} - \sqrt{2x^2-1}$$

Thus, the given expression can be written as

$$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$$

But $(a+b)^6 + (a-b)^6 = 2[a^6 + {}^6C_2a^4b^2 + {}^6C_4a^2b^4 + b^6]$
Therefore,

$$\left(\sqrt{2x^2+1} + \sqrt{2x^2-1}\right)^6 + \left(\sqrt{2x^2+1} - \sqrt{2x^2-1}\right)^6$$

$$= 2[(2x^2+1)^3 + 15(2x^2+1)^2(2x^2-1) + 15(2x^2+1)\times(2x^2-1)^2 + (2x^2-1)^3]$$

which is a polynomial of degree 6.

5. (3) It is given that 6^{th} term in the expansion of $\left(\frac{1}{x^{8/3}} + x^2\log_{10}x\right)^8$ is 5600; therefore,

$${}^8C_5\left(x^2\log_{10}x\right)^5\left(\frac{1}{x^{8/3}}\right)^3 = 5600$$

$$\text{or} \quad 56x^{10}(\log_{10}x)^5\frac{1}{x^8} = 5600$$

$$\text{or} \quad x^2(\log_{10}x)^5 = 100$$

$$\text{or} \quad x^2(\log_{10}x)^5 = 10^2(\log_{10}10)^5$$

$$\text{or} \quad x = 10$$

6. (2) $T_5 = {}^nC_4\,a^{n-4}(-2b)^4$
and $T_6 = {}^nC_5\,a^{n-5}(-2b)^5$
As $T_5 + T_6 = 0$, we get

$${}^nC_4\,2^4\,a^{n-4}\,b^4 = {}^nC_5\,2^5\,a^{n-5}\,b^5$$

$$\text{or} \quad \frac{a^{n-4}b^4}{a^{n-5}\,b^5} = \frac{n!2^5}{5!(n-5)!}\cdot\frac{4!\,(n-4)!}{n!2^4}$$

$$\text{or} \quad \frac{a}{b} = \frac{2(n-4)}{5}$$

7. (1) $T_{r+1} = {}^{4n-2}C_r(ix)^r$
T_{r+1} is negative, if i^r is negative and real.

$$i^r = -1$$

$$\Rightarrow \quad r = 2, 6, 10, \ldots \text{ which form an A.P.}$$

$$0 \le r \le 4n-2$$

$$4n - 2 = 2 + (r-1)\,4$$

$$\Rightarrow \quad r = n$$

The required number of terms is n.

8. (4) General term in the expansion of $(\sqrt{2} + \sqrt[3]{3} + \sqrt[6]{5})^{10}$ is

$$\frac{10!}{a!b!c!}(\sqrt{2})^a(\sqrt[3]{3})^b(\sqrt[6]{5})^c \text{ where } a+b+c = 10.$$

For rational term, we have the following:

Value of a, b, c	Value of term
$a = 4, b = 0, c = 6$	$\frac{10!}{4!0!6!}(\sqrt{2})^4(\sqrt[3]{3})^0(\sqrt[6]{5})^6 = 4200$
$a = 10, b = 0, c = 0$	$\frac{10!}{10!0!0!}(\sqrt{2})^{10}(\sqrt[3]{3})^0(\sqrt[6]{5})^0 = 32$
$a = 4, b = 6, c = 0$	$\frac{10!}{4!6!0!}(\sqrt{2})^4(\sqrt[3]{3})^6(\sqrt[6]{5})^0 = 7560$

9. (2) By the given condition,

$$84 = T_6 = T_{5+1}$$

$$= {}^7C_5\left(2^{\log_2\sqrt{9^{x-1}+7}}\right)^2\left(\frac{1}{2^{\frac{1}{5}\log_2\left(3^{x-1}+1\right)}}\right)^5$$

$$= 21\times 2^{\log_2(9^{x-1}+7)}\,2^{-\log_2(3^{x-1}+1)}$$

$$\text{or} \quad 4 = 2^{\log_2\frac{9^{x-1}+7}{3^{x-1}+1}} = \frac{9^{x-1}+7}{3^{x-1}+1}$$

$$\text{or} \quad (3^{x-1})^2 - 4\times 3^{x-1} + 3 = 0$$

$$\text{or} \quad (3^{x-1}-1)(3^{x-1}-3) = 0$$

$$\Rightarrow \quad 3^{x-1} = 1 \text{ or } 3$$

$$\Rightarrow \quad 3^{x-1} = 3^0 \text{ or } 3^1$$

$$\Rightarrow \quad x - 1 = 0 \text{ or } 1$$

$$\Rightarrow \quad x = 1, 2$$

10. (2) $\left(x + \frac{1}{x} + x^2 + \frac{1}{x^2}\right)^{15} = \left(\frac{x^3 + x + x^4 + 1}{x^2}\right)^{15}$

$$= \frac{a_0 + a_1x + a_2x^2 + \cdots + a_{60}x^{60}}{x^{30}}$$

Hence, the total number of terms is 61.

11. (2) $(1+2)^n - (1-2)^n = 2[{}^nC_1\times 2 + {}^nC_3\times 2^3 + {}^nC_5\times 2^5 + \ldots]$

$$\therefore \quad {}^nC_1\times 2 + {}^nC_3\times 2^3 + {}^nC_5\times 2^5 + \ldots = \frac{3^n - (-1)^n}{2}$$

12. (2) $(4x^2+1)^n = \sum_{r=0}^{n} a_r (1+x^2)^{n-r} x^{2r}$

$\Rightarrow \left(1+\frac{3x^2}{1+x^2}\right)^n = \sum_{r=0}^{n} a_r \left(\frac{x^2}{1+x^2}\right)^r$

$\therefore \quad a_r = {}^nC_r 3^r$

$\therefore \quad \sum_{r=0}^{n} a_r = \sum_{r=0}^{n} {}^nC_r 3^r = (1+3)^n = 4^n$

13. (1) $\frac{2^{4n}}{15} = \frac{(15+1)^n}{15}$

$= \frac{({}^nC_0 15^n + {}^nC_1 15^{n-1} + \cdots + {}^nC_{n-1}15 + {}^nC_n)}{15}$

$=$ Integer $+ \frac{1}{15}$

Hence, the fractional part of $\frac{2^{4n}}{15}$ is $\frac{1}{15}$.

14. (1) $p = (8+3\sqrt{7})^n = {}^nC_0 8^n + {}^nC_1 8^{n-1}(3\sqrt{7}) + \ldots$

Let $p_1 = (8-3\sqrt{7})^n = {}^nC_0 8^n - {}^7C_1 8^{n-1}(3\sqrt{7}) + \ldots$

$p + p_1 = 2({}^nC_0 8^n + {}^nC_2 8^{n-2}(3\sqrt{7})^2 + \cdots) =$ even integer

p_1 clearly belongs to (0, 1)

$\Rightarrow \quad [p] + f + p_1 =$ even integer

$\Rightarrow \quad f + p_1 =$ integer

$f \in (0, 1), p_1 \in (0, 1)$

$\Rightarrow \quad f + p \in (0, 2)$

$\Rightarrow \quad f + p_1 = 1$

$\Rightarrow \quad p_1 = 1 - f$

Now, $p(1-f) = pp_1 = [(8+3\sqrt{7})^n (8-3\sqrt{7})]^n = 1$

15. (1) $3^{256} - 3^{12} = 3^{12}(3^{244} - 1)$

$= (1+8)^6((1+8)^{122} - 1)$

$= (1+8\lambda)(1+8\mu - 1)$

$= 8\mu(1+8\lambda)$

So, number is divisible by 8.

16. (2) $(\sqrt{3}+\sqrt{2})^6 + (\sqrt{3}-\sqrt{2})^6$

$= 2({}^6C_0(\sqrt{3})^6 + {}^6C_2(\sqrt{3})^4(\sqrt{2})^2 + {}^6C_4(\sqrt{3})^2(\sqrt{2})^4 + {}^6C_6(\sqrt{2})^6)$

$= 970$

Since $0 < (\sqrt{3}-\sqrt{2})^6 < 1$, the smallest integer larger than $(\sqrt{3}+\sqrt{2})^6$ is 970.

17. (3) $(1+x)^{21} + (1+x)^{22} + \cdots + (1+x)^{30}$

$= (1+x)^{21}\left[\frac{(1+x)^{10}-1}{(1+x)-1}\right] = \frac{1}{x}\left[(1+x)^{31} - (1+x)^{21}\right]$

$\Rightarrow$ Coefficient of x^5 in the given expression

$=$ Coefficient of x^5 in $\left\{\frac{1}{x}\left[(1+x)^{31} - (1+x)^{21}\right]\right\}$

$=$ Coefficient of x^6 in $[(1+x)^{31} - (1+x)^{21}]$

$= {}^{31}C_6 - {}^{21}C_6$

18. (4) Let $(x^3+2x^2+x+4)^{15} = a_0 + a_1 x + a_2 x^2 + \ldots$

Differentiating w.r.t. x, we get

$15(x^3+2x^2+x+4)^{14}(3x^2+4x+1)$

$= a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

Again, differentiating w.r.t x, we get

$15\{(x^3+2x^2+x+4)^{14}(6x+4) + 14(x^3+2x^2+x+4)^{13}(3x^2+4x+1)^2\}$

$= 2a_2 + 6a_3 x + \ldots$

Now, putting $x = 0$, we get

$15(4 \times 4^{14} + 14 \times 4^{13}) = 2a_2$

$\Rightarrow \quad 2a_2 = 15 \times 4^{13} \times (16 + 14)$

$\Rightarrow \quad a_2 = (15)^2 \times 2^{26}$

which is a perfect square.

19. (2) We have $T_{r+1} = {}^{29}C_r 3^{29-r}(7x)^r = ({}^{29}C_r \times 3^{29-r} \times 7^r)x^r$

Coefficient of $(r+1)^{\text{th}}$ term is ${}^{29}C_r \times 3^{29-r} \times 7^r$

and coefficient of r^{th} term is ${}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

From given condition,

${}^{29}C_r \times 3^{29-r} \times 7^r = {}^{29}C_{r-1} \times 3^{30-r} \times 7^{r-1}$

$\Rightarrow \quad \frac{{}^{29}C_r}{{}^{29}C_{r-1}} = \frac{3}{7}$

$\Rightarrow \quad \frac{30-r}{r} = \frac{3}{7}$ or $r = 21$

20. (3) $\left(\frac{1}{x^2} - x^3\right)^n$

General term $T_{r+1} = \frac{n!}{r!(n-r)!}(-1)^{n-r} x^{5r-2n}$

If $5r - 2n = 5$, then $5r = 2n + 5$ or $r = \frac{2n}{5} + 1$

If $5r - 2n = 10$, then $5r = 2n + 10$ or $r = \frac{2n}{5} + 2$

x^5 and x^{10} terms occurs if $n = 5k$

Given that sum of x^5 and x^{10} is zero.

$\Rightarrow \quad \frac{5k!}{(2k+1)!(3k-1)!} - \frac{5k!}{(2k+2)!(3k-2)!} = 0$

or $\quad \frac{1}{3k-1} - \frac{1}{2k+2} = 0$

or $\quad k = 3 \Rightarrow n = 15$

21. (3) $(1+2x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$

or $\quad [(1+x)^2]^n = \sum_{r=0}^{2n} a_r x^r$

or $\quad (1+x)^{2n} = \sum_{r=0}^{2n} a_r x^r$

or $\quad \sum_{r=0}^{2n} {}^{2n}C_r x^r = \sum_{r=0}^{2n} a_r x^r$

or $\quad a_r = {}^{2n}C_r$

22. (2) $T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r}\left(\frac{-k}{x^2}\right)^r = {}^{10}C_r x^{5-5r/2}(-k)^r$

For this to be independent of x, r must be 2, so that

${}^{10}C_2 k^2 = 405$ or $k = \pm 3$

23. (3) The given sigma is the expansion of

$[(x-3)+2]^{100} = (x-1)^{100} = (1-x)^{100}$

Therefore, x^{53} will occur in T_{54}.

$T_{54} = {}^{100}C_{53}(-x)^{53}$

Therefore, the coefficient is $-{}^{100}C_{53}$.

24. (3) For $\left(ax^2 + \left(\frac{1}{bx}\right)\right)^{11}$,

$$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx}\right)^r = {}^{11}C_r\, a^{11-r}\, \frac{1}{b^r}\, x^{22-3r}$$

For x^7,

$22 - 3r = 7$

or $3r = 15$

or $r = 5$

$\Rightarrow$ $T_6 = {}^{11}C_5\, a^6\, \frac{1}{b^5}\, x^7$

$\Rightarrow$ Coefficient of x^7 is ${}^{11}C_5\, \frac{a^6}{b^5}$

Similarly, coefficient of x^{-7} in $\left(ax - \left(\frac{1}{bx^2}\right)\right)^{11}$ is ${}^{11}C_6\, \frac{a^5}{b^6}$.

Given that ${}^{11}C_5\, \frac{a^6}{b^5} = {}^{11}C_6\, \frac{a^5}{b_6}$

$\Rightarrow$ $a = \frac{1}{b}$

or $ab = 1$

25. (4) $(x-2)^5 (x+1)^5$

$= [{}^5C_0 x^5 - {}^5C_1 x^4 \times 2 + \cdots][{}^5C_0 + {}^5C_1 x + \cdots]$

$\Rightarrow$ Coefficient of x^5

$= {}^5C_0\, {}^5C_5 - {}^5C_1 \times 2 \times {}^5C_4 + {}^5C_2 \times 2^2 \times {}^5C_3$
$\quad - {}^5C_3 \times 2^3 \times {}^5C_2 + {}^5C_4 \times 2^4 \times {}^5C_1 - {}^5C_5 \times 2^5 \times {}^5C_0$

$= 1 - 5 \times 5 \times 2 + 10 \times 10 \times 4 - 10 \times 10 \times 8$
$\quad + 5 \times 5 \times 16 - 32$

$= -81$

26. (2) $\left(1 + \sqrt{a} + \frac{1}{\sqrt{a}-1}\right)^{-30}$

$= \left(\frac{a}{\sqrt{a}-1}\right)^{-30}$

$= \left(\frac{\sqrt{a}-1}{a}\right)^{30}$

$= \frac{1}{a^{30}}(1-\sqrt{a})^{30}$

$= \frac{1}{a^{30}}\{{}^{30}C_0 - {}^{30}C_1\sqrt{a} + \cdots + {}^{30}C_{30}(\sqrt{a})^{30}\}$

There is no term independent of a.

27. (1) We rewrite the given expression as $[1 + x^2(1-x)]^8$ and expand by using the binomial theorem. We have,

$$[1 + x^2(1-x)]^8 = {}^8C_0 + {}^8C_1 x^2 (1-x) + {}^8C_2 x^4 (1-x)^2 + {}^8C_3 x^6 (1-x)^3 + {}^8C_4 x^8 (1-x)^4 + {}^8C_5 x^{10} (1-x)^5 + \cdots$$

The two terms which contain x^{10} are ${}^8C_4\, x^8\, (1-x)^4$ and ${}^8C_5 x^{10} (1-x)^5$.

Thus, the coefficient of x^{10} in the given expression is given by ${}^8C_4$ [coefficient of x^2 in the expansion of $(1-x)^4$] + ${}^8C_5$

$= {}^8C_4\,(6) + {}^8C_5 = \frac{8!}{4!4!}(6) + \frac{8!}{3!5!}$

$= (70)(6) + 56 = 476$

28. (3) We have,

$(1+x)^{101}(1-x+x^2)^{100}$

$= (1+x)((1+x)(1-x+x^2))^{100}$

$= (1+x)(1+x^3)^{100}$

$= (1+x)\{C_0 + C_1 x^3 + C_2 x^6 + \cdots + C_{100} x^{300}\}$, where $C_r = {}^{100}C_r$

$= (1+x)\sum_{r=0}^{n} {}^nC_r x^{3r} = \sum_{r=0}^{n} {}^nC_r x^{3r} + \sum_{r=0}^{n} {}^nC_r x^{3r+1}$

Hence, there will be no term containing $3r + 2$.

29. (2) $(1 + x^3 - x^6)^{30}$

$= \{1 + x^3(1-x^3)\}^{30}$

$= {}^{30}C_0 + {}^{30}C_1 x^3 (1-x^3) + {}^{30}C_2 x^6 (1-x^3)^2 + \cdots$

Obviously, each term will contain x^{3m}, $m \in N$. But 28 is not divisible by 3. Therefore, there will be no term containing x^{28}.

30. (3) $a^{10}b^{10}c^{10}d^{10}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10}$

Therefore the required coefficient is equal to the coefficient of $a^{-2}b^{-6}c^{-1}d^{-1}$ in $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10}$, which is given by

$$\frac{10!}{2!6!1!1!} = \frac{10 \times 9 \times 8 \times 7}{2} = 2520$$

31. (3) $\left(1 + x + \frac{7}{x}\right)^{11} = \left(\frac{7 + x + x^2}{x}\right)^{11}$

So, we have to find the coefficient of x^{11} in $(7 + x + x^2)^{11}$

Now $(7 + x + x^2)^{11}$

$= [(7 + x) + x^2)]^{11}$

$= {}^{11}C_0 (7+x)^{11} + {}^{11}C_1 (7+x)^{10} x^2 + {}^{11}C_2 (7+x)^9 x^4 \ldots + \ldots$

$\therefore$ Required coefficient $= {}^{11}C_0 + {}^{11}C_1 \times {}^{10}C_9 \times 7 + {}^{11}C_2 \times {}^9C_7 \times 7^2 + {}^{11}C_3 \times {}^8C_5 \times 7^3 + {}^{11}C_4 \times {}^7C_3 \times 7^4 + {}^{11}C_5 \times {}^6C_1 \times 7^5$

$$= \sum_{r=0}^{5} {}^{11}C_r\, {}^{11-r}C_{11-2r}\, 7^r$$

32. (3) $(1 - x - x^3 + x^4)^8 = (1-x)^8 (1-x^3)^8$

$= (1 - {}^8C_1 x + {}^8C_1 x^2 - {}^8C_3 x^3 + \ldots)$
$\quad \times (1 - {}^8C_1 x^3 + {}^8C_2 x^6 - {}^8C_3 x^9 + \ldots)$

$\therefore$ Coefficient of $x^7 = -{}^8C_7 - {}^8C_1 \times {}^8C_4 - {}^8C_1 \times {}^8C_2$

$= -792$

33. (2) Required sum of coefficients

= coefficient of t^{13} in $(1+t)^{11}(1+t^2-t)^{10}$

= coefficient of t^{13} in $(1+t)(1+t^3)^{10}$

= coefficient of t^{13} in $((1+t^3)^{10} + t(1+t^3)^{10})$

= coefficient of t^{12} in $(1+t^3)^{10}$

$= {}^{10}C_4$

34. (4) The general term in the expansion of $(1 - x + y)^{20}$ is $\frac{20!}{r!s!t!} 1^r (-x)^s (y)^t$, where $r + s + t = 20$

For x^2y^3, we have the term

$$\frac{20!}{15!2!3!} 1^{15} (-x)^2 (y)^3$$

Hence, the coefficient of x^2y^3 is

$$\frac{20!}{15!2!3!}$$

35. (4) As $L \neq 0$

$\therefore \quad m = 2 + 3 + 4 = 9$

$\therefore \quad L = \dfrac{9!}{2!3!4!}$

Now coefficient of $a^4b^4c = \dfrac{9!}{4!4!1!} = \dfrac{L}{2}$

36. (2) We have

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \cdots + (x+2)^{n-1}$$

$$= \frac{(x+3)^n - (x+2)^n}{(x+3)-(x+2)} = (x+3)^n - (x+2)^n$$

$$\left(\because \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a^1 + x^{n-3}a^2 + \cdots + a^{n-1}\right)$$

Therefore, coefficient of x^r in the given expression is equal to coefficient of x^r in $[(x+3)^n - (x+2)^n]$, which is given by

$${}^nC_r 3^{n-r} - {}^nC_r 2^{n-r} = {}^nC_r (3^{n-r} - 2^{n-r})$$

37. (2) a_1 = coefficient of x in $(1 + 2x + 3x^2)^{10}$

= coefficient of x in $((1+2x) + 3x^2)^{10}$

= coefficient of x in $({}^{10}C_0(1+2x)^{10} + {}^{10}C_1(1+2x)^9(3x^2) + \cdots)$

= coefficient of x in ${}^{10}C_0(1+2x)^{10}$

$= {}^{10}C_0 2 \cdot {}^{10}C_1 = 20$

38. (2) $f(x) = 1 - x + x^2 - x^3 + \cdots - x^{15} + x^{16} - x^{17} = \dfrac{1-x^{18}}{1+x}$

$\Rightarrow \quad f(x-1) = \dfrac{1-(x-1)^{18}}{x}$

Therefore, required coefficient of x^2 is equal to coefficient of x^3 in $1 - (x-1)^{18}$, which is given by ${}^{18}C_3 = 816$.

39. (2) $\dfrac{f(x)}{1-x} = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots$

$\Rightarrow \quad a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$

$= (1-x)(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + \cdots)$

Comparing the coefficient of x^n on both the sides,

$a_n = b_n - b_{n-1}$.

40. (1) Coefficient of x^n in $\left(1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots + \dfrac{x^n}{n!}\right)^3$

$= $ Coefficient of x^n in $\left(1 + x + \dfrac{x^2}{2!} + \ldots \dfrac{x^n}{n!} + \ldots\right)^3$

(as higher powers of x are not considered while calculating the coefficient of x^n)

$= $ Coefficient x^n in $e^{3x} = \dfrac{3^n}{n!}$

41. (1) To get sum of coefficients put $x = 0$. Given that sum of coefficients is

$2^n = 64$

or $\quad n = 6$

The greatest binomial coefficient is ${}^6C_3$.

Now given that

$T_4 - T_3 = 6 - 1 = 5$

$\Rightarrow \quad {}^6C_3(3^{-x/4})^3(3^{5x/4})^3 - {}^6C_2(3^{-x/4})^4(3^{5x/4})^2 = 5$

which is satisfied by $x = 0$.

42. (3) $(1 + x + x^2 + x^3)^5 = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{15}x^{15}$

Putting $x = 1$ and $x = -1$ alternatively, we have

$a_0 + a_1 + a_2 + a_3 + \cdots + a_{15} = 4^5 \quad (1)$

$a_0 - a_1 + a_2 - a_3 + \cdots - a_{15} = 0 \quad (2)$

Adding (1) and (2), we have

$2(a_0 + a_2 + a_4 + \cdots + a_{14}) = 4^5$

or $\quad a_0 + a_2 + a_4 + \cdots + a_{14} = 2^9 = 512$

43. (3) Sum of coefficients in $(1 - x\sin\theta + x^2)^n$ is $(1 - \sin\theta + 1)^n$ (putting $x = 1$)

This sum is greatest when $\sin\theta = -1$, then maximum sum is 3^n.

44. (1) We know that the sum of the coefficients in a binomial expansion is obtained by replacing each variable by '1' in the given expression. Therefore, sum of the coefficients in $(a+b)^n$ is given by $(1+1)^n$.

$\therefore \quad 4096 = 2^n$

or $\quad 2^n = 2^{12}$ or $n = 12$

Hence, n is even. So, the greatest coefficient is ${}^nC_{n/2}$, i.e., ${}^{12}C_6 = 924$.

45. (2) Given series is ${}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \cdots + {}^{20}C_8$

$$= \frac{1}{2}(2\cdot{}^{20}C_0 + 2\,{}^{20}C_1 + \cdots 2\cdot{}^{20}C_8)$$

$$= \frac{1}{2}\Big[\big({}^{20}C_0 + {}^{20}C_1 + \cdots + {}^{20}C_8 + {}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11} + \ldots + {}^{20}C_{20}\big) - ({}^{20}C_9 + {}^{20}C_{10} + {}^{20}C_{11})\Big]$$

$$= \frac{1}{2}\Big[2^{20} - 2\cdot{}^{20}C_9 - {}^{20}C_{10}\Big]$$

$$= 2^{19} - \frac{(2\cdot{}^{20}C_9 + {}^{20}C_{10})}{2}$$

46. (1) We know that

$$(1-1)^{20} = {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots + {}^{20}C_{10} - {}^{20}C_{11} + {}^{20}C_{12} - \cdots + {}^{20}C_{20} = 0$$

$$2({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots - {}^{20}C_9) + {}^{20}C_{10} = 0 \quad [\because {}^{20}C_{20} = {}^{20}C_0, {}^{20}C_{19} = {}^{20}C_1, \text{ etc.}]$$

$$\Rightarrow \quad {}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \cdots - {}^{20}C_9 + {}^{20}C_{10}$$

$$= -\frac{1}{2}{}^{20}C_{10} + {}^{20}C_{10} = \frac{1}{2}{}^{20}C_{10}$$

47. (3) Put $x = \omega, \omega^2$

$(3 + \omega + \omega^2)^{2010} = a_0 + a_1\omega + a_2\omega^2 + \cdots$

and $\quad (3 + \omega^2 + \omega^4) = a_0 + a_1\omega^2 + a_2\omega^4 + \ldots$

$\Rightarrow \quad 2^{2010} = a_0 + a_1\omega + a_2\omega^2 + a_3 + a_4\omega + \cdots \quad (1)$

and $\quad 2^{2010} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 + \cdots \quad (2)$

Adding (1) and (2), we have

$2 \times 2^{2010} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 - \cdots$

or $\quad 2^{2010} = a_0 - \dfrac{1}{2}a_1 - \dfrac{1}{2}a_2 + a_3 - \dfrac{1}{2}a_4 - \dfrac{1}{2}a_5 + a_6 \cdots$

48. (3) $\displaystyle\sum_{k=1}^{\infty}\sum_{r=0}^{k}\frac{1}{3^k}({}^kC_r) = \sum_{k=1}^{\infty}\left(\frac{1}{3^k}\left(\sum_{r=0}^{k}{}^kC_r\right)\right)$

$$= \sum_{k=1}^{\infty}\left(\frac{2^k}{3^k}\right)$$

$$= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots \infty$$

$$= \frac{2/3}{1 - \frac{2}{3}} = 2$$

49. (1) $\sum_{r=0}^{10}(r)\,{}^{20}C_r = \sum_{r=1}^{10} 20 \times {}^{19}C_{r-1}$

$= 20\,({}^{19}C_0 + {}^{19}C_1 + \ldots + {}^{19}C_{10})$

$= 20\,({}^{19}C_0 + {}^{19}C_1 + \ldots + {}^{19}C_{10})$

$= 20\left(\frac{1}{2}\times 2^{19} + {}^{19}C_{10}\right)$

$= 20\,(2^{18} + {}^{19}C_{10})$

50. (4) $(1+\omega)^n = {}^nC_0 + {}^nC_1\,\omega + \cdots$

$= ({}^nC_0 + {}^nC_3 + \cdots) + ({}^nC_1 + {}^nC_4 + \cdots)\left(\frac{-1+\sqrt{3}\,i}{2}\right)$

$+ ({}^nC_2 + {}^nC_5 + \cdots)\left(\frac{-1-\sqrt{3}\,i}{2}\right)$

$= ({}^nC_0 + {}^nC_3 + \cdots) - \frac{1}{2}({}^nC_1 + {}^nC_2 + {}^nC_4 + {}^nC_5 \ldots)$

$+ \frac{i\sqrt{3}}{2}({}^nC_1 - {}^nC_2 + {}^nC_4 - {}^nC_5 + \cdots)$

Equating the modulus, we get $|(-\omega^2)^n| = 1$.

51. (1) $\sum_{r=1}^{n+1}\left(\sum_{k=1}^{n} {}^kC_{r-1}\right) = \sum_{r=1}^{n+1}\left(\sum_{k=1}^{n}\left({}^{k+1}C_r - {}^kC_r\right)\right)$

$= \sum_{r=1}^{n+1}\left({}^{n+1}C_r - {}^1C_r\right)$

$= 2^{n+1} - 2$

52. (3) $\sum_{0\le i\le j\le 10} ({}^{10}C_j)\left({}^jC_i\right)$

$= {}^{10}C_1({}^1C_0 + {}^1C_1) + {}^{10}C_2\,({}^2C_0 + {}^2C_1 + {}^2C_2) + \ldots$

$+ {}^{10}C_{10}\,({}^{10}C_0 + {}^{10}C_1 + \ldots + {}^{10}C_{10})$

$= {}^{10}C_1\,(2) + {}^{10}C_2\,(2^2) + {}^{10}C_3(2^3) + \ldots + {}^{10}C_{10}\,(2^{10})$

$= (2+1)^{10} - 1$

$= 3^{10} - 1$

53. (1) ${}^{1000}C_{50} + {}^{999}C_{49} + {}^{998}C_{48} + \ldots + {}^{950}C_0$

= coefficient of x^{950} in $[(1+x)^{950} + (1+x)^{951} + \ldots + (1+x)^{1000}]$

= coefficient of x^{950} in $\frac{(1+x)^{950}((1+x)^{51}-1)}{(1+x)-1}$

= coefficient of x^{950} in $\frac{(1+x)^{1001} - (1+x)^{950}}{x}$

= coefficient of x^{951} in $(1+x)^{1001}$

$= {}^{1001}C_{951} = {}^{1001}C_{50}$

54. (2) Let $\alpha - x - 1 = t$.

Then $\sum_{r=0}^{n} a_r(1-t)^r = \sum_{r=0}^{n} b_r t^r$

$\Rightarrow$ b_n = coefficient of t^n in $\sum_{r=0}^{n} a_r(1-t)^r$

= coefficient of t^n in $a_n(1-t)^n$

$= (-1)^n \times a_n$

55. (4) In the given equation, put $x - 3 = y$.

$\therefore$ $\sum_{r=0}^{2n} a_r\,(1+y)^r = \sum_{r=0}^{2n} b_r\,(y)^r$

$\Rightarrow$ $a_0 + a_1\,(1+y) + \cdots + a_{n-1}\,(1+y)^{n-1} + (1+y)^n$

$+ (1+y)^{n+1} + \ldots + (1+y)^{2n}$

$= \sum_{r=0}^{2n} b_r y^r$ [Using $a_k = 1,\ \forall\ k \ge n$]

Equating the coefficients of y^n on both sides, we get

${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$

$\Rightarrow$ $\left({}^{n+1}C_{n+1} + {}^{n+1}C_n\right) + {}^{n+2}C_n + \ldots + {}^{2n}C_n = b_n$

[Using ${}^nC_n = {}^{n+1}C_{n+1} = 1$]

$\Rightarrow$ $b_n = {}^{n+2}C_{n+1} + {}^{n+2}C_n + \ldots + {}^{2n}C_n$

Combing the terms in similar way, we get

$\Rightarrow$ $b_n = {}^{2n}C_{n+1} + {}^{2n}C_n = {}^{2n+1}C_{n+1}$

56. (3) $\sum_{r=2}^{10} {}^rC_2 \cdot {}^{10}C_r = \sum_{r=2}^{10} \frac{r(r-1)}{2} \cdot \frac{10!}{r!(10-r)!}$

$= \frac{1}{2}\sum_{r=2}^{10} \frac{10!}{(r-2)!(10-r)!}$

$= \frac{10\times 9}{2}\sum_{r=2}^{10} \frac{8!}{(r-2)!(10-r)!}$

$= 45\sum_{r=2}^{10} {}^8C_{r-2}$

$= 45 \times 2^8$

$= 11520$

57. (4) Given,

$\frac{{}^{n+1}C_{r+1}}{{}^nC_r} = \frac{11}{6}$ or $\frac{\frac{n+1}{r+1}\times {}^nC_r}{{}^nC_r} = \frac{11}{6}$

or $6n + 6 = 11r + 11$ or $6n - 11r = 5$ (1)

Also,

$\frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{6}{3}$ or $\frac{\frac{n}{r}\times {}^{n-1}C_{r-1}}{{}^{n-1}C_{r-1}} = \frac{6}{3}$ or $n = 2r$ (2)

From (1) and (2), $r = 5$ and $n = 10$,

$\therefore$ $nr = 50$

58. (2) Here $a = {}^nC_r$, $b = {}^nC_{r+1}$ and $c = {}^nC_{r+2}$.

Put $n = 2$, $r = 0$, then option (2) holds the condition, i.e.,

$n = \frac{2ac + ab + bc}{b^2 - ac}$

59. (3) For algebraically the least coefficient, we first find the numerically the greatest term for x = 1.

Using $r \le \frac{n+1}{1+\left|\frac{a}{bx}\right|}$, we get

$r \le \frac{19+1}{1+\left|\frac{2}{3}\right|}$ $\Rightarrow r \le 12$

Therefore, T_{13} and T_{12} have numerically the greatest coefficients.

T_{12} is algebraically the least as it is followed by negative sign.

60. (2) Let,

$$S = \frac{{}^nC_0}{n} + \frac{{}^nC_1}{n+1} + \frac{{}^nC_2}{n+2} + \cdots + \frac{{}^nC_n}{2n}$$

$$= {}^nC_0\int_0^1 x^{n-1}dx + {}^nC_1\int_0^1 x^n dx + \cdots + {}^nC_n\int_0^1 x^{2n-1}dx$$

$$= \int_0^1 [{}^nC_0x^{n-1} + {}^nC_1x^n + \cdots + {}^nC_nx^{2n-1}]dx$$

$$= \int_0^1 x^{n-1}(1+x)^n dx$$

$$= \int_1^2 x^n(x-1)^{n-1}dx$$

61. (4) $\displaystyle\sum_{r=1}^{n}(-1)^{r+1}\frac{{}^nC_r}{(r+1)} = \frac{1}{n+1}\sum_{r=1}^{n}(-1)^{r+1}\,{}^{n+1}C_{r+1}$

$$= \frac{1}{n+1}(0 - 1 + (n+1)) = \frac{n}{n+1}$$

62. (4) $\displaystyle\sum_{r=0}^{n}\left(\frac{r+1}{r+1} + \frac{1}{r+1}\right){}^nC_r$

$$= \sum_{r=0}^{n}{}^nC_r + \sum_{r=0}^{n}\frac{{}^nC_r}{r+1}$$

$$= 2^n + \frac{1}{n+1}\cdot\sum_{r=0}^{n}{}^{n+1}C_{r+1}$$

$$= 2^n + \frac{2^{n+1}-1}{n+1}$$

$$= \frac{(n+3)\,2^n - 1}{n+1}$$

$$= \frac{2^8-1}{6}$$

$$\Rightarrow \quad n = 5$$

63. (3) $\displaystyle\sum_{r=0}^{10}(-1)^r\cdot 4^{10-r}\cdot{}^{30}C_r\cdot{}^{30-r}C_{10-r}$

$$= \sum_{r=0}^{10}(-1)^r\cdot 4^{10-r}\cdot\frac{30!}{(30-r)!r!}\cdot\frac{(30-r)!}{20!(10-r)!}$$

$$= \frac{30!}{20!}\sum_{r=0}^{10}(-1)^r\cdot 4^{10-r}\cdot\frac{1}{r!(10-r)!}$$

$$= \frac{30!}{10!\times 20!}\sum_{r=0}^{10}(-1)^r\cdot 4^{10-r}\cdot\frac{10!}{r!(10-r)!}$$

$$= {}^{30}C_{10}\sum_{r=0}^{10}(-1)^r\cdot 4^{10-r}\cdot{}^{10}C_r$$

$$= {}^{30}C_{10}(4-1)^{10}$$

$$= {}^{30}C_{10}\cdot 3^{10}$$

64. (4) $\displaystyle\sum_{r=0}^{10}r\,{}^{10}C_r 3^r(-2)^{10-r} = 10\sum_{r=0}^{10}{}^9C_{r-1}3^r(-2)^{10-r}$

$$= 10\times 3\sum_{r=0}^{10}{}^9C_{r-1}3^{r-1}(-2)^{10-r}$$

$$= 30(3-2)^9 = 30$$

65. (3) As we know that ${}^nC_0 - {}^nC_1^2 + {}^nC_2^2 - {}^nC_3^2 + \cdots + (-1)^n\,{}^nC_n^2 = 0$ (if n is odd) and in the question $n = 15$ (odd). Hence, sum of given series is 0.

66. (3) $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_{n-1}x^{n-1} + C_nx^n$ (1)

$(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots + C_{n-1}x + C_n$ (2)

Multiplying Eqs. (1) and (2) and equating the coefficient of x^{n-2}, we get

$C_0C_2 + C_1C_3 + C_2C_4 + \cdots + C_{n-2}C_n$

= Coefficient of x^{n-2} in $(1+x)^{2n}$

$= {}^{2n}C_{n-2}$

$$= \frac{(2n)!}{(n-2)!(n+2)!}$$

67. (2) The given expression is the coefficient of x^4 in

${}^4C_0(1+x)^{404} - {}^4C_1(1+x)^{303} + {}^4C_2(1+x)^{202} - {}^4C_3(1+x)^{101} + {}^4C_4$

= Coefficient of x^4 in $[(1+x)^{101} - 1]^4$

= Coefficient of x^4 in $({}^{101}C_1x + {}^{101}C_2x^2 + \cdots)^4$

$= (101)^4$

68. (1) $\displaystyle\sum_{r=0}^{40}r\,{}^{40}C_r\,{}^{30}C_r = 40\sum_{r=0}^{40}{}^{39}C_{r-1}\,{}^{30}C_r$

$$= 40\sum_{r=0}^{40}{}^{39}C_{r-1}\,{}^{30}C_{30-r}$$

$$= 40\,{}^{39+30}C_{r-1+30-r}$$

$$= 40\,{}^{69}C_{29}$$

69. (1) $\dfrac{r\times 2^r}{(r+2)!} = \dfrac{(r+2-2)2^r}{(r+2)!}$

$$= \frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!}$$

$$= -\left(\frac{2^{r+1}}{(r+2)!} - \frac{2^r}{(r+1)!}\right)$$

$$= -(V(r) - V(r-1))$$

$$\Rightarrow \quad \sum_{r=1}^{15}\frac{r\times 2^r}{(r+2)!} = -(V(15) - V(0))$$

$$= -\left(\frac{2^{16}}{17!} - \frac{2}{2!}\right)$$

$$= 1 - \frac{2^{16}}{(17)!}$$

70. (3) $t_{r+1} = (-1)^r(n-r+2)\,{}^nC_r\,2^{n-r+1}$

$$= (n+2)\,2^{n+1}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r - 2^{n+1}(-1)^r r\,{}^nC_r\left(\frac{1}{2}\right)^r$$

$$= (n+2)\,2^{n+1}\,{}^nC_r\left(-\frac{1}{2}\right)^r + 2^n n\,{}^{n-1}C_{r-1}\left(-\frac{1}{2}\right)^{r-1}$$

$$\therefore \text{Sum} = (n+2)\,2^{n+1}\left\{{}^nC_0 - {}^nC_1 \times \frac{1}{2} + {}^nC_2 \times \left(\frac{1}{2}\right)^2 - \ldots\right\}$$

$$+ n\,2^n\left\{{}^{n-1}C_0 - {}^{n-1}C_1 \times \frac{1}{2} + {}^{n-1}C_2 \times \left(\frac{1}{2}\right)^2 + \ldots\right\}$$

$$= (n+2)\,2^{n+1}\left(1-\frac{1}{2}\right)^n + n\,2^n\left(1-\frac{1}{2}\right)^{n-1}$$

$$= 2(n+2) + 2n$$

$$= 4n + 4$$

71. (3) Here,

$$T_r = (-1)^r \frac{{}^{50}C_r}{r+2}$$

$$= (-1)^r (r+1)\frac{{}^{50}C_r}{(r+1)(r+2)}$$

$$= (-1)^r (r+1)\frac{{}^{52}C_{r+2}}{51\times 52}$$

$$= (-1)^r \frac{[(r+2)-1]{}^{52}C_{r+2}}{51\times 52}$$

$$= (-1)^r \frac{[52\,{}^{51}C_{r+1} - {}^{52}C_{r+2}]}{51\times 52}$$

$$= \frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51\times 52}$$

$$\sum_{r=0}^{50}(-1)^r \frac{{}^{50}C_r}{r+2}$$

$$= \sum_{r=0}^{50}\frac{[-52\,{}^{51}C_{r+1}(-1)^{r+1} - {}^{52}C_{r+2}(-1)^{r+2}]}{51\times 52}$$

$$= -52\,\frac{(1-1)^{51} - {}^{51}C_0}{51\times 52} - \frac{(1-1)^{52} - {}^{52}C_0 + {}^{52}C_1}{51\times 52}$$

$$= \frac{1}{51} - \frac{1}{52}$$

$$= \frac{1}{51\times 52}$$

Alternate solution:

$$(1-x)^n = \sum_{r=0}^{n} {}^nC_r(-1)^r x^r$$

or $$x(1-x)^n = \sum_{r=0}^{n}(-1)^r\,{}^nC_r x^{r+1}$$

Integrating both sides within the limits 0 to 1, we get

$$\int_0^1 x(1-x)^n\,dx = \sum_{r=0}^{n}(-1)^r\,\frac{{}^nC_r}{r+2}$$

or $$\sum_{r=0}^{n}(-1)^r\frac{{}^nC_r}{r+2} = \int_0^1 x(1-x)^n\,dx$$

$$= \int_0^1 (1-x)x^n\,dx \qquad \text{(Replace } x \text{ by } 1-x\text{)}$$

$$= \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\Big|_0^1$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{(n+1)(n+2)}$$

Now put $n = 50$.

72. (4) $(1-x)^n(1+x)^n = \sum_{r=0}^{n} a_r x^r (1-x)^n (1-x)^{n-r}$

or $$(1-x+2x)^n = \sum_{r=0}^{n} a_r x^r (1-x)^{n-r}$$

or $$\sum_{r=0}^{n} {}^nC_r (1-x)^{n-r}(2x)^r = \sum_{r=0}^{n} a_r x^r (1-x)^{n-r}$$

Comparing general term, we get $a_r = {}^nC_r\,2^r$.

73. (3) $\sum_{r=0}^{300} a_r \times x^r = (1+x+x^2+x^3)^{100}$

Clearly, 'a_r' is the coefficient of x^r in the expansion of $(1+x+x^2+x^3)^{100}$.

Replacing x by $1/x$ in the given equation, we get

$$\sum_{r=0}^{300} a_r\left(\frac{1}{x}\right)^r = \frac{1}{x^{300}}(x^3+x^2+x+1)^{100}$$

or $$\sum_{r=0}^{300} a_r x^{300-r} = (1+x+x^2+x^3)^{100}$$

Here, a_r represents the coefficient of x^{300-r} in $(1+x+x^2+x^3)^{100}$.

Thus, $a_r = a_{300-r}$

Let $$I = \sum_{r=0}^{300} r \times a_r$$

$$= \sum_{r=0}^{300}(300-r)a_{300-r}$$

$$= \sum_{r=0}^{300}(300-r)a_r$$

$$= 300\sum_{r=0}^{300} a_r - \sum_{r=0}^{300} r a_r$$

$\Rightarrow \quad 2I = 300a$

or $\quad I = 150a$

74. (4) $\sum_{r=0}^{20} r(20-r)\times({}^{20}C_r)^2 = \sum_{r=0}^{20} r\times {}^{20}C_r(20-r)\times {}^{20}C_{20-r}$

or $$\sum_{r=0}^{20} 20\,{}^{19}C_{r-1}\times 20\times {}^{19}C_{19-r}$$

$$= 400\sum_{r=0}^{20} {}^{19}C_{r-1}\times {}^{19}C_{19-r}$$

$$= 400 \times \text{coefficient of } x^{18} \text{ in } (1+x)^{19}(1+x)^{19}$$

$$= 400 \times {}^{38}C_{18}$$

$$= 400 \times {}^{38}C_{20}$$

75. (1) $f(x) = {}^{40}C_1 \cdot x(1-x)^{39} + 2 \cdot {}^{40}C_2 \times x^2 (1-x)^{38} + 3 \cdot {}^{40}C_3 \times x^3(1-x)^{37} + \ldots + 40 \cdot {}^{40}C_{40} \times x^{40}$

$T_r = r \cdot {}^{40}C_r \cdot x^r (1-x)^{40-r}$

$= 40x \cdot {}^{39}C_{r-1} \cdot x^{r-1}(1-x)^{40-r}$

$\therefore \quad f(x) = 40x\,(x+1-x)^{39}$

$= 40x$

76. (4) $S = \underset{0\le i<j\le n}{\Sigma\Sigma}\left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j}\right)$

$= \underset{0\le i<j\le n}{\Sigma\Sigma}\left(\frac{n-i}{{}^nC_{n-i}} + \frac{n-j}{{}^nC_{n-j}}\right)$

$= n \underset{0\le i<j\le n}{\Sigma\Sigma}\left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right) - S$

$\Rightarrow \quad S = \frac{n}{2} \underset{0\le i<j\le n}{\Sigma\Sigma}\left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right)$

$= \frac{n}{2}\left(\sum_{r=0}^{n-1} \frac{n-r}{{}^nC_r} + \sum_{r=1}^{n} \frac{r}{{}^nC_r}\right)$

$= \frac{n}{2}\left(\sum_{r=0}^{n} \frac{n}{{}^nC_r}\right) = \frac{n^2 a}{2}$

77. (1) Given term can be written as

$(1+x)^2 (1-x)^{-2} = (1+2x+x^2)\,[1+2x+3x^2+\cdots + (n-1)\times x^{n-2} + nx^{n-1} + (n+1)x^n + \cdots]$

Coefficient of x^n is $(n+1+2n+n-1) = 4n$.

78. (1) $1 + n\left(1-\frac{1}{x}\right) + \frac{n(n+1)}{2!}\left(1-\frac{1}{x}\right)^2 + \cdots \infty$

$= 1 - n\left[-\left(1-\frac{1}{x}\right)\right] + \frac{-n(-n-1)}{2!}\left[-\left(1-\frac{1}{x}\right)\right]^2 + \cdots \infty$

$= \left[1-\left(1-\frac{1}{x}\right)\right]^{-n}$

$= x^n$

79. (3) $\sum_{k=1}^{\infty} k\left(1-\frac{1}{n}\right)^{k-1}$

$= 1 + 2\left(1-\frac{1}{n}\right)^1 + 3\left(1+\frac{1}{n}\right)^2 + \cdots$

$= 1 + 2t + 3t^2 + \cdots$

$= (1-t)^{-2}$

$= \left[1-\left(1-\frac{1}{n}\right)\right]^{-2} = \left(\frac{1}{n}\right)^{-2} = n^2$

80. (4) $\left[\sqrt{1+x^2} - x\right]^{-1} = \frac{1}{\sqrt{1+x^2}-x} \times \frac{\left(\sqrt{1+x^2}+x\right)}{\left(\sqrt{1+x^2}+x\right)}$

$= \frac{\sqrt{1+x^2}+x}{1+x^2-x^2} = x + \sqrt{1+x^2} = x + \left(1+x^2\right)^{1/2}$

$= x + 1 + \frac{1}{2}x^2 + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{x^4}{2!} + \cdots$

Therefore, the coefficient of x^4 is $-1/8$.

81. (4) Let,

$(1+y)^n = 1 + \frac{1}{3}x + \frac{1\times 4}{3\times 6}x^2 + \frac{1\times 4\times 7}{3\times 6\times 9}x^3 + \ldots$

$= 1 + ny + \frac{n(n-1)}{2!}y^2 + \cdots$

Comparing the terms, we get

$ny = \frac{1}{3}x, \frac{n(n-1)}{2!}y^2 = \frac{1\times 4}{3\times 6}x^2$

Solving, $n = -1/3$, $y = -x$. Hence, the given series is $(1-x)^{-1/3}$

82. (1) Let the given series be identical with

$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \cdots \infty$

$\therefore \quad nx = \frac{1}{4}$ or $n^2x^2 = \frac{1}{16}$

Also, $\frac{n(n-1)}{2}x^2 = \frac{3}{32}$

or $\frac{2n}{n-1} = \frac{\frac{1}{16}}{\frac{3}{32}} = \frac{2}{3}$

or $3n = n - 1$

or $2n = -1$

or $n = -\frac{1}{2}$

$\Rightarrow \quad x = -\frac{1}{2}$

$\therefore \quad$ Required sum $= \left(1-\frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$

83. (4) Required value is

$\left(1-\frac{2x}{1+x}\right)^{-n} = \left(\frac{1+x-2x}{1+x}\right)^{-n} = \left(\frac{1-x}{1+x}\right)^{-n} = \left(\frac{1+x}{1-x}\right)^{n}$

84. (4) $(1+2x+3x^2+\cdots)^{-3/2} = [(1-x)^{-2}]^{-3/2}$

$= (1-x)^3 = 1 - 3x + 3x^2 - x^3$

Therefore, coefficient of x^5 is 0.

85. (4) $(1+x+x^2+\cdots)^2 = ((1-x)^{-1})^2 = (1-x)^{-2}$

$= 1 + 2x + 3x^2 + \cdots$

Therefore, coefficient of x^n is $n+1$.

86. (2) T_{r+1} in $(1+x)^n$ is

$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^r$

For first negative term,

$n - r + 1 < 0$

or $\frac{27}{5} - r + 1 < 0$ or $r > \frac{32}{5}$

Thus, first negative term occurs when $r = 7$.

87. (4) $\frac{(1+x)^{3/2} - \left(1+\frac{1}{2}x\right)^3}{(1-x)^{1/2}} = \frac{\left(1+\frac{3}{2}x+\frac{3}{8}x^2\right) - \left(1+\frac{3}{2}x+3\frac{x^2}{4}\right)}{(1-x)^{1/2}}$

$= \frac{-3}{8}x^2(1-x)^{-1/2}$

$= -\frac{3}{8}x^2\left(1+\frac{x}{2}\right) = -\frac{3}{8}x^2$

(4) $\dfrac{1}{(1-ax)(1-bx)} = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$

But $(1-ax)^{-1}(1-bx)^{-1} = (1 + ax + a^2x^2 + \cdots) \times (1 + bx + b^2x^2 + \cdots)$

$\Rightarrow$ Coefficient of x^n is

$b^n + ab^{n-1} + a^2b^{n-2} + \cdots + a^{n-1}b + a^n = \dfrac{b^{n+1} - a^{n+1}}{b-a}$

$\Rightarrow a_n = \dfrac{b^{n+1} - a^{n+1}}{b-a}$

(3) $\dfrac{(x^2+x+1)(1-x)}{(1-x)^2} = (1-x^3)(1-x)^{-2}$

$= (1-x^3)(1 + 2x + 3x^2 + \ldots)$

Now, $a_r = (r+1) - (r-2) = 3$

But $a_1 = 2$

So, $\sum_{r=1}^{50} a_r = 2 + 49 \times 3 = 149$

(2) We have,

$(1-x)^{-n} = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots$

and $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots$

Hence,

$a_0 + a_1 + a_2 + \cdots + a_r$

= Coefficient of x^r in the product of the two series

= Coefficient of x^r in $(1-x)^{-n}(1-x)^{-1}$

= Coefficient of x^r in $(1-x)^{-(n+1)}$

$= \dfrac{(n+1)(n+2)\cdots(n+r)}{r!} = {}^{n+r}C_n$

Multiple Correct Answers Type

1. (1), (3)

Inclusion of $\log x$ implies $x > 0$.

Now, 3rd term in the expansion is

$T_{2+1} = {}^5C_2\, x^{5-2}\, (x^{\log_{10} x})^2 = 1000000$ (given)

or $x^{3+2\log_{10} x} = 10^5$

Taking logarithm of both sides, we get

$(3 + 2\log_{10} x)\log_{10} x = 5$

or $2y^2 + 3y - 5 = 0$, where $\log_{10} x = y$

or $(y-1)(2y+5) = 0$ or $y = 1$ or $-5/2$

or $\log_{10} x = 1$ or $-5/2$

$\therefore\ x = 10^1 = 10$ or $10^{-5/2}$

2. (3), (4)

$$E = \left[\sqrt{9^{x-1} + 7} + \frac{1}{(3^{x-1}+1)^{1/5}}\right]^7$$

Given, $T_6 = 84$

$\Rightarrow\ {}^7C_5 (\sqrt{9^{x-1}+7})^{7-5} \left(\dfrac{1}{(3^{x-1}+1)^{1/5}}\right)^5 = 84$

or $\quad {}^7C_5 (9^{x-1} + 7) \cdot \dfrac{1}{(3^{x-1}+1)} = 84$

or $\quad 9^{x-1} + 7 = 4(3^{x-1} + 1)$

or $\quad 3^{2x} - 12 \cdot 3^x + 27 = 0$

or $\quad (3^x - 3)(3^x - 9) = 0$

or $\quad 3^x = 3, 9$

or $\quad x = 1, 2.$

3. (1), (4)

It is given that the fourth term in the expansion of $\left(ax + \dfrac{1}{x}\right)^n$ is $\dfrac{5}{2}$; therefore,

$${}^nC_3\,(ax)^{n-3}\left(\frac{1}{x}\right)^3 = \frac{5}{2} \Rightarrow {}^nC_3\, a^{n-3}\, x^{n-6} = \frac{5}{2} \qquad (1)$$

$\Rightarrow\ n = 6$ [$\because$ R.H.S. is independent of x]

Putting $n = 6$ in (1), we get ${}^6C_3\, a^3 = \dfrac{5}{2}$ or $a^3 = \dfrac{1}{8}$ or $a = \dfrac{1}{2}$

4. (1), (2), (3)

We have

$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + \cdots + {}^nC_n a^n$

$= [{}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots] + [{}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots]$

or $\quad (x+a)^n = P + Q \qquad (1)$

Similarly,

$(x-a)^n = P - Q \qquad (2)$

(i) $(1) \times (2) \Rightarrow P^2 - Q^2 = (x^2 - a^2)^n$

(ii) Squaring (1) and (2) and subtracting (2) from (1), we get

$4PQ = (x+a)^{2n} - (x-a)^{2n}$

(iii) Squaring (1) and (2) and adding,

$2(P^2 + Q^2) = (x+a)^{2n} + (x-a)^{2n}$

5. (1), (3), (4)

$I + f = (4 + \sqrt{15})^n$

Let $f' = (4 - \sqrt{15})^n$. Then $0 < f' < 1$

$I + f = {}^nC_0\,4^n + {}^nC_1\,4^{n-1}\sqrt{15} + {}^nC_2\,4^{n-2}\,15 + {}^nC_3\,4^{n-3}\,(\sqrt{15})^3 + \cdots$

$f' = {}^nC_0\,4^n - {}^nC_1\,4^{n-1}\sqrt{15} + {}^nC_2\,4^{n-2}\cdot 15 - {}^nC_3\,4^{n-3}\,(\sqrt{15})^3 + \cdots$

$\therefore\quad I + f + f' = 2({}^nC_0\,4^n + {}^nC_2\,4^{n-2} \times 15 + \cdots) =$ even integer

$\because\quad 0 < f + f' < 2 \Rightarrow f + f' = 1 \Rightarrow 1 - f = f'$

Thus, I is an odd integer. Now,

$1 - f = f' = (4 - \sqrt{15})^n$

$(I + f)(1 - f) = (I + f)f' = 1$

6. (1), (2), (3)

We have,

${}^{49}C_{39}\,p^{10}\,q^{39} = {}^{49}C_{40}\,p^9\,q^{40}$

$\Rightarrow\quad p = \dfrac{{}^{49}C_{40}}{{}^{49}C_{39}}q$

$\Rightarrow\quad p = \dfrac{49 - 40 + 1}{40}q$

$\Rightarrow\quad p = \dfrac{1}{4}q$

7. (1), (2), (3), (4)

To get the sum of coefficients, we put $x = 1$.

Then, sum of coefficients in $(1 + ax - 2x^2)^n$ is $(a-1)^n$.

Obviously, when $a > 1$, sum is positive for any n.

8. (2), (3), (4)

L.H.S. $= (1 + 2x^2 + x^4)(1 + C_1x + C_2x^2 + C_3x^3 + \cdots)$

R.H.S. $= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$

Comparing the coefficients of $x, x^2, x^3, \ldots$

$a_1 = C_1,\ a_2 = C_2 + 2,\ a_3 = C_3 + 2C_1 \qquad (1)$

Now, $2a_2 = a_1 + a_3$ (A.P.)

$\Rightarrow\quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2\,{}^nC_1)$ [Using (1)]

or $2\,\frac{n(n-1)}{2}+4=3n+\frac{n(n-1)(n-2)}{6}$

or $n^3-9n^2+26n-24=0$

or $(n-2)(n^2-7n+12)=0$

or $(n-2)(n-3)(n-4)=0$

or $n=2,3,4$

9. (1), (3)

$(1-y)^m(1+y)^n$

$=(1-{}^mC_1y+{}^mC_2y^2-\cdots)(1+{}^nC_1y+{}^nC_2y^2+\cdots)$

$=1+(n-m)y+\left\{\frac{m(m-1)}{2}+\frac{n(n-1)}{2}-mn\right\}y^2+\cdots$

Given,

$a_1=10$

$\Rightarrow \quad a_1=n-m=10 \qquad (1)$

$a_2=\frac{m^2+n^2-m-n-2mn}{2}=10$

$(m-n)^2-(m+n)=20$

$\Rightarrow \quad m+n=80 \qquad (2)$

Solving (1) and (2), we get $m=35$, $n=45$.

10. (1), (4)

Middle term is $\left(\frac{n}{2}+1\right)$th or $(4+1)^{\text{th}}$ or T_5

$\Rightarrow \quad T_5={}^8C_4\left(\frac{x}{2}\right)^4\times 2^4=1120$

or $\frac{8\times7\times6\times5}{1\times2\times3\times4}x^4=1120$

or $x^4=\frac{1120}{70}=16$

or $(x^2+4)(x^2-4)=0$

or $x=\pm 2 \qquad (\because x\in R)$

11. (1)

We have

$\frac{17}{4}+3\sqrt{2}=\frac{1}{4}(9+8+12\sqrt{2})=\frac{1}{4}(3+2\sqrt{2})^2$

$\therefore \quad 3-\sqrt{\frac{17}{4}+3\sqrt{2}}=3-\frac{1}{2}(3+2\sqrt{2})=\frac{3}{2}-\sqrt{2}$

Hence, the 10th term of $\left(3-\sqrt{\frac{17}{4}+3\sqrt{2}}\right)^{20}=\left(\frac{3}{2}-\sqrt{2}\right)^{20}$ is

${}^{20}C_9\left(\frac{3}{2}\right)^{20-9}(-\sqrt{2})^9$

which is an irrational number.

12. (1), (2), (3)

$(x\sin p+x^{-1}\cos p)^{10}$

The general term in the expansion is

$T_{r+1}={}^{10}C_r(x\sin p)^{10-r}(x^{-1}\cos p)^r$

For the term independent of x, we have $10-2r=0$ or $r=5$.

Hence, the independent term is

${}^{10}C_5\sin^5p\cos^5p={}^{10}C_5\frac{\sin^5 2p}{32}$

which is the greatest when $\sin 2p=1$.

The least value of ${}^{10}C_5\frac{\sin^5 2p}{32}$ is $-\frac{10!}{2^5(5!)^2}$ when

$\sin 2p=-1$ or $p=(4n-1)\frac{\pi}{4}, n\in Z.$

Sum of coefficient is $(\sin p+\cos p)^{10}$, when $x=1$

or $(1+\sin 2p)^5$, which is least when $\sin 2p=-1$.

Hence, least sum of coefficients is zero. Greatest sum of coefficient occurs when $\sin 2p=1$. Hence, greatest sum is $2^5=32$.

13. (1), (2), (3), (4)

Let T_5 be numerically the greatest term in the expansion of $(1+x/3)^{10}$.

Then, $\left|\frac{T_5}{T_4}\right|\geq 1$ and $\left|\frac{T_6}{T_5}\right|\leq 1$

Now, $\frac{T_{r+1}}{T_r}=\frac{10-r+1}{r}\frac{x}{3}$

$\Rightarrow \quad \left|\frac{7}{4}\times\frac{x}{3}\right|\geq 1$ and $\left|\frac{6}{5}\times\frac{x}{3}\right|\leq 1$

$\Rightarrow \quad |x|\geq\frac{12}{7}$ and $|x|\leq\frac{5}{2}$ (1)

$\Rightarrow \quad \frac{12}{7}\leq|x|\leq\frac{5}{2}$

$\Rightarrow \quad x\in\left[-\frac{5}{2},-\frac{12}{7}\right]\cup\left[\frac{12}{7},\frac{5}{2}\right]$

14. (1), (4)

$\left(\frac{1002}{1000}\right)^{1000}\times\frac{1}{1002}=\left(1+\frac{2}{1000}\right)^{1000}\times\frac{1}{1002}$

$<\frac{e^2}{1002}$

<1

We know that $\left(1+\frac{2}{n}\right)^n$ is monotonically increasing function of n and $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=e^2$.

$\Rightarrow \quad 1002^{999}<1000^{1000}$

Also, $\left(\frac{1002}{1000}\right)^{1000}\times\frac{1}{1000^2}=\left(1+\frac{2}{1000}\right)^{1000}\times\frac{1}{1000^2}$

$<\frac{e^2}{1000^2}$

<1

$\Rightarrow \quad 1002^{1000}<1000^{1002}$

15. (1), (3)

$\sum_{r=0}^{n}\frac{r}{{}^nC_r}=\sum_{r=0}^{n}\frac{n-r}{{}^nC_{n-r}}=\frac{n}{2}\sum_{r=0}^{n}\frac{1}{{}^nC_r}$

$\therefore \quad \frac{n}{2}=\frac{n^2-3n+3}{2}$

or $n=1,3$

16. **(1), (3), (4)**

$${}^{n}C_1 + {}^{n+1}C_2 + {}^{n+2}C_3 + \cdots + {}^{n+m-1}C_m$$
$$= {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \cdots + {}^{n+m-1}C_{n-1}$$
$$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \cdots + (1+x)^{n+m-1}$$
$$= \text{Coefficient of } x^{n-1} \text{ in } (1+x)^n \left[\frac{(1+x)^m - 1}{(1+x)-1}\right]$$
$$= \text{Coefficient of } x^{n-1} \text{ in } \frac{(1+x)^{m+n} - (1+x)^n}{x}$$
$$= \text{Coefficient of } x^n \text{ in } \left[(1+x)^{m+n} - (1+x)^n\right]$$
$$= {}^{m+n}C_n - 1$$

Similarly, we can prove

$${}^{m}C_1 + {}^{m+1}C_2 + {}^{m+2}C_3 + \cdots + {}^{m+n-1}C_n = {}^{m+n}C_m - 1$$

17. **(1), (3)**

$$\left(x^2 + 1 + \frac{1}{x^2}\right)^n$$
$$= {}^{n}C_0 + {}^{n}C_1\left(x^2 + \frac{1}{x^2}\right) + {}^{n}C_2\left(x^2 + \frac{1}{x^2}\right)^2 + \cdots + {}^{n}C_n\left(x^2 + \frac{1}{x^2}\right)^n$$

This contains terms having $x^0, x^2, x^4, \ldots x^{2n}, x^{-2}, x^{-4}, \ldots, x^{-2n}$

coefficient of constant term $= {}^{n}C_0 + ({}^{n}C_2)(2) + ({}^{n}C_4)({}^{4}C_2) + ({}^{n}C_6)({}^{6}C_3) + \cdots \neq 2^{n-1}$. Coefficient of x^{2n-2} is ${}^{n}C_{n-1} = n$

coefficient of x^2 is ${}^{n}C_1 + ({}^{n}C_3)({}^{3}C_1) + ({}^{n}C_5)({}^{5}C_2) + \cdots > n$

18. **(1), (2), (3)**

General term is ${}^{6561}C_r \; 7^{\frac{6561-r}{3}} \; 11^{\frac{r}{9}}$.

To make the term free of radical sign, r should be a multiple of 9.

$\therefore \quad r = 0, 9, 18, 27, \ldots, 6561.$

Hence, there are 730 rational terms. The greatest binomial coefficients are

$${}^{6561}C_{\frac{6561-1}{2}} \text{ and } {}^{6561}C_{\frac{6561+1}{2}} \text{ or } {}^{6561}C_{3280} \text{ and } {}^{6561}C_{3281}.$$

Now, 3280 are 3281 are not a multiple of 3; hence, both terms involving greatest binomial coefficients are irrational.

19. **(2), (3)**

For $n = 2m$, the given expression is

$$C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots (-1)^{n-1}(C_0 + C_1 + \cdots + C_{n-1})$$
$$= C_0 - (C_0 + C_1) + (C_0 + C_1 + C_2) - (C_0 + C_1 + C_2 + C_3) + \cdots - (C_0 + C_1 + \cdots + C_{2m-1})$$
$$= -(C_1 + C_3 + C_5 + \cdots + C_{2m-1})$$
$$= -(C_1 + C_3 + C_5 + \cdots + C_{n-1}) = -2^{n-1}$$

20. **(1), (2)**

We have $(a+b)^n$

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \cdot \frac{b}{a} = 1$$

$\therefore \quad (n+1)b - br = ar$

$\therefore \quad r = \dfrac{(n+1)b}{a+b}$ is integer

Also, considering $(b+a)^n$.

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \cdot \frac{a}{b} = 1$$

$\therefore \quad r = \dfrac{(n+1)a}{a+b}$ is integer

21. **(1), (2), (4)**

$$(1 + z^2 + z^4)^8 = C_0 + C_1 z^2 + C_2 z^4 + \cdots + C_{16} z^{32} \quad (1)$$

Putting $z = i$, where $i = \sqrt{-1}$,

$$(1 - 1 + 1)^8 = C_0 - C_1 + C_2 - C_3 + \cdots + C_{16}$$

or $\quad C_0 - C_1 + C_2 - C_3 + \cdots + C_{16} = 1$

Also, putting $z = \omega$,

$$(1 + \omega^2 + \omega^4)^8 = C_0 + C_1\omega^2 + C_2\omega^4 + \cdots + C_{16}\omega^{32} \quad (2)$$

or $\quad C_0 + C_1\omega^2 + C_2\omega + C_3 + \cdots + C_{16}\omega^2 = 0$

Putting $x = \omega^2$,

$$(1 + \omega^4 + \omega^8)^8 = C_0 + C_1\omega^4 + C_2\omega^8 + \cdots + C_{16}\omega^{64} \quad (3)$$

or $\quad C_0 + C_1\omega + C_2\omega^2 + \cdots + C_{16}\omega = 0 \quad (4)$

Putting $x = 1$,

$$3^8 = C_0 + C_1 + C_2 + \cdots + C_{16}$$

Adding (2), (3) and (4), we have

$$3(C_0 + C_3 + \cdots + C_{15}) = 3^8$$

or $\quad C_0 + C_3 + \cdots + C_{15} = 3^7$

Similarly, first multiplying (1) by z and then putting $1, \omega, \omega^2$ and adding, we get

$$C_1 + C_4 + C_7 + C_{10} + C_{13} + C_{16} = 3^7$$

Multiplying (1) by z^2 and then putting $1, \omega, \omega^2$ and adding, we get

$$C_2 + C_5 + C_8 + C_{11} + C_{14} = 3^7$$

22. **(1), (2), (4)**

$$f(m) = \sum_{i=0}^{m} \binom{30}{30-i}\binom{20}{m-i} = \sum_{i=0}^{m} \binom{30}{i}\binom{20}{m-i} = {}^{50}C_m$$

$f(m)$ is greatest when $m = 25$. Also,

$$f(0) + f(1) + \cdots + f(50) = {}^{50}C_0 + {}^{50}C_1 + {}^{50}C_2 + \cdots + {}^{50}C_{50} = 2^{50}$$

Also, ${}^{50}C_m$ is not divisible by 50 for all m as 50 is not a prime number

$$\sum_{m=0}^{50} (f(m))^2 = ({}^{50}C_0)^2 + ({}^{50}C_1)^2 + ({}^{50}C_2)^2 + \cdots + ({}^{50}C_{50})^2$$
$$= {}^{100}C_{50}$$

23. **(1), (2), (4)**

$$\frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}$$
$$= \frac{(n-1)(n-2)\cdots(n-m+1)(n-m)\cdots 2 \cdot 1}{(n-m)!(m-1)!}$$
$$= {}^{n-1}C_{m-1}$$
$$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$$
$$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^n (1+x)^{-1}$$

Now,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_{m-1}x^{m-1} + \cdots + C_n x^n \quad (1)$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots + (-1)^{m-1} x^{m-1} + \cdots \quad (2)$$

Collecting the coefficients of x^{m-1} in the product of (1) and (2), we get

$$(-1)^{m-1} C_0 + (-1)^{m-2} C_1 + \cdots + C_{m-1}$$
$$= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1}$$
$$= {}^{n-1}C_{m-1}$$

$\therefore \quad C_0 - C_1 + C_2 - \cdots + (-1)^{m-1} C_{m-1}$
$$= {}^{n-1}C_{m-1}(-1)^{m-1}$$
$$= \frac{(n-1)(n-2)\cdots(n-m+1)}{(m-1)!}(-1)^{m-1}$$

24. (1), (3)

$$S = \sum_{r=0}^{n} (pr+2)\cdot {}^nC_r$$

$$\therefore \quad = \sum_{r=0}^{n} (p(n-r)+2).{}^nC_r \qquad (\because \ {}^nC_r = {}^nC_{n-r})$$

Adding these, we get

$$2S = (pn+4)\sum_{r=0}^{n} {}^nC_r$$

$$\Rightarrow \quad S = (pn+4)\times 2^{n-1}$$

So, $n = 7$ and $7p + 4 = 25$ or $p = 3$.

25. (1), (2), (3)

$$\left(x+\frac{1}{x}+1\right)^6 = \sum_{r=0}^{6} {}^6C_r \left(x+\frac{1}{x}\right)^r$$

(1) $a_0 = 1 + {}^6C_2 \cdot {}^2C_1 + {}^6C_4 \cdot {}^4C_2 + {}^6C_6 \cdot {}^6C_3$
$= 1 + 30 + 90 + 20 = 141$

(2) $a_5 = {}^6C_5 \times$ Coefficient of x^5 in $\left(x+\frac{1}{x}\right)^5$
$= {}^6C_5 \cdot {}^5C_0 = 6$

(3) Putting, $x = 1$
$a_0 + (a_1 + b_1) + (a_2 + b_2) + \ldots + (a_6 + b_6) = 3^6$

$$\therefore \quad \sum_{i=1}^{6} (a_i + b_i) = 3^6 - 141 = 588$$

26. (1), (2)

${}^nC_0 \times {}^{2n}C_r - {}^nC_1 \times {}^{2n-2}C_r + {}^nC_2 \times {}^{2n-4}C_r + \ldots$

= Coefficient of x^r in $({}^nC_0 (1+x)^{2n} - {}^nC_1(1+x)^{2n-2}$
$+ {}^nC_2(1+x)^{2n-4} + \ldots)$

= Coefficient of x^r in $({}^nC_0 ((1+x)^2)^n - {}^nC_1((1+x)^2)^{n-1}$
$+ {}^nC_2((1+x)^2)^{n-2} + \ldots)$

= Coefficient of x^r in $((1+x)^2 - 1)^n$

= Coefficient of x^r in $(2x + x^2)^n$

General term, $T_{p+1} = {}^nC_p(2x)^{n-p}(x^2)^p = {}^nC_p 2^{n-p} x^{n+p}$

For x^r, $n + p = r$. So, $p = r - n$.

$\therefore$ Coefficient of $x^r = {}^nC_{r-n} \times 2^{2n-r}$

This is non-zero if $r \geq n$ and zero if $r < n$.

27. (1), (2), (3), (4)

$2 \times {}^{40}C_2 + 6 \times {}^{40}C_3 + 12 \times {}^{40}C_4 + 20 \times {}^{40}C_5 + \ldots + 1560 \times {}^{40}C_{40}$

$$= \sum_{r=2}^{40} (r^2 - r)\cdot {}^{40}C_r$$

$$= \sum_{r=2}^{40} r(r-1)\cdot {}^{40}C_r$$

$$= \sum_{r=2}^{40} 40(r-1)\cdot {}^{39}C_{r-1}$$

$$= \sum_{r=2}^{40} 40\times 39\cdot {}^{38}C_{r-2}$$

$= 40 \times 39 \times 2^{38}$

$= 5 \times 39 \times 2^{41}$

Linked Comprehension Type

1. (2), 2. (4), 3. (3)

The coefficient of the 2nd, 3rd, and 4th terms in the expansion are ${}^mC_1$, ${}^mC_2$, and ${}^mC_3$, which are given in A.P. Hence,

$2\,{}^mC_2 = {}^mC_1 + {}^mC_3$

or $\quad \frac{2m(m-1)}{2!} = m + \frac{m(m-1)(m-2)}{3!}$

or $\quad m(m^2 - 9m + 14) = 0$

or $\quad m(m-2)(m-7) = 0$

or $\quad m = 7 \qquad (\because m \neq 0$ or 2 as 6th term is given equal to 21)

Now, 6th term in the expansion, when $m = 7$, is

$${}^7C_5\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}}\right]^{7-5} \times \left[\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^5 = 21$$

$$\Rightarrow \quad \frac{7\times 6}{2!}\, 2^{\log(10-3^x)} \times 2^{(x-2)\log 3} = 21$$

or $\quad 2^{\log(10-3^x) + (x-2)\log 3} = 1 = 2^0$

or $\quad \log(10 - 3^x) + (x-2)\log 3 = 0$

or $\quad \log(10 - 3^x)(3)^{(x-2)} = 0$

or $\quad (10 - 3^x) \times 3^x \times 3^{-2} = 1$

or $\quad 10 \times 3^x - (3^x)^2 = 9$

or $\quad (3^x)^2 - 10 \times 3^x + 9 = 0$

or $\quad (3^x - 1)(3^x - 9) = 0$

$\Rightarrow \quad 3^x = 1, 9$

$\Rightarrow \quad x = 0, 2$

When $x = 2$,

$$\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}} + \sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m = [1+1]^7 = 128$$

When $x = 0$,

$$\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}} + \sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^m$$

$$= \left[\sqrt{\left\{2^{\log 9}\right\}} + \sqrt[5]{\left\{2^{-2\log 3}\right\}}\right]^7$$

$$= \left[2^{\frac{\log 9}{2}} + \frac{1}{2^{\frac{\log 9}{5}}}\right]^7 > 2^7$$

Hence, the minimum value is 128.

4. (2), 5. (3), 6. (3)

2nd term is ${}^nC_1 x^{n-1} a = 240$ (1)

3rd term is ${}^nC_2 x^{n-2} a^2 = 720$ (2)

4th term is ${}^nC_3 x^{n-3} a^3 = 1080$ (3)

Multiplying (1) and (3) and dividing by the square of (2), we get

$$\frac{{}^nC_1 \times {}^nC_3}{({}^nC_2)^2} = \frac{240\times 1080}{(720)^2}$$

or $\quad \frac{n \times n(n-1)(n-2)(2!)^2}{n^2(n-1)^2 \times 3!} = \frac{1}{2}$

or $\quad 4(n-2) = 3(n-1) \qquad (\because n \neq 1)$

or $\quad n = 5$

Putting $n = 5$, from (1) and (2), we get

$5x^4 a = 240$ and $10x^3 a^2 = 720$

$$\Rightarrow \quad \frac{(5x^4a)^2}{10x^3a^2} = \frac{(240)^2}{720}$$

or $x^5 = 32$

or $x = 2$

$\therefore \quad a = \dfrac{240}{5x^4} = \dfrac{48}{2^4} = 3$

Hence, $x = 2$, $a = 3$ and $n = 5$.

$(x - a)^n = (2 - 3)^5 = -1$

Also,

$$(2 + 3)^5 = 2^5 + {}^5C_1 2^4 \times 3 + {}^5C_2 2^3 \times 3^2 + {}^5C_3 2^2 \times 3^3 + {}^5C_4 2 \times 3^4 + {}^5C_5 3^5$$

$$= 32 + 240 + 720 + 1080 + 810 + 243$$

Hence, least value of the term is 32.

Sum of odd numbered terms is $32 + 720 + 810 = 1562$.

7. (3), 8. (1), 9. (3)

$a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100} = 0$ has roots ${}^{99}C_0$, ${}^{99}C_1$, ${}^{99}C_2$, ..., ${}^{99}C_{99}$.

or $a_0 + a_1x + a_2x^2 + \cdots + a_{99}x^{99} + x^{100}$

$= (x - {}^{99}C_0)(x - {}^{99}C_1)(x - {}^{99}C_2) \cdots (x - {}^{99}C_{99})$

Now, sum of roots is

$${}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 + \cdots + {}^{99}C_{99} = -\frac{a_{99}}{\text{coefficient of } x^{100}}$$

or $a_{99} = -2^{99}$

Also, sum of product of roots taken two at a time is

$$\frac{a_{99}}{\text{coefficient of } x^{100}}$$

$$\therefore \quad \sum\sum_{0 \le i < j \le 99} {}^{99}C_i \, {}^{99}C_j = \frac{\left(\sum_{i=0}^{99}\sum_{j=0}^{99} {}^{99}C_i \, {}^{99}C_j\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{\left(\sum_{i=0}^{99} {}^{99}C_i 2^{99}\right) - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{99}2^{99} - \sum_{i=0}^{99}({}^{99}C_i)^2}{2}$$

$$= \frac{2^{198} - {}^{198}C_{99}}{2}$$

$$({}^{99}C_0)^2 + ({}^{99}C_1)^2 + \cdots + ({}^{99}C_{99})^2$$

$$= ({}^{99}C_0 + {}^{99}C_1 + {}^{99}C_2 \cdots + {}^{99}C_{99})^2 - 2\sum\sum_{0 \le i < j \le 99} {}^{99}C_i \, {}^{99}C_j$$

$$= (-a_{99})^2 - 2a_{98}$$

$$= a_{99}^2 - 2a_{98}$$

10. (2) We have

$a + b + c = 2^{20}$

Now, $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= 2^{20}(1 + \omega)^{20}(1 + \omega^2)^{20}$

$= 2^{20}$

11. (2) $(a - b)^2 + (b - c)^2 + (c - a)^2$

$= 2(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= 2$

12. (2) We have $(a + b + c + d)^6$

Sum of coefficients which contains all of a, b, c and d

= Number of ways of distributing six distinct objects in four boxes such that no box remains empty

$= 4^6 - {}^4C_1 3^6 + {}^4C_2 2^6 - {}^4C_1 1^6 = 1560$

13. (3) Sum of coefficients which contains a but not b

= Number of ways of distributing six distinct objects in three boxes (a, c, d)

− Number of ways of distributing six distinct objects in two boxes (c, d)

$= 3^6 - 2^6 = 665$

14. (4) Sum of coefficients which contain both a and b

= Number of ways of distributing six distinct objects in four boxes (a, b, c, d)

− Number of ways of distributing six distinct objects in three boxes (a, c, d)

− Number of ways of distributing six distinct objects in three boxes (b, c, d)

+ Number of ways of distributing six distinct objects in two boxes (c, d)

$= 4^6 - 2 \times 3^6 + {}^2C_2 \times 2^6 = 2702$

15. (2), 16. (2), 17. (3)

In P, general term of the series is

$$T(r) = \frac{{}^{50+r}C_r(2r - 1)}{{}^{50}C_r(50 + r)}$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(1 - \frac{50 - r + 1}{50 + r}\right)$$

$$= \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50 - r + 1}{50 + r}\right)$$

Now,

$$\frac{{}^{50+r}C_r}{{}^{50}C_r}\left(\frac{50 - r + 1}{50 + r}\right)$$

$$= \frac{(50 - r + 1)(50 + r)!r!(50 - r)!}{r!50!(50 + r)50!}$$

$$= \frac{(50 - r + 1)!(50 + r - 1)!}{50!50!}$$

$$= \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}}$$

$$\Rightarrow \quad T(r) = \frac{{}^{50+r}C_r}{{}^{50}C_r} - \frac{{}^{50+r-1}C_{r-1}}{{}^{50}C_{r-1}} = V(r) - V(r - 1)$$

where $V(r) = \dfrac{{}^{50+r}C_r}{{}^{50}C_r}$

Now, sum of the given series

$$P = \sum_{r=1}^{50} T(r) = V(50) - V(0)$$

$$= \frac{{}^{100}C_{50}}{{}^{50}C_{50}} - \frac{{}^{50}C_0}{{}^{50}C_0} = {}^{100}C_{50} - 1$$

Also,

$$Q = \sum_{r=0}^{50}({}^{50}C_r)^2 = {}^{50}C_0^2 + {}^{50}C_1^2 + {}^{50}C_2^2 + \cdots + {}^{50}C_{50}^2$$

$$= {}^{100}C_{50}$$

$\Rightarrow \quad P - Q = -1$

We know that

$$C_0^2 - C_1^2 + C_2^2 + \cdots + (-1)^n C_n^2 = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^n {}^nC_{n/2}, & \text{if } n \text{ is even} \end{cases}$$

$$\Rightarrow \quad \sum_{r=0}^{100} (-1)^r \left({}^{100}C_r\right)^2 = (-1)^{100}\, {}^{100}C_{50} = {}^{100}C_{50}$$

$\Rightarrow \quad P - R = -1$

$Q + R = 2^{100}C_{50} = 2P + 2$

18. (2), 19. (1), 20. (3)

Let, $(1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^r$ (1)

Replacing x by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^{20} = \sum_{r=0}^{40} a_r \left(\frac{1}{x}\right)^r$$

or $(1 + x + x^2)^{20} = \sum_{r=0}^{40} a_r x^{40-r}$ (2)

Since (1) and (2) are same series, coefficient of x^r in (1) = coefficient of x^r in (2)

$\Rightarrow \quad a_r = a_{40-r}$

In (1), putting $x = 1$, we get

$$3^{20} = a_0 + a_1 + a_2 + \cdots + a_{40}$$
$$= (a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20} + (a_{21} + a_{n+2} + \cdots + a_{40})$$
$$= 2(a_0 + a_1 + a_2 + \cdots + a_{19}) + a_{20} \qquad (\because \ a_r = a_{40-r})$$

or $a_0 + a_1 + a_2 + \cdots + a_{19} = \frac{1}{2}(3^{20} - a_{20}) = \frac{1}{2}(9^{10} - a_{20})$

Also,

$$a_0 + 3a_1 + 5a_2 + \cdots + 81a_{40}$$
$$= (a_0 + 81a_{40}) + (3a_1 + 79a_{39}) + \cdots + (39a_{19} + 43a_{21}) + 41a_{20}$$
$$= 82(a_0 + a_1 + a_2 + \cdots + a_{19}) + 41a_{20}$$
$$= 41(9^{10} - a_{20}) + 41a_{20}$$
$$= 41 \times 3^{20}$$

$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots$ suggests that we have to multiply the two expansions.

Replacing x by $-1/x$ in (1), we get

$$\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)^{20} = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} - \cdots + \frac{a_{40}}{x^{40}}$$

$\Rightarrow \quad (1 - x + x^2)^{20} = a_0 x^{40} - a_1 x^{39} + a_2 x^{38} - \cdots + a_{40}$ (3)

Clearly,

$a_0^2 - a_1^2 + a_2^2 + \cdots + a_{40}^2$ is the coefficient of x^{40} in $(1 + x + x^2)^{20}(1 - x + x^2)^{20}$

= Coefficient of x^{40} in $(1 + x^2 + x^4)^{20}$

In $(1 + x^2 + x^4)^{20}$, replace x^2 by y, then the coefficient of y^{20} in $(1 + y + y^2)^{20}$ is a_{20}.

Hence, $a_0^2 - a_1^2 + a_2^2 - \cdots + a_{40}^2 = a_{20}$

or $(a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 + (-a_{21}^2 + \cdots + a_{40}^2) = a_{20}$

or $2(a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2) + a_{20}^2 = a_{20}$

or $a_0^2 - a_1^2 + a_2^2 - \cdots - a_{19}^2 = \frac{a_{20}}{2}[1 - a_{20}]$

Matrix Match Type

1. a → r; b → s; c → q; d → p.

a. If $T(r+1)$ is greatest, then $\frac{n}{1+\left|\frac{a}{bx}\right|} \le r \le \frac{n+1}{1+\left|\frac{a}{bx}\right|}$ for $(a + bx)^n$.

For $x = 1$ and $r = 11$, $\frac{n}{1+\frac{3}{5}} \le 11 \le \frac{n+1}{1+\frac{3}{5}}$

$\Rightarrow \quad 5n \le 88 \le 5(n+1)$

$\Rightarrow \quad n = 17$

b. If $T(r+1)$ is greatest, then $\left(\frac{r}{n-r+1}\right)\frac{a}{b} < x < \left(\frac{r+1}{n-r}\right)\frac{a}{b}$ for $(a + bx)^n$.

For $n = 25$ and $r = 20$, $2 < x < \frac{63}{25}$

c. We have $\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \cdot \frac{bx}{a}$ for $(a + bx)^n$

For $n = 20$ and $r = 10$, $\frac{11}{10} \cdot \frac{5x}{3} = 1$ or $x = 6/11$

d. We have $\frac{n}{1+\left|\frac{a}{bx}\right|} \le r \le \frac{n+1}{1+\left|\frac{a}{bx}\right|}$ for $(a + bx)^n$.

For $x = 2$ and $r = 8$ (as $T(9)$ will be largest),

$$\frac{n}{1+\frac{3}{10}} \le 8 \le \frac{n+1}{1+\frac{3}{10}}$$

$\Rightarrow \quad 10n \le 104 \le 10(n+1)$

$\Rightarrow \quad n = 10$

2. a → q; b → p; c → s; d → t.

a. $\sum_{r=0}^{14} \frac{(15-r)}{r+1} {}^{15}C_r \, {}^{20}C_{m-r}$

$$= \sum_{r=0}^{14} \frac{(15-r)}{r+1} \frac{15!}{r!(15-r)!} {}^{20}C_{m-r}$$

$$= \sum_{r=0}^{14} \frac{15!}{(r+1)!(14-r)!} {}^{20}C_{m-r}$$

$$= \sum_{r=0}^{14} {}^{15}C_{r+1} \, {}^{20}C_{m-r}$$

$$= {}^{35}C_{m+1}$$

Which has maximum value ${}^{35}C_{18}$.

b. ${}^{35}C_0 - {}^{35}C_1 + {}^{35}C_2 - {}^{35}C_3 + \ldots + (-1)^r {}^{35}C_r + \ldots$

= Coefficient of x^r in $({}^{35}C_0 - {}^{35}C_1 x + {}^{35}C_2 x^2 - {}^{35}C_3 x^3 + \ldots + (-1)^r {}^{35}C_r + \ldots) \times (1 + x + x^2 + x^3 + \ldots + x^r + \ldots)$

= Coefficient of x^r in $(1-x)^{35}(1-x)^{-1}$

= Coefficient of x^r in $(1-x)^{34}$

$= (-1)^r {}^{34}C_r$

Required maximum value is ${}^{34}C_{17}$.

c. ${}^{31}C_{r-5} + 5\times{}^{31}C_{r-4} + 10\times{}^{31}C_{r-3} + 10\times{}^{31}C_{r-2} + 5\times{}^{31}C_{r-1} + {}^{31}C_r$

$= {}^5C_5\times{}^{31}C_{r-5} + {}^5C_4\times{}^{31}C_{r-4} + {}^5C_3\times{}^{31}C_{r-3} + {}^5C_2\times{}^{31}C_{r-2} + {}^5C_1\times{}^{31}C_{r-1} + {}^5C_0\times{}^{31}C_r$

= Coefficient of x^r in $(1+x)^5(1+x)^{31}$

= Coefficient of x^r in $(1+x)^{36}$

$= {}^{36}C_r$

Which has maximum value of ${}^{36}C_{18}$.

d. $\sum_{r=0}^{k} 2^{k-r}\cdot(-1)^r\cdot{}^{37}C_r\cdot{}^{37-r}C_{37-k}$

$= \sum_{r=0}^{k} 2^{k-r}\cdot(-1)^r\cdot\frac{37!}{(37-r)!r!}\frac{(37-r)!}{(37-k)!(k-r)!}$

$= \frac{37!}{k!(37-k)!}\sum_{r=0}^{k} 2^{k-r}\cdot(-1)^r\cdot\frac{k!}{r!(k-r)!}$

$= {}^{37}C_k\sum_{r=0}^{k}{}^kC_r 2^{k-r}\cdot(-1)^r$

$= {}^{37}C_k(2-1)^k$

$= {}^{37}C_k$

Which has maximum value ${}^{37}C_{18}$.

3. (2) a. We know that

${}^mC_r + {}^mC_{r-1}\times{}^nC_1 + {}^mC_{r-2}\times{}^nC_2 + \ldots + {}^nC_r = {}^{m+n}C_r$

So, if $m, n > r$ and $m + n < r$, then

${}^mC_r + {}^mC_{r-1}\times{}^nC_1 + {}^mC_{r-2}\times{}^nC_2 + \ldots + {}^nC_r = 0$

b. If $m, n > r$ then

${}^mC_r + {}^mC_{r-1}\times{}^nC_1 + {}^mC_{r-2}\times{}^nC_2 + \ldots + {}^nC_r = {}^{m+n}C_r$

c. If $m, n < r < m + n$ then

${}^mC_m\times{}^nC_{r-m} + {}^mC_{m-1}\times{}^nC_{r-m+1} + {}^mC_{m-2}\times{}^nC_{r-m+2} + \ldots + {}^mC_{r-n}\times{}^nC_n = {}^{m+n}C_r$

d. If $m < r < n$

${}^mC_m\times{}^nC_{r-m} + {}^mC_{m-1}\times{}^nC_{r-m+1} + {}^mC_{m-2}\times{}^nC_{r-m+2} + \ldots + {}^nC_r = {}^{m+n}C_r$

Numerical Value Type

1. (8) Let the three consecutive coefficients be ${}^nC_{r-1} = 28$, ${}^nC_r = 56$ and ${}^nC_{r+1} = 70$,

so that $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r} = \frac{56}{28} = 2$ and

$\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1} = \frac{70}{56} = \frac{5}{4}$

This gives $n + 1 = 3r$ and $4n - 5 = 9r$. Therefore,

$\frac{4n-5}{n+1} = 3$ or $n = 8$

2. (3) $(1 + 0.00002)^{50000} = \left(1 + \frac{1}{50000}\right)^{50000}$.

Now we know that $2 \le \left(1+\frac{1}{n}\right)^n < 3\ \forall n \ge 1$

$\Rightarrow$ Least integer is 3

3. (4) $T_2 = {}^nC_1\ (a^{1/13})^{n-1}\cdot a\sqrt{a} = 14a^{5/2}$

or $n\cdot a^{\frac{n-1}{13}} = 14a$

or $n\cdot a^{\frac{n-14}{13}} = 14$

or $\frac{n-14}{13} = 0$

or $n = 14$

$\Rightarrow \frac{{}^{14}C_3}{{}^{14}C_2} = \frac{14!}{3!11!}\frac{2!\cdot12!}{14!} = \frac{12}{3} = 4$

4. (6) $T_{r+1} = {}^nC_r\,(x^2)^{n-r}(-1)^r x^{-r}$

$= {}^nC_r\,x^{2n-3r}(-1)^r$

Constant term $= {}^nC_r\,(-1)^r$ if $2n = 3r$

i.e., coefficient of $x = 0$

Hence, ${}^nC_{2n/3}\,(-1)^{2n/3} = 15 = {}^6C_4$

$\Rightarrow n = 6$

5. (4) $\left(5^{\frac{2}{5}\log_5\sqrt{4^x+44}} + \frac{1}{5^{\log_5\sqrt[3]{2^{x-1}+7}}}\right)^8$

$= \left((\sqrt{4^x+44})^{2/5} + \left(\frac{1}{\sqrt[3]{2^{x-1}+7}}\right)\right)^8$

$= \left((4^x+44)^{1/5} + \frac{1}{(2^{x-1}+7)^{1/3}}\right)^8$

Now $T_4 = T_{3+1} = {}^8C_3\,((4^x+44)^{1/5})^{8-3}\frac{1}{((2^{x-1}+7)^{1/3})^3}$

Given $336 = {}^8C_3\left(\frac{4^x+44}{2^{x-1}+7}\right)$

Let $2^x = y$

$\Rightarrow 336 = {}^8C_3\left(\frac{y^2+44}{(y/2)+7}\right)$

or $336 = \frac{8\times7\times6}{3\times2\times1}\left(\frac{2(y^2+44)}{y+14}\right)$

$\Rightarrow y^2 - 3y + 2 = 0$

or $y = 0, 2$

or $y = 2$ or $x = 4$

6. (4) We have b = coefficient of x^3 in

$\left((1+x+2x^2+3x^3)+4x^4\right)^4$

= coefficient of x^3 in $[{}^4C_0(1+x+2x^2+3x^3)^4\,(4x^4)^0 + {}^4C_1(1+x+2x^2+3x^3)^3\,(4x^4)^1 + \cdots]$

= coefficient of x^3 in $(1+x+2x^2+3x^3)^4 = a$

Hence, $4a/b = 4$.

7. (35) $(1+7)^{83} + (7-1)^{83} = (1+7)^{83} - (1-7)^{83}$

$= 2[{}^{83}C_1\cdot7 + {}^{83}C_3\cdot7^3 + \ldots + {}^{83}C_{83}\cdot7^{83}]$

$= (2\cdot7\cdot83) + 49I$ where I is an integer

Now $14\times83 = 1162$

$\therefore \frac{1162}{49} = 23\frac{35}{49}$

$\therefore$ Remainder is 35

8. (0) $1 + 2 + 2^2 + 2^3 + \ldots + 2^{1999}$

$= \frac{1(2^{2000}-1)}{2-1}$

$= 2^{2000} - 1$

$= (1-5)^{1000} - 1$

$= 1 - {}^{1000}C_1 \cdot 5 + {}^{1000}C_2 \cdot 5^2 + \cdots + {}^{1000}C_{1000} \cdot 5^{1000} - 1$

which is divisible by 5.

9. (6) $(1 - 2x + 5x^2 - 10x^3)\,[{}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots]$

$= 1 + a_1 x + a_2 x^2 + \cdots$

$\Rightarrow \quad a_1 = n - 2$ and $a_2 = \dfrac{n(n-1)}{2} - 2n + 5$

Given that $a_1^2 = 2a_2$

$\Rightarrow \quad (n-2)^2 = n(n-1) - 4n + 10$

or $\quad n^2 - 4n + 4 = n^2 - 5n + 10$

or $\quad n = 6$

10. (1) $\displaystyle\sum_{k=0}^{4}\left(\frac{3^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)$

$$= \sum_{k=0}^{4}\left(\frac{4!}{(4-k)!\,k!}\,3^{4-k}\cdot x^k \cdot \frac{1}{4!}\right)$$

$$= \sum_{k=0}^{4}\frac{{}^4C_k \cdot 3^{4-k}\cdot x^k}{4!}$$

$$= \frac{(3+x)^4}{4!}$$

According to the question,

$$\frac{(3+x)^4}{4!} = \frac{32}{3}$$

or $\quad (3+x)^4 = 256$

or $\quad x + 3 = 4$

or $\quad x = 1$

11. (0) $E = (1 + x + x^2 + x^3 + x^4)^{199}\,(x-1)^{201}$

$$= \left(\frac{1-x^5}{1-x}\right)^{199}(x-1)^{201}$$

$$= -(1-x)^2\,(1-x^5)^{199}$$

$$= -(1 - 2x + x^2)\,(1 - {}^{199}C_1 x^5 + {}^{199}C_2 x^{10} - \ldots)$$

$\therefore$ Coefficient x^{103} in $E = 0$

12. (102) $({}^{101}C_0 - {}^{101}C_1 x + {}^{101}C_2 x^2 - \ldots - {}^{101}C_{101}x^{101})$

$(1 + x + x^2 + \ldots + x^{100})^{101}$

$$= (1-x)^{101}\cdot\left(\frac{1-x^{101}}{1-x}\right)^{101}$$

$$= (1 - x^{101})^{101}$$

$\therefore$ Number of different terms = 102

13. (495) $\left(\log\left(x^{\log x}\right) - \log_{x^2} 100\right)^{12}$

$$= \left((\log x)^2 - \frac{1}{\log x}\right)^{12}$$

$\therefore$ Constant term $= {}^{12}C_4((\log x)^2)^4\left(-\dfrac{1}{\log x}\right)^8$

$$= {}^{12}C_4$$

$$= \frac{12\times 11\times 10\times 9}{24}$$

$$= 55 \times 9 = 495$$

14. (220) $\displaystyle\sum_{r=0}^{3} {}^8C_r\,({}^5C_{r+1} - {}^4C_r) = \sum_{r=0}^{3} {}^8C_r\,{}^4C_{r+1}$

$= {}^8C_0 \times {}^4C_1 + {}^8C_1 \times {}^4C_2 + {}^8C_2 \times {}^4C_3 + {}^8C_3 \times {}^4C_4$

= coefficient of x^3 in $(1+x)^4(1+x)^8$

= coefficient of x^3 in $(1+x)^{12}$

$= {}^{12}C_3 = 220$

15. (5151)

Here, $T_r = \dfrac{r\cdot {}^{101}C_r}{{}^{101}C_{r-1}} = \dfrac{r\cdot(101-r+1)}{r} = 102 - r$

$\therefore$ Given sum $= \displaystyle\sum_{r=1}^{101}(102-r)$

$= 101 + 100 + 99 + \ldots + 2 + 1$

$= 5151$

16. (9) $f(n) = {}^nC_0 a^{n-1} - {}^nC_1 a^{n-2} + {}^nC_2 a^{n-3} + \cdots + (-1)^{n-1}\,{}^nC_{n-1}a^0$

$$= \frac{1}{a}\left({}^nC_0\,a^n - {}^nC_1\,a^{n-1} + {}^nC_2\,a^{n-2} + \cdots + (-1)^{n-1}C_{n-1}\,a\right)$$

$$= \frac{1}{a}\left((a-1)^n - (-1)^n\,{}^nC_n\right)$$

$$= \frac{1}{a}\left(\left(3^{\frac{n}{223}} - (-1)^n\right)\right)$$

$$f(x) = \frac{3^{\frac{n}{223}} - (-1)^n}{\left(3^{\frac{1}{223}} + 1\right)}$$

$$\Rightarrow \quad f(2007) = \frac{3^{\frac{2007}{223}} + 1}{3^{\frac{1}{223}} + 1}$$

$$\Rightarrow \quad f(2008) = \frac{3^{\frac{2008}{223}} - 1}{3^{\frac{1}{223}} + 1}$$

$$\Rightarrow \quad f(2007) + f(2008) = \frac{3^{\frac{2007}{223}} + 3^{\frac{2008}{223}}}{3^{\frac{1}{223}} + 1}$$

$$= \frac{3^9 + 3^{9+\frac{1}{223}}}{3^{\frac{1}{223}} + 1}$$

$$= 3^9\,\frac{\left(1 + 3^{\frac{1}{223}}\right)}{\left(1 + 3^{\frac{1}{223}}\right)} = 3^9$$

$\Rightarrow \quad 3^9 = 3^k$ then $k = 9$

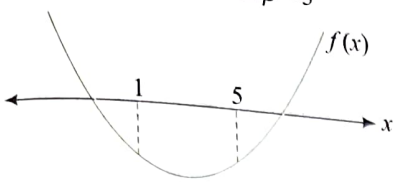
17. (5) We have $1 + \displaystyle\sum_{r=1}^{10}(3^r\cdot {}^{10}C_r + r\cdot {}^{10}C_r)$

$$= 1 + \sum_{r=1}^{10} 3^r\cdot {}^{10}C_r + 10\sum_{r=1}^{10} {}^9C_{r-1}$$

$$= 1 + 4^{10} - 1 + 10\times 2^9$$

$$= 4^{10} + 5\times 2^{10} = 2^{10}\,(4^5 + 5)$$

$= 2^{10}(\alpha \cdot 4^5 + \beta)$, so $\alpha = 1$ and $\beta = 5$



Now $f(1) < 0$ and $f(5) < 0$

$f(1) < 0 \Rightarrow -k^2 < 0 \Rightarrow k \neq 0$

and $f(5) < 0$

$\Rightarrow 16 - k^2 < 0$

or $k^2 - 16 > 0$

$\Rightarrow k \in (-\infty, 4) \cup (4, \infty)$

Hence, the smallest positive integral value of $k = 5$.

18. (1) $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r \left(\sum\limits_{t=0}^{r-1} {}^rC_t \cdot 3^t \right) = \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{1}{5^n} \cdot {}^nC_r (4^r - 3^r)$

$= \lim\limits_{n\to\infty} \frac{1}{5^n} \left(\sum\limits_{r=1}^{n} {}^nC_r\, 4^r - \sum\limits_{r=1}^{n} {}^nC_r\, 3^r \right)$

$= \lim\limits_{n\to\infty} \frac{1}{5^n} (5^n - 4^n) = 1$

19. (5) $\sum\limits_{r=0}^{n} \left(\frac{r+2}{r+1} \right) {}^nC_r = \sum\limits_{r=0}^{n} \left(1 + \frac{1}{r+1} \right) {}^nC_r$

$= \sum\limits_{r=0}^{n} {}^nC_r + \sum\limits_{r=0}^{n} \frac{{}^nC_r}{r+1}$

$= \sum\limits_{r=0}^{n} {}^nC_r + \frac{1}{(n+1)} \sum\limits_{r=0}^{n} {}^{n+1}C_{r+1}$

$= 2^n + \frac{1}{(n+1)} (2^{n+1} - 1)$

$= \frac{2^n(n+1) + 2^{n+1} - 1}{n+1}$

$= \frac{2^n(n+3) - 1}{n+1}$

$\therefore \quad \frac{2^n(n+3) - 1}{n+1} = \frac{2^8 - 1}{6}$

Hence, $n = 5$

20. (3) $S_n = {}^{2n}C_n$

$\Rightarrow \quad \frac{S_{n+1}}{S_n} = \frac{{}^{2n+2}C_{n+1}}{{}^{2n}C_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$

$= \frac{2(2n+1)}{n+1} = 4 - \frac{2}{n+1}$

$\therefore \quad \left[\frac{S_{n+1}}{S_n} \right]_{\max} = 3$

21. (1) Given sum

$= $ Coefficient of x^{40} in ${}^{40}C_0 (1+x)^{100} - {}^{40}C_1 (1+x)^{99} + {}^{40}C_2 (1+x)^{96} + \ldots + {}^{40}C_{40} (1+x)^{60}$

$= $ Coefficient of x^{40} in $(1+x)^{60}({}^{40}C_0 (1+x)^{40} - {}^{40}C_1 (1+x)^{39} + {}^{40}C_2 (1+x)^{38} + \ldots + {}^{40}C_{40})$

$= $ Coefficient of x^{40} in $(1+x)^{60}(1+x-1)^{40}$

$= $ Coefficient of x^{40} in $(1+x)^{60} x^{40}$

$= 1$

22. (211) $\sum\limits_{0 \le i < j \le 5} \sum \left({}^5C_j \right) \left({}^jC_i \right)$

$= {}^5C_1 \cdot {}^1C_0 + {}^5C_2 ({}^2C_0 + {}^2C_1) + \ldots + {}^5C_5 ({}^5C_0 + {}^5C_1 + \ldots + {}^5C_4)$

$= {}^5C_1 (2^1 - 1) + {}^5C_2 (2^2 - 1) + \ldots + {}^5C_5 (2^5 - 1)$

$= ({}^5C_1 \cdot 2^1 + {}^5C_2 \cdot 2^2 + \ldots + {}^5C_5 \cdot 2^5) - ({}^5C_1 + {}^5C_2 + \ldots + {}^5C_5)$

$= [(1+2)^5 - 1] - [2^5 - 1]$

$= 3^5 - 2^5$

$= 243 - 32 = 211$

23. (40) $\left(1 - x - x^2\right)^{20} = \sum\limits_{r=0}^{40} a_r \cdot x^r$

Differentiating both sides w.r.t. x, we get

$$20(1 - x - x^2)^{19}(-1 - 2x) = \sum_{r=0}^{40} r a_r \cdot x^{r-1} \qquad \ldots(1)$$

Putting x = 1 into (1), we get

$$20(1 - 1 - 1)^{19}(-1 - 2) = \sum_{r=0}^{40} r a_r$$

$\Rightarrow \quad 60 = a_1 + 2a_2 + 3a_3 + \ldots + 40a_{40} \qquad \ldots(2)$

Putting $x = -1$ into (1), we get

$$20(1 + 1 - 1)^{19}(-1 + 2) = \sum_{r=0}^{40} r a_r (-1)^{r-1}$$

$\Rightarrow \quad 20 = a_1 - 2a_2 + 3a_3 + \ldots - 40a_{40} \qquad \ldots(3)$

Adding (2) and (3), we get

$80 = 2a_1 + 6a_3 + \ldots + 78a_{39}$

$\therefore \quad a_1 + 3a_3 + \ldots + 39a_{39} = 40$

24. (2499)

$$\sum_{r=1}^{49} \frac{2r^2 - 48r + 1}{(50-r) \cdot {}^{50}C_r} = \sum_{r=1}^{49} \frac{(r+1)^2 - r(50-r)}{(50-r) \cdot {}^{50}C_r}$$

$$= \sum_{r=1}^{49} \left(\frac{(r+1)^2}{(50-r) \cdot {}^{50}C_r} - \frac{r}{{}^{50}C_r} \right)$$

$$= \sum_{r=1}^{49} \left(\frac{r+1}{(50-r)\left(\frac{{}^{50}C_r}{r+1} \right)} - \frac{r}{{}^{50}C_r} \right)$$

$$= \sum_{r=1}^{49} \left(\frac{r+1}{{}^{50}C_{r+1}} - \frac{r}{{}^{50}C_r} \right)$$

$$= \frac{50}{{}^{50}C_{50}} - \frac{1}{{}^{50}C_1}$$

$$= 50 - \frac{1}{50}$$

Archives

JEE Main

Single Correct Answer Type

1. (2) $8^{2n} - (62)^{2n+1} = (1+63)^n - (63-1)^{2n+1}$

$= (1+63)^n + (1-63)^{2n+1}$

$= [1 + {}^nC_1 63 + {}^nC_2 (63)^2 + \cdots + (63)^n]$
$+ [1 - {}^{(2n+1)}C_1 63 + {}^{(2n+1)}C_2 (63)^2 + \cdots - 63^{2n+1}]$
$= 2 + 63K$

Hence, the remainder is 2.

2. (2) $S_1 = \sum_{j=1}^{10} j(j-1)\frac{10!}{j(j-1)(j-2)!(10-j)!}$

$= 90\sum_{j=2}^{10}\frac{8!}{(j-2)!(8-(j-2))!}$

$= 90\sum_{j=2}^{10} {}^8C_{j-2} = 90 \times 2^8$

$S_2 = \sum_{j=1}^{10} j\frac{10!}{j(j-1)!(9-(j-1))!}$

$= 10\sum_{j=1}^{10}\frac{9!}{(j-1)!(9-(j-1))!}$

$= 10\sum_{j=1}^{10} {}^9C_{j-1} = 10 \times 2^9$

$S_3 = \sum_{j=1}^{10}[j(j-1)+j]{}^{10}C_j$

$= \sum_{j=1}^{10} j(j-1){}^{10}C_j + \sum_{j=1}^{10} j \cdot {}^{10}C_j$

$= 90 \times 2^8 + 10 \times 2^9 = 55 \times 2^9$

3. (4) $(1 - x - x^2 + x^3)^6$
$= (1-x)^6(1-x^2)^6$
$= (1 - {}^6C_1x + {}^6C_2x^2 - {}^6C_3x^3 + {}^6C_4x^4 - {}^6C_5x^5 + {}^6C_6x^6)$
$(1 - {}^6C_1x^2 + {}^6C_2x^4 - {}^6C_3x^6 + \cdots)$

$\Rightarrow$ coefficient of x^7 is ${}^6C_1 \cdot {}^6C_3 - {}^6C_3 \cdot {}^6C_2 + {}^6C_5 \cdot {}^6C_1$
$= 6 \times 20 - 20 \times 15 + 6 \times 6 = -144$

4. (1) $(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n}$
$= 2[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + {}^{2n}C_5(\sqrt{3})^{2n-5} + \cdots]$
= an irrational number

5. (3) $\left[\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right]^{10}$

$= \left[\frac{(x^{1/3}+1)(x^{2/3}-x^{1/3}+1)}{(x^{2/3}-x^{1/3}+1)} - \frac{(x^{1/2}-1)(x^{1/2}+1)}{x^{1/2}(x^{1/2}-1)}\right]^{10}$

$(\because \quad x+1 = (x^{1/3})^3 + 1^3)$

$= [(x^{1/3}+1) - (1+x^{-1/2})]^{10} = [x^{1/3} - x^{-1/2}]^{10}$

$\Rightarrow \quad T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r}(-x^{-1/2})^r$

$\Rightarrow \quad T_{r+1} = {}^{10}C_r x^{\frac{10-r}{3}-\frac{r}{2}}(-1)^r$

$\Rightarrow \quad T_{r+1} = {}^{10}C_r x^{\frac{20-5r}{6}}(-1)^r$

For constant term,
$20 - 5r = 0$
$\Rightarrow \quad r = 4$
$\Rightarrow \quad T_5 = {}^{10}C_4 = 210$

6. (4) $(1 + ax + bx^2)(1-2x)^{18}$
$= 1(1-2x)^{18} + ax(1-2x)^{18} + bx^2(1-2x)^{18}$

Coefficient of x^3 : $(-2)^3 {}^{18}C_3 + a(-2)^2 {}^{18}C_2 + b(-2){}^{18}C_1 = 0$

$\frac{4 \times (17 \times 16)}{(3 \times 2)} - 2a \cdot \frac{17}{2} + b = 0 \quad (1)$

Coefficient of x^4 : $(-2)^4 {}^{18}C_4 + a(-2)^3 {}^{18}C_3 + b(-2)^2 {}^{18}C_2 = 0$

$(4 \times 20) - 2a \cdot \frac{16}{3} + b = 0 \quad (2)$

From equations (1) and (2), we get

$4\left(\frac{17 \times 8}{3} - 20\right) + 2a\left(\frac{16}{3} - \frac{17}{2}\right) = 0$

$\Rightarrow \quad a = 16$

$\Rightarrow \quad b = \frac{2 \times 16 \times 16}{3} - 80 = \frac{272}{3}$

7. (1) $(1 - 2\sqrt{x})^{50} = C_0 - C_1 2\sqrt{x} + C_2(2\sqrt{x})^2 - \ldots + C_{50}(2\sqrt{x})^{50}$
$(1 + 2\sqrt{x})^{50} = C_0 + C_1(2\sqrt{x}) + C_2(2\sqrt{x})^2 + \ldots + C_{50}(2\sqrt{x})^{50}$

Putting $x = 1$ and adding, we get

$\therefore \quad \frac{3^{50}+1}{2} = C_0 + C_2(2)^2 + \ldots$

8. (3) Theoretically the number of terms are $2n + 1$ (i.e. odd).
But given that number of terms is 28.
So considering number of terms $= {}^{n+2}C_2 = 28$ (Here we are ignoring clubbing of terms)
$\therefore \quad n = 6$
$\therefore$ Sum of coefficient $= 3^n = 3^6 = 729$ (Putting $x = 1$)

9. (1) We have
$({}^{21}C_1 + {}^{21}C_2 \ldots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 \ldots {}^{10}C_{10})$
$= \frac{1}{2}[{}^{21}C_0 + {}^{21}C_1 + {}^{21}C_2 + \ldots + {}^{21}C_{21}] - 2^{10}$
$= 2^{20} - 2^{10}$

10. (1) $(x + \sqrt{x^3-1})^5 + (x - \sqrt{x^3-1})^5$
$= 2\left[{}^5C_0 x^5 + {}^5C_2 x^3(\sqrt{x^3-1})^2 + {}^5C_4 x(\sqrt{x^3-1})^4\right]$
$= 2\left[x^5 + 10x^3(x^3-1) + 5x(x^3-1)^2\right]$
$= 2[x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 5x]$
$\therefore$ Sum of the coefficients of odd degree terms
$= 2(1 - 10 + 5 + 5) = 2$

JEE Advanced

Single Correct Answer Type

1. (4) A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$.
$\therefore \quad A_r = {}^{10}C_r, B_r = {}^{20}C_r, C_r = {}^{30}C_r$
$\therefore \quad \sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$

$= B_{10}\sum_{r=1}^{10} A_r B_r - C_{10}\sum_{r=1}^{10}(A_r)^2$

$= B_{10}\sum_{r=1}^{10} {}^{10}C_r \, {}^{20}C_r - C_{10}\sum_{r=1}^{10}({}^{10}C_r)^2$

$$= B_{10}\sum_{r=1}^{10} {}^{10}C_r \; {}^{20}C_{20-r} - C_{10}\sum_{r=1}^{10}({}^{10}C_r)^2$$

$$= B_{10}\left[\left(\sum_{r=0}^{10} {}^{10}C_r \; {}^{20}C_{20-r}\right)-1\right] - C_{10}\left[\left(\sum_{r=0}^{10}({}^{10}C_r)^2\right)-1\right]$$

$$= B_{10}[{}^{30}C_{20} - 1] - C_{10}[{}^{20}C_{10} - 1]$$

$$(\because \quad {}^{n}C_0^2 + {}^{n}C_1^2 + {}^{n}C_2^2 + \ldots + {}^{n}C_n^2 = {}^{2n}C_n)$$

$$= [B_{10}\,{}^{30}C_{20} - C_{10}\,{}^{20}C_{10}] + [C_{10} - B_{10}]$$

$$= [{}^{20}C_{10}\,{}^{30}C_{20} - {}^{30}C_{10}\,{}^{20}C_{10}] + [C_{10} - B_{10}]$$

$$= C_{10} - B_{10}$$

2. (3) $2x_1 + 3x_2 + 4x_3 = 11$

Possibilities are (0, 1, 2); (1, 3, 0); (2, 1, 1); (4, 1, 0).

$\therefore$ Required coefficients

$$= ({}^4C_0 \times {}^7C_1 \times {}^{12}C_2) + ({}^4C_1 \times {}^7C_3 \times {}^{12}C_0) + ({}^4C_2 \times {}^7C_1 \times {}^{12}C_1) + ({}^4C_4 \times {}^7C_1 \times 1)$$

$$= (1 \times 7 \times 66) + (4 \times 35 \times 1) + (6 \times 7 \times 12) + (1 \times 7)$$

$$= 462 + 140 + 504 + 7 = 1113$$

Numerical Value Type

1. (6) Let T_{r-1}, T_r, T_{r+1} are three consecutive terms of $(1+x)^{n+5}$

$T_{r-1} = {}^{n+5}C_{r-2}\,(x)^{r-2}$, $T_r = {}^{n+5}C_{r-1}\,x^{r-1}$, $T_{r+1} = {}^{n+5}C_r x^r$,

where, ${}^{n+5}C_{r-2} : {}^{n+5}C_{r-1} : {}^{n+5}C_r = 5 : 10 : 14$.

So $\dfrac{{}^{n+5}C_{r-2}}{5} = \dfrac{{}^{n+5}C_{r-1}}{10} = \dfrac{{}^{n+5}C_r}{14}$

Comparing first two results we have $n - 3r = -9$ (1)

Comparing last two results we have $5n - 12r = -30$ (2)

From equation (1) and (2), $n = 6$

2. (8) x^9 can be formed in following eight ways.

i.e. $x^9, x^{1+8}, x^{2+7}, x^{3+6}, x^{4+5}, x^{1+2+6}, x^{1+3+5}, x^{2+3+4}$

and coefficient in each case is 1.

$\therefore$ Coefficient of $x^9 = 1 + 1 + 1 + \ldots$ 8 times

$= 8$

3. (5) Coefficient of x^2 in expansion

$$= 1 + {}^3C_2 + {}^4C_2 + {}^5C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2.m^2 \quad [\text{as } {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$$

$$= ({}^3C_3 + {}^3C_2) + {}^4C_2 + {}^5C_2 + \ldots + {}^{49}C_2 + {}^{50}C_2.m^2$$

$$= ({}^4C_3 + {}^4C_2) + \ldots + {}^{50}C_2\,m^2$$

$$= {}^5C_3 + \ldots + {}^{49}C_2 + {}^{50}C_2\,m^2$$

$$= {}^{50}C_3 + {}^{50}C_2\,m^2 + {}^{50}C_2 - {}^{50}C_2$$

$$= {}^{51}C_3 + {}^{50}C_2\,(m^2 - 1)$$

$$= (3n+1)^{51}C_3 \text{ (given)}$$

$\therefore$ $3n \cdot \dfrac{51}{3}\,{}^{50}C_2 = {}^{50}C_2\,(m^2 - 1)$

$$\frac{m^2 - 1}{51} = n$$

Value of n is 5.

4. (646) $X = \displaystyle\sum_{r=1}^{10} r \cdot ({}^{10}C_r)^2 = \sum_{r=1}^{10} r \cdot {}^{10}C_r \cdot {}^{10}C_r$

$$= 10 \cdot \sum_{r=1}^{10} {}^9C_{r-1} \cdot {}^{10}C_{10-r} = 10 \cdot {}^{19}C_9$$

Now, $\dfrac{X}{1430} = \dfrac{10 \cdot {}^{19}C_9}{1430} = \dfrac{{}^{19}C_9}{143} = \dfrac{{}^{19}C_9}{11 \times 13}$

$$= \frac{19 \times 17 \times 16}{8} = 19 \times 34 = 646$$

Chapter 9

Concept Application Exercises

Exercise 9.1

1. (a) In assignment (a), each number is positive and less than 1
Sum of probabilities

$$= 0.1 + 0.01 + 0.05 + 0.03 + 0.01 + 0.2 + 0.6 = 1$$

Thus, this assignment is valid.

(b) In assignment (b), each number is positive and less than 1.

$$\text{Sum of probabilities} = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7} + \frac{1}{7}$$

$$= 7 \times \frac{1}{7} = 1$$

Thus, this assignment is valid.

(c) In assignment (c), each number is positive and less than 1
Sum of probabilities

$$= 0.1 + 0.2 + 0.3 + 0.4 + 0.5 + 0.6 + 0.7$$
$$= 2.8 \neq 1$$

Thus, the assignment is not valid.

(d) In assignment, two numbers are negative. So, it is not valid assignment.

(e) In assignment, $p(\omega_7) = \frac{15}{14} > 1$. So, it is not valid assignment.

2. (a) $P(X \text{ is a prime number}) = P(2) + P(3) + P(5) + P(7)$

$$= 0.23 + 0.12 + 0.20 + 0.07$$
$$= 0.62$$

(b) $P(X \text{ is a number greater than } 4) = P(5) + P(6) + P(7) + P(8)$

$$= 0.20 + 0.08 + 0.07 + 0.05$$
$$= 0.4$$

3. There are 366 days in a leap year. Also, there are 52 weeks and 2 days.

The possible combinations of two days are as follows:

Sunday-Monday,
Monday-Tuesday,
Tuesday-Wednesday,
Wednesday-Thursday,
Thursday-Friday,
Friday-Saturday,
Saturday-Sunday

Now, let E be the event in which leap year will have 53 Fridays or 53 Saturdays.

We observe that there are three possibilities for event E, viz. Thursday-Friday, Friday-Saturday, and Saturday-Sunday.

Since each pair of 2 days is equally likely,

$$P(E) = \frac{1}{7} + \frac{1}{7} + \frac{1}{7} = \frac{3}{7}$$

4. Since the probabilities of the faces are proportional to the numbers on them, we can take the probabilities of faces 1, 2, …, 6 as k, $2k$, …, $6k$, respectively. Since one of the faces must occur, we have

$k + 2k + 3k + 4k + 5k + 6k = 1$ or $k = 1/21$

Therefore, the probability of occurence of an even number is $2k + 4k + 6k = 12k = 12/21 = 4/7$.

5. There are 52 cards in the pack and each card has equal probability of getting drawn that is $\frac{1}{52}$.

There are four king and four ace cards.

So, probability of drawing either an ace or a king

$$= \frac{1}{52} + \frac{1}{52} + \ldots + \frac{1}{52} (8 \text{ times})$$

$$= \frac{8}{52} = \frac{2}{13}$$

6. Sample space $= \{Y_1, Y_2, Y_3, R_1, R_2, B\}$, where Y stands from yellow, R for red and B for blue.

Now, $P(Y_1) = P(Y_2) = P(Y_3) = P(R_1) = P(R_2) = P(B) = \frac{1}{6}$.

$$\therefore \quad P(Y) = P(Y_1) + P(Y_2) + P(Y_3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$$

$$P(R) = P(R_1) + P(R_2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

And $P(B) = \frac{1}{6}$

Exercise 9.2

1. Total number of cases $n(S) = 36$

Since $\omega^{ab} = 1$, ab must be multiple of 3.

So, at least one of a and b must be multiple of 3.

If none of the dices shows 3 or 6, number of cases is $4 \times 4 = 16$.

So, number of cases in which at least one of the dices shows 3 or 6 is $36 - 16 = 20$.

$\therefore$ Required probability $= \frac{20}{36} = \frac{5}{9}$

2. Required probability is $1 - P(\text{all letters in right envelope}) = 1 - 1/n!$

(As there are total number of $n!$ ways in which letters can take envelopes and just one way in which they have corresponding envelopes.)

3. Total number of ways is 36. Favorable cases are (1, 4), (2, 3), (3, 2), (4, 1), (1, 5), (2, 4), (3, 3), (4, 2), (5, 1). Total number of favorable cases is 9.

Hence, the required probability is $9/36 = 1/4$.

4. If both integers are even, then product is even. If both integers are odd, then product is odd. If one integer is odd and other is even, then product is even. Therefore, the required probability is 2/3.

5. The total number of ways in which 2 integers can be chosen from the given 20 integers is ${}^{20}C_2$.

The sum of the selected numbers is odd if exactly one of them is even and only one is odd. Therefore, the favorable number of outcomes is ${}^{10}C_1 \times {}^{10}C_1$.

Therefore, the required probability is

$$\frac{{}^{10}C_1 \times {}^{10}C_1}{{}^{20}C_2} = \frac{10}{19}$$

6. The required probability is

$$\frac{{}^{3}C_3 + {}^{7}C_3 + {}^{4}C_3}{{}^{14}C_3} = \frac{1 + 35 + 4}{14 \times 13 \times 2} = \frac{40}{14 \times 26} = \frac{10}{91}$$

The total number of ways is $n = 6^5$. A total of 12 in 5 throw can be obtained in following two ways:

(i) one blank and four 3's can be obtained in ${}^5C_1 = 5$ ways.

(ii) three 2's and two 3's can be obtained in ${}^5C_2 = 10$ ways.

Hence, the required probability is $15/6^5 = 5/2592$.

Total number of ways of arranging 11 letters is (11)! The number of selection of 4 letters to be placed between R and E from remaining 9 is ${}^9C_4$. These four letters can be permuted in 4! ways. Now R and E can be interchanged in 2! ways.

$$\underbrace{(R\bullet\bullet\bullet\bullet}_{\text{4 letters}}\ \underbrace{E)\bullet\bullet\bullet\bullet\bullet}_{\text{5 letters}}$$

Hence, the number of favorable ways is ${}^9C_4 \times 4! \times 6! \times 2!$

Therefore, the required probability is

$$\frac{11!}{{}^9P_4 \times 6! \times 2!}$$

9. Total number of numbers formed by the digits 1, 2, 3, 4, 5 without repetition is 5! = 120. Now we know that a number is divisible by 4 if the number placed at its last two digits is divisible by 4. So that last two digits can be 12, 24, 32 or 52, i.e., they can be filled in 4 ways. There are 3! = 6 ways of filling the other three places.

Therefore, the favorable number of ways is $4 \times 6 = 24$.

Hence, the required probability is

$$P = \frac{\text{Favorable no. of cases}}{\text{Exhaustive no. of cases}} = \frac{24}{120} = \frac{1}{5}$$

10. In an 8-floor house, there are 7 floors above the ground floor.

Each person can leave the cabin at any of the seven floors, i.e., each person can leave the cabin in 7 ways. Thus, total number of ways into which 5 persons can leave the cabin is 7^5. Now number of the ways of leaving the cabin by 5 person each at different floor is ${}^7P_5$.

Hence, the required probability is ${}^7P_5/7^5$.

11. Let each of the two friends have n daughters. Then probability that all the tickets go to the daughters of A is

$$\frac{{}^nC_3}{{}^{2n}C_3} = \frac{n-2}{4(2n-1)} = \frac{1}{20}$$

$$\Rightarrow \quad n = 3$$

12. The number of ways of selecting 4 socks from 24 socks is $n(S) = {}^{24}C_4$.

The number of ways of selecting 4 socks from different pairs is

$$n(E) = {}^{12}C_4 \times 2^4$$

$$\Rightarrow \quad P(E) = \frac{{}^{12}C_4 \times 2^4}{{}^{24}C_4}$$

Hence, the probability of getting at least one pair is

$$1 - \frac{{}^{12}C_4 \times 2^4}{{}^{24}C_4}$$

13. Total number of ways is 7! Favorable number of ways is 7! – 2 (6!). Hence, the probability is

$$\frac{7! - 2(6!)}{7!} = 1 - \frac{2}{7} = \frac{5}{7}$$

14. Five tickets out of 50 can be drawn in ${}^{50}C_5$ ways. Since $x_1 < x_2 < x_3 < x_4 < x_5$ and $x_3 = 30$, we have $x_1, x_2 < 30$, i.e., x_1 and x_2 should come from tickets numbered 1 and 29 and this may happen in ${}^{29}C_2$ ways. Remaining ways, i.e., $x_4, x_5 > 30$, should come from 20 tickets numbered 31 to 50 in ${}^{20}C_2$ ways.

So, favorable number of cases is ${}^{29}C_2\ {}^{20}C_2$. Hence, required probability is

$$\frac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5}$$

15. We have to find the probability that out of 26 cards drawn at random from the pack of cards, 13 will be red and 13 black. The total number of different ways to draw 26 cards from 52 is ${}^{52}C_{26}$. The favorable ways are those in which there will be 13 cards drawn from 26 red cards and 13 from 26 black cards. 13 red cards may be drawn in ${}^{26}C_{13}$ different ways and 13 black cards also in ${}^{26}C_{13}$ different ways. Therefore, total number of favorable ways is equal to ${}^{26}C_{13}\ {}^{26}C_{13}$. And consequently the required probability is

$$P = \frac{{}^{26}C_{13}\ {}^{26}C_{13}}{{}^{52}C_{26}}$$

$$= \frac{\left(\frac{(26)!}{(13)!(13)!}\frac{(26)!}{(13)!(13)!}\right)}{\left(\frac{(52)!}{(26)!(26)!}\right)} = \frac{\{(26)!\}^4}{\{(13)!\}^4\,(52)!}$$

16. Given set of numbers is {1, 2, ..., 9} in which 4 are even and 5 are odd, so in the given product it is not possible to arrange to substract only even number from odd number, there must be at least one factor involving substraction of an odd number from another odd number. So at least one of the factor is even. Hence, product is always even.

$\therefore$ Required probability = 1

17. $2^m + 2^n + 1 = (3-1)^m + (3-1)^n + 1$

$= 3k + (-1)^m + (-1)^n + 1$

This is divisible by 3 if both m and n are even.

$\therefore$ Required probability $= \frac{{}^{50}C_2}{{}^{100}C_2} = \frac{50 \times 49}{100 \times 99} = \frac{49}{198}$

18. The total number of ways of choosing two numbers out of 1, 2, 3, ..., 30 is ${}^{30}C_2 = 435$.

Now, 30 natural numbers are categorized as

$3k$ type: 3, 6, 9, ... 30 → 10 numbers

$3k + 1$ type: 1, 4, 7, ..., 28 → 10 numbers

$3k + 2$ type: 2, 5, 8, ..., 29 → 10 numbers

$a^2 - b^2$ is divisible by 3 iff either both of a and b are divisible by 3 or none of a and b is divisible by 3.

So, number of favourable cases = ${}^{10}C_2 + {}^{20}C_2 = 235$

$\therefore$ Required probability $= \frac{235}{435} = \frac{47}{87}$

19. Each ball can be put into any one of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes is 3^{12}. Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways for the first box. Now, remaining 9 balls can be put in the remaining 2 boxes in 2^9 ways. So, the total number of ways in which 3 balls are put in the first box and the remaining balls in other two boxes is ${}^{12}C_3 \times 2^9$. Hence, required probability is ${}^{12}C_3 \times 2^9/3^{12}$.

Exercise 9.3

1. $P(A) = 0.5$, $P(A \cap B) \le 0.3$

So, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ gives

$P(B) = P(A \cup B) - P(A) + P(A \cap B) \le 1 + 0.3 - 0.5 = 0.8$

$[\because \quad P(A \cup B) \le 1]$

Hence, $P(B) = 0.9$ is not possible

2. $P(A^c) = 1 - P(A)$ or $P(A) = 1 - P(A^c) = 1 - 2/3 = 1/3$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad 3/4 = 1/3 + P(B) - 1/4 \Rightarrow P(B) = 2/3$

$$P(A \cap B^c) = P(A) - P(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$P((A^c \cap B) = P(B) - P(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

3. We have,

$$P(\bar{A} \cap \bar{B}) = \frac{1}{3}$$

$$\Rightarrow \quad P(\overline{A \cup B}) = \frac{1}{3}$$

$$\Rightarrow \quad 1 - P(A \cup B) = \frac{1}{3}$$

$$\text{or} \quad P(A \cup B) = \frac{2}{3}$$

$$\text{or} \quad P(A) + P(B) - P(A \cap B) = \frac{2}{3}$$

$$\Rightarrow \quad p + 2p - \frac{1}{2} = \frac{2}{3}$$

$$\text{or} \quad p = \frac{7}{18}$$

4. Consider the following events:

A: A student is passed in Mathematics

B: A student is passed in Statistics

Then,

$$P(A) = \frac{70}{125}, P(B) = \frac{55}{125}, P(A \cap B) = \frac{30}{125}$$

Required probability is

$$P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

$$= \frac{70}{125} + \frac{55}{125} - \frac{60}{125} = \frac{65}{125} = \frac{13}{25}$$

5. Here,

$$P(R) \ \frac{10}{100} = 0.1, P(F) = \frac{5}{100} = 0.05$$

$$P(F \cap R) = \frac{3}{100} = 0.03$$

Therefore, the required probability is

$$P(R) + P(F) - 2P(F \cap R) = 0.1 + 0.05 - 2(0.03)$$

$$= 0.09$$

6. $P(C) = p$; $P(A) = 2p$; $P(B) = 2p$

$\therefore \quad 5p = 1$ or $p = 1/5$

So, $\quad P(B \text{ or } C) = P(B) + P(C) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$

7. Since $P(A \cup B \cup C) \geq 0.75$, we have

$$0.75 \leq P(A \cup B \cup C) \leq 1$$

or $\quad 0.75 \leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) \leq 1$

or $\quad 0.75 \leq 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09 \leq 1$

or $\quad 0.75 \leq 1.23 - P(B \cap C) \leq 1$

or $\quad -0.48 \leq -P(B \cap C) \leq -0.23$

or $\quad 0.23 \leq P(B \cap C) \leq 0.48$

Exercises

Singe Correct Answer Type

1. (1) $p^2 + 2p + 4p - 1 = 1$ (Exhaustive)

$p^2 + 6p - 2 = 0$

or $\quad p = -3 \pm \sqrt{11}$

$= \sqrt{11} - 3$

2. (2) $P(E) + P(E') = 1$

$\therefore \quad 1 + \lambda + \lambda^2 + (1 + \lambda)^2 = 1$

$\Rightarrow \quad 2\lambda^2 + 3\lambda + 1 = 0$

$\Rightarrow \quad (2\lambda + 1)(\lambda + 1) = 0$

$\Rightarrow \quad \lambda = -1, -\frac{1}{2}$

For $\lambda = -1$, $P(E) = 1 + (-1) + (-1)^2 = 1$ (not possible)

For $\lambda = -\frac{1}{2}$, $P(E) = 1 + \left(-\frac{1}{2}\right) + \left(-\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

3. (2) Since the sum of the noted numbers is 6, the number on the balls are either 1, 2, 3 or 2, 2, 2.
So, total number of cases = 3! + 1 = 7
So, the required probability = 1/7

4. (3) If A draws card higher than B, then the number of favorable cases is $(n-1) + (n-2) + \ldots + 3 + 2 + 1$ (as when B draws card number 1, then A can draw any card from 2 to n and so on). Therefore, the required probability is

$$\frac{\frac{n(n-1)}{2}}{n^2} = \frac{n-1}{2n}$$

5. (1) L and W can be filled at 14 places in 2^{14} ways.

$\therefore \quad n(S) = 2^{14}$

Now 13 L's and 1 W can be arranged at 14 places in 14 ways.
Hence, $n(A) = 14$.

$\therefore \quad p = \frac{14}{2^{14}} = \frac{7}{2^{13}}$

6. (4) A person can have his/her birthday on any one of the seven days of the week. So 5 persons can have their birthdays in 7^5 ways. Out of 5, three persons can have their birthdays on days other than Sundays in 6^3 ways and other 2 on Sundays. Hence, the required probability is

$$= \frac{{}^5C_2 \times 6^3}{7^5} = \frac{10 \times 6^3}{7^5}$$

(Note that 2 persons can be selected out of 5 in ${}^5C_2$ ways.)

7. (1) Required probability $= \dfrac{\text{No. of favorable cases}}{\text{Total no. of exhaustive cases}}$

$$= \frac{3}{3 \times 3 \times 3} = \frac{1}{9}$$

8. (4) The number of ways of arranging n numbers is $n!$ In each order obtained, we must now arrange the digits 1, 2, ..., k as group and the $n - k$ remaining digits. This can be done in $(n - k + 1)!$ ways. Therefore, the probability for the required event is $(n - k + 1)!/n!$

9. (2) The total number of cases is $11!/(2! \times 2!)$. The number of favorable cases is $[11!/(2! \times 2!)] - 9!$ Therefore, the required probability is

$$= 1 - \frac{9! \times 4}{11!} = \frac{53}{55}$$

10. (1) The number of ways in which 20 people can be divided into two equal groups is

$$n(S) = \frac{20!}{10!\ 10!\ 2!}$$

The number of ways in which 18 people can be divided into groups of 10 and 8 is

$$n(E) = \frac{18!}{10!\ 8!}$$

$$\therefore \quad P(E) = \frac{18!}{10!8!}\frac{10!10!2}{20!} = \frac{10\times 9\times 2}{20\times 19} = \frac{9}{19}$$

11. (2) Total number of ways of distribution is 4^5.

$\therefore \quad n(S) = 4^5$

Total number of ways of distribution so that each child gets at least one game is

$$4^5 - {}^4C_1\,3^5 + {}^4C_2\,2^5 - {}^4C_3 = 1024 - 4\times 243 + 6\times 32 - 4 = 240$$

$\therefore \quad n(E) = 240$

Therefore, the required probability is

$$\frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$$

12. (1) Out of 9 socks, 2 can be drawn in ${}^9C_2$ ways. Therefore, the total number of cases is ${}^9C_2$. Two socks drawn from the drawer will match if either both are brown or both are blue. Therefore, favorable number of cases is ${}^5C_2 + {}^4C_2$. Hence, the required probability is

$$= \frac{{}^5C_2 + {}^4C_2}{{}^9C_2} = \frac{4}{9}$$

13. (2) The total number of ways in which four-figure numbers can be formed is 4! = 24. A number is divisible by 5 if at unit's place we have 5. Therefore, unit's place can be filled in one way and the remaining 3 places can be filled with the other digits in 3! ways. Hence, total number of numbers divisible by 5 is 3! = 6. So, the required probability is 6/24 = 1/4.

14. (1) Since each ball can be placed in any one of the 3 boxes, therefore there are 3 ways in which a ball can be placed in any one of the three boxes. Thus, there are 3^{12} ways in which 12 balls can be placed in 3 boxes. The number of ways in which 3 balls out of 12 can be put in the box is ${}^{12}C_3$. The remaining 9 balls can be placed in 2 boxes in 2^9 ways. So, required probability is

$$\frac{{}^{12}C_3}{3^{12}}\,2^9 = \frac{110}{9}\left(\frac{2}{3}\right)^{10}$$

15. (4) The total number of ways of choosing 11 players out of 15 is ${}^{15}C_{11}$. A team of 11 players containing at least 3 bowlers can be chosen in the following mutually exclusive ways:

(I) Three bowlers out of 5 bowlers and 8 other players out of the remaining 10 players.

(II) Four bowlers out of 5 bowlers and 7 other players out of the remaining 10 players.

(III) Five bowlers out of 5 bowlers and 6 other players out of the remaining 10 players.

So, required probability is

$$P(\text{I}) + P(\text{II}) + P(\text{III}) = \frac{{}^5C_3\times{}^{10}C_8}{{}^{15}C_{11}} + \frac{{}^5C_4\times{}^{10}C_7}{{}^{15}C_{11}} + \frac{{}^5C_5\times{}^{10}C_6}{{}^{15}C_{11}}$$

$$= \frac{1260}{1365} = \frac{12}{13}$$

16. (3) Required probability = 1 P (G_5 and G_7 are consecutive)

$$= 1 - \frac{{}^4C_3}{{}^7C_5} = \frac{17}{21}$$

17. (2) Since lines are more than telegrams, ${}^NC_M$ are those lines where telegrams will be sent.

So, number of favourable cases = ${}^NC_M \times M!$

Total number of cases = N^M

[As first telegram can go in any one of n lines]

So, required probability = $\dfrac{{}^NC_M \times M!}{N^M}$

18. (2) There are 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The last two digits can be dialled in ${}^{10}P_2 = 90$ ways out of which only one way is favorable; thus, the required probability is 1/90.

19. (1) For each toss, there are four choices:

(i) A gets head, B gets head

(ii) A gets tail, B gets head

(iii) A gets head, B gets tail

(iv) A gets tail, B gets tail

Thus, exhaustive number of ways is 4^{50}. Out of the four choices listed above, (iv) is not favorable to the required event in a toss. Therefore, favorable number of cases is 3^{50}. Hence, the required probability is $(3/4)^{50}$.

20. (2) Let A denote the event that there is an odd man out in a game. The total number of possible cases is 2^m. A person is odd man out if he is alone in getting a head or a tail.

The number of ways in which there is exactly one tail (head) and the rest are heads (tails) is ${}^mC_1 = m$. Thus, the number of favourable ways is $m + m = 2m$. Therefore,

$$P(A) = \frac{2m}{2^m} = \frac{m}{2^{m-1}}$$

21. (1) The total number of ways in which $2n$ boys can be divided into two equal groups is

$$= \frac{(2n)!}{(n!)^2 2!}$$

Now, the number of ways in which $2n - 2$ boys other than the two tallest boys can be divided into two equal groups is

$$= \frac{(2n-2)!}{((n-1)!)^2 2!}$$

Two tallest boys can be put in different groups in ${}^2C_1$ ways. Hence, the required probability is

$$= \frac{2\dfrac{(2n-2)!}{((n-1)!)^2 2!}}{\dfrac{(2n)!}{(n!)^2 2!}}$$

$$= \frac{n}{2n-1}$$

22. (4) The total number of ways in which papers of 4 students can be checked by seven teachers is 7^4. The number of ways of choosing two teachers out of 7 is ${}^7C_2$. The number of ways in which they can check four papers is 2^4. But this includes two ways in which all the papers will be checked by a single teacher. Therefore, the number of ways in which 4 papers can be checked by exactly two teachers is $2^4 - 2 = 14$. Therefore, the number of favorable ways is $({}^7C_2)\ (14) = (21)\ (14)$. Thus, the required probability is $(21)(14)/7^4 = 6/49$.

23. (2) We must have,

$$0 \le P(A) \le 1$$

$$\Rightarrow \quad 0 \le \frac{3x+1}{3} \le 1$$

$$\Rightarrow \quad \frac{-1}{3} \le x \le \frac{2}{3} \qquad (1)$$

and $0 \le P(B) \le 1$

$$\Rightarrow \quad 0 \le \frac{1-x}{4} \le 1$$

$$\Rightarrow \quad -3 \le x \le 1$$

and $0 \le P(A \cup B) = P(A) + P(B) \le 1$ (2)

$$\Rightarrow \quad 0 \le \frac{3x+1}{3} + \frac{1-x}{4} \le 1$$

$$\Rightarrow \quad \frac{-7}{9} \le x \le \frac{5}{9} \qquad (3)$$

From (1), (2) and (3), we get $x \in \left[\frac{-1}{3}, \frac{5}{9}\right]$.

24. (3) We have,

$$x + \frac{100}{x} > 50$$

or $x^2 + 100 > 50x$

or $(x-25)^2 > 525$

$\Rightarrow$ $x - 25 < -\sqrt{525}$ or $x - 25 > \sqrt{525}$

$\Rightarrow$ $x < 25 - \sqrt{525}$ or $x > 25 + \sqrt{525}$

As x is a positive integer and $\sqrt{525} = 22.91$, we must have $x \le 2$ or $x \ge 48$. Thus, the favorable number of cases is $2 + 53 = 55$. Hence, the required probability is $55/100 = 11/20$.

25. (3) Since it is known that each time a different digit appears, total number of cases is 6!

The sum in the first three throws is 12 if we get (1, 5, 6), (2, 4, 6) or (3, 4, 5) in any order.

So, number of cases for first three throws $= 3 \times 3!$

The remaining three throws have 3! ways.

$\therefore$ Number of favourable cases $= 3 \times 3! \times 3!$

$\therefore$ Required probability $= \dfrac{3 \times 3! \times 3!}{6!} = \dfrac{3}{20}$

26. (3) $x^2 + 2(a+4)x - 5a + 64 \ge 0$

If $D < 0$, then

$$(a+4)^2 - (-5a + 64) < 0$$

or $a^2 + 13a - 48 < 0$

or $(a+16)(a-3) < 0$

$\Rightarrow$ $-16 < a < 3 \Leftrightarrow -5 \le a \le 2$

Then, the favorable cases is equal to the number of integers in the interval $[-5, 2]$, i.e., 8.

Total number of cases is equal to the number of integers in the interval $[-5, 30]$, i.e., 36.

Hence, the required probability is $8/36 = 2/9$.

27. (3) Given that 5 and 6 have appeared on two of the dice, the sample space reduces to $6^4 - 2 \times 5^4 + 4^4$ (inclusion–exclusion principle). Also, the number of favorable cases are $4! = 24$. So, the required probability is $24/302 = 12/151$.

28. (2) The prime digits are 2, 3, 5, 7. If we fix 2 at first place, then other $2n - 1$ places are filled by all four digits. So the total number of cases is 4^{2n-1}.

Now, sum of 2 consecutive digits is prime when consecutive digits are (2, 3) or (2, 5). Then 2 will be fixed at all alternative places.

2		2		2		2	...	...	2	

So favorable number of cases is 2^n. Therefore, probability is

$$\frac{2^n}{4^{2n-1}} = 2^n \, 2^{-4n+2} = 2^2 \, 2^{-3n} = \frac{4}{2^{3n}}$$

29. (3) When 4 points are selected, we get one intersecting point. So, probability is

$$\frac{{}^nC_4}{({}^nC_2 - n)\,{}^nC_2}$$

30. (4) Three-digit numbers are 100, 101, ..., 999. Total number of such numbers is 900. The three-digit numbers (which have all same digits) are 111, 222, 333, ..., 999. Favorable number of cases is 9. Therefore, the required probability is $9/900 = 1/100$.

31. (3) Given,

$$7a - 9b = 0 \text{ or } b = \frac{7}{9}a$$

Hence, number of pairs (a, b) can be (9, 7); (18, 14); (27, 21); (36, 28). Hence, the required probability is $4/{}^{39}C_2 = 4/741$.

32. (3) The total number of mapping is n^n. The number of one-one mapping is $n!$. Hence, the probability is

$$\frac{n!}{n^n} = \frac{3}{32} = \frac{4!}{4^4}$$

Comparing, we get $n = 4$.

33. (1) The number of composite numbers in 1 to 30 is $n(S) = 19$.

The number of composite numbers when divided by 5 leaves a remainder is $n(E) = 14$. Therefore, the required probability is 14/19.

34. (2) Team totals must be 0, 1, 2, ..., 39. Let the teams be $T_1 T_2 \ldots T_{40}$, so that T_i loses to T_j for $i < j$. In other words, this order uniquely determines the result of every game. There are 40! such orders and 780 games, so 2^{780} possible outcomes for the games. Hence, the probability is $40!/2^{780}$.

35. (1) We have,

$$n(S) = {}^{64}C_3$$

Let 'E' be the event of selecting 3 squares which form the letter 'L'.

The number of ways of selecting squares consisting of 4 unit squares is $7 \times 7 = 49$.

Each square with 4 unit squares form 4 L-shapes consisting of 3 unit squares. Therefore,

$$n(E) = 7 \times 7 \times 4 = 196$$

$$\therefore \quad P(E) = \frac{196}{{}^{64}C_3}$$

36. (1) The total number of ways of making the second draw is ${}^{10}C_5$.

The number of draws of 5 balls containing 2 balls common with first draw of 6 balls is ${}^6C_2 \, {}^4C_3$. Therefore, the probability is

$$\frac{{}^6C_2 \, {}^4C_3}{{}^{10}C_5} = \frac{5}{21}$$

37. (2) A number has exactly 3 factors if the number is squares of a prime number. Squares of 11, 13, 17, 19, 23, 29, 31 are 3-digit numbers. Hence, the required probability is 7/900.

38. (3) Suppose, there exist three rational points or more on the circle $x^2+y^2+2gx+2fy+c=0$. Therefore, if (x_1, y_1), (x_2, y_2), and (x_3, y_3) are those three points, then

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \quad (1)$$
$$x_2^2+y_2^2+2gx_2+2fy_1+c=0 \quad (2)$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \quad (3)$$

Solving Eqs. (1), (2) and (3), we will get g, f, c as rational. Thus, center of the circle $(-g, -f)$ is a rational point. Therefore, both the coordinates of the center are rational numbers. Obviously, the possible values of p are 1, 2. Similarly, the possible values of q are 1, 2. Thus, for this case, (p, q) may be chosen in 2×2, i.e., 4 ways. Now, (p, q) can be, without restriction, chosen in 6×6, i.e., 36 ways.

Hence, the probability that at the most two rational points exist on the circle is $(36-4)/36 = 32/36 = 8/9$.

39. (1) The total number of ways of selecting 3 integers from 20 natural numbers is ${}^{20}C_3 = 1140$. Their product is a multiple of 3 means at least one number is divisible by 3. The numbers which are divisible by 3 are 3, 6, 9, 12, 15, 18 and the number of ways of selecting at least one of them is ${}^6C_1 \times {}^{14}C_2 + {}^6C_2 \times {}^{14}C_1 + {}^6C_3 = 776$. Hence, the required probability is $776/1140 = 194/285$.

40. (3) We have,

$$n(S) = 5^5$$

For computing favorable outcomes, 2 boxes which are to remain empty, can be selected in ${}^5C_2$ ways and 5 marbles can be placed in the remaining 3 boxes in groups of 221 or 311 in

$$3!\left[\frac{5!}{2!2!2!}+\frac{5!}{3!2!}\right] = 150 \text{ ways}$$

$$\Rightarrow \quad n(A) = {}^5C_2 \times 150$$

Hence,

$$P(E) = {}^5C_2 \times \frac{150}{5^5} = \frac{60}{125} = \frac{12}{25}$$

41. (1) $n(S) = {}^{10}C_7 = 120$

$$n(E) = {}^5C_4 \times {}^3C_2 \times {}^2C_1$$

$$\therefore \quad P(E) = \frac{5 \times 3 \times 2}{120} = \frac{1}{4}$$

42. (4) Since there are r cars in N places, total number of selection of places out of $N-1$ places for $r-1$ cars (except the owner's car) is

$${}^{N-1}C_{r-1} = \frac{(N-1)!}{(r-1)!(N-r)!}$$

If neighboring places are empty, then $r-1$ cars must be parked in $N-3$ places. So, the favorable number of cases is

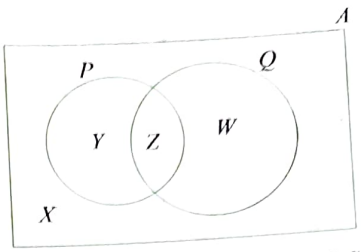
$${}^{N-3}C_{r-1} = \frac{(N-3)!}{(r-1)!(N-r-2)!}$$

Therefore, the required probability is

$$= \frac{(N-3)!}{(r-1)!(N-r-2)!} \times \frac{(r-1)!(N-r)!}{(N-1)!}$$

$$= \frac{(N-r)(N-r-1)}{(N-1)(N-2)} = \frac{{}^{N-r}C_2}{{}^{N-1}C_2}$$

43. (3)



Sets P and Q are subsets of set A. Sets P and Q are dividing set A in four regions X, Y, Z and W.

$\therefore$ Total number of cases

= Number of ways in which n elements can be distributed in four regions

$= 4^n$

If set $(P \cap Q)$ has exactly m elements, we can select these m elements in ${}^nC_m$ ways. Remaining $n-m$ elements can be distributed in the regions X, Y and W in 3^{n-m} ways.

$\therefore$ Number of favourable ways $= {}^nC_m \times 3^{n-m}$

$\therefore$ Required probability $= \dfrac{{}^nC_m \times 3^{n-m}}{4^n}$

44. (3) $f'(x) = 3x^2 + 2ax + b$

$y = f(x)$ is increasing

$\Rightarrow$ $f'(x) \geq 0, \forall x$ and for $f'(x) = 0$ should not form an interval

$\Rightarrow$ $(2a)^2 - 4 \times 3 \times b \leq 0 \Rightarrow a^2 - 3b \leq 0$

This is true for exactly 16 ordered pairs (a, b), $1 \leq a, b \leq 6$, namely (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6), and (4, 6). Thus, the required probability is $16/36 = 4/9$

45. (3) Given limit,

$$\lim_{x\to 0}\left(\frac{a^x+b^x}{2}\right)^{\frac{2}{x}}$$

$$= \left(\lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{2}\right)^{\frac{2}{a^x+b^x-2}}\right)^{\lim\limits_{x\to 0}\left(\frac{a^x-1+b^x-1}{x}\right)}$$

$$= e^{\log ab} = ab = 6$$

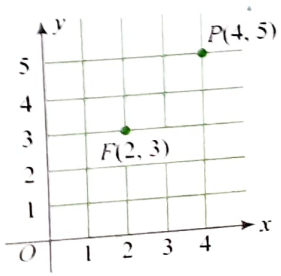
Total number of possible ways in which a, b can take values is $6 \times 6 = 36$. Total possible ways are (1, 6), (6, 1), (2, 3), (3, 2). The total number of possible ways is 4. Hence, the required probability is $4/36 = 1/9$.

46. (2) $n(S) = \dfrac{9!}{4!.5!} = 126$

$n(A) = O$ to F and F to P

$$= \frac{5!}{2!\times 3!} \times \frac{4!}{2!\times 2!} = 60$$

$$\Rightarrow \quad P(A) = \frac{60}{126} = \frac{10}{21}$$



Multiple Correct Answers Type

1. **(1), (2), (3), (4)**

We have,

P(exactly one of A, B occurs)

$= P[(A \cap \bar{B}) \cup (\bar{A} \cap B)]$

$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$

$= P(A) - P(A \cap B) + P(B) - P(A \cap B)$
$= P(A) + P(B) - 2P(A \cap B)$
$= P(A \cup B) - P(A \cap B)$

Also,

P(exactly one of A, B occurs)
$= [1 - P(\bar{A} \cap \bar{B})] - [1 - P(\bar{A} \cup \bar{B})]$
$= P(\bar{A} \cup \bar{B}) - P(\bar{A} \cap \bar{B})] = P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

2. (2), (3), (4)

Roots of $x^2 + px + q = 0$ will be real if $p^2 \geq 4q$.

The possible selections are as follows:

p	q
1	–
2	1
3	1, 2
4	1, 2, 3, 4
5	1, 2, 3, 4, 5, 6
6	1, 2, …, 9
7	1, 2, …, 10
8	1, 2, …, 10
9	1, 2, …, 10
10	1, 2, …, 10
Total	62

Therefore, number of favourable ways is 62 and total number of ways is $10^2 = 100$. Hence, the required probability is
$62/100 = 31/50$.

The probability that the roots are imaginary is
$1 - 31/50 = 19/50$.

Roots are equal when $(p, q) \equiv (2, 1), (4, 4), (6, 9)$. The probability that the roots are real and equal is 3/50. Hence, probability that the roots are real and distinct is 3/5.

3. (1), (2), (3), (4)

$A \subseteq A \cup B$

$\Rightarrow \quad P(A) \leq P(A \cup B)$

$\Rightarrow \quad P(A \cup B) \geq \frac{3}{4}$

Also, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
$\geq P(A) + P(B) - 1$
$= \frac{3}{4} + \frac{5}{8} - 1 = \frac{3}{8}$

Now, $A \cap B \subseteq B$

$\Rightarrow \quad P(A \cap B) \leq P(B) = \frac{5}{8}$

$\therefore \quad \frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$ (1)

and $P(A \cap B') = P(A) - P(A \cap B)$

or $\frac{3}{4} - \frac{5}{8} \leq P(A \cap B') \leq \frac{3}{4} - \frac{3}{8}$

or $\frac{1}{8} \leq P(A \cap B') \leq \frac{3}{8}$

$\therefore \quad P(A \cap B) = P(B) - P(A' \cap B)$ [Using Eq. (1)]

or $\frac{3}{8} \leq P(B) - P(A' \cap B) \leq \frac{5}{8}$

or $0 \leq P(A' \cap B) \leq \frac{1}{4}$

4. (1), (3)

Given that A and B are mutually exclusive events.

$\therefore \quad A \cap B = \phi$

$\Rightarrow \quad A \subseteq \bar{B}$ and $B \subseteq \bar{A}$

$\Rightarrow \quad P(A) \leq P(\bar{B})$ and $P(B) \leq P(\bar{A})$

5. (1), (3), (4)

$$P(E) = \frac{{}^{2n}C_n}{2^{2n}} = \frac{(2n)!}{n!\,n!\,2^n\,2^n}$$
$$= \frac{1 \times 2 \times 3 \times \cdots \times (2n)}{n!\,n!\,2^n\,2^n}$$
$$= \frac{1 \times 3 \times 5 \cdots \times (2n-1)}{n!\;2^n}$$

Now,

$$\prod_{r=1}^{n}\left(\frac{2r-1}{2r}\right) = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)}$$
$$= \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{(1 \times 2 \times 3 \times \cdots \times n)2^n}$$

$$\sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^n}\right)^2 = \frac{1}{2^n 2^n}\sum_{r=0}^{n}({}^nC_r)^2$$
$$= \frac{1}{2^n 2^n}\,{}^{2n}C_n$$

Also, $$\frac{\sum_{r=0}^{n}({}^nC_r)^2}{\left(\sum_{r=0}^{2n}{}^{2n}C_r\right)} = \frac{{}^{2n}C_n}{2^{2n}}$$

6. (1), (3)

Let p_1, p_2 be the chances of happening of the first and second events, respectively. Then according to the given conditions, we have

$p_1 = p_2^2$

and $\frac{1-p_1}{p_1} = \left(\frac{1-p_2}{p_2}\right)^3$

Hence,

$\frac{1-p_2^2}{p_2^2} = \left(\frac{1-p_2}{p_2}\right)^3$ or $p_2(1+p_2) = (1-p_2)^2$

or $3p_2 = 1$ or $p_2 = \frac{1}{3}$ and so $p_1 = \frac{1}{9}$

7. (1), (2)

Let the number of red and blue balls be r and b, respectively. Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^rC_2}{{}^{r+b}C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^bC_2}{{}^{r+b}C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^rC_1 \times {}^bC_1}{{}^{r+b}C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1 = 5p_2$ and $p_3 = 6p_2$. Therefore,
$r(r-1) = 5b(b-1)$ and $2br = 6b(b-1)$
$\Rightarrow$ $r = 6, b = 3$

8. **(1), (2), (3)**
Here total number of cases is ${}^8C_2 = 28$.
1. Number of favorable case is 13.
For $2 \rightarrow 6$ choices
For $1 \rightarrow 7$ choices
2. For $6 \rightarrow 5$ choices
For $7 \rightarrow 6$ choices
For $8 \rightarrow 7$ choices
3. For $1 \rightarrow 4$ choices (2, 4, 6, 8)
For $2 \rightarrow 6$ choices (3, 4, 5, 6, 7, 8)
For $3 \rightarrow 3$ choices (4, 6, 8)
For $4 \rightarrow 4$ choices (5, 6, 7, 8)
For $5 \rightarrow 2$ choices (6, 8)
For $6 \rightarrow 2$ choices (7, 8)
For $7 \rightarrow 1$ choice (8)

Alternate solution:

a. $\dfrac{{}^8C_2 - {}^6C_2}{{}^8C_2} = \dfrac{13}{28}$

b. $\dfrac{{}^8C_2 - {}^5C_2}{{}^8C_2} = \dfrac{9}{14}$

c. $\dfrac{{}^8C_2 - {}^4C_2}{{}^8C_2} = \dfrac{11}{14}$

nked Comprehension Type

1. (3), **2. (4), 3. (4)**
Let in 8 coupons S, U, R, F appears x_1, x_2, x_3, x_4 times. Then $x_1 + x_2 + x_3 + x_4 = 8$, where $x_1, x_2, x_3, x_4 \geq 0$.

We have to find non-negative integral solutions of the equation. The total number of such solutions is
${}^{8+4-1}C_{4-1} = {}^{11}C_3 = 165$.

If a person gets at least one free packet, then he must get each coupon at least once, which is equal to number of positive integral solutions of the equation. The number of such solutions is ${}^{8-1}C_{4-1} = {}^7C_3 = 35$. Then, the probability that he gets exactly one free packet is
$(35 - 1)/165 = 34/165$
The probability that he gets two free packets is $1/{}^{11}C_3 = 1/165$.

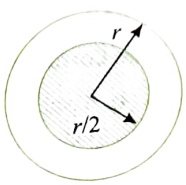
4. (3) x can be 2, 3, 4, 5, 6. The number of ways in which sum of 2, 3, 4, 5, 6 can occur is given by the coefficients of x^2, x^3, x^4, x^5, x^6 in
$(3x + 2x^2 + x^3)(x + 2x^2 + 3x^3) = 3x^2 + 8x^3 + 14x^4 + 8x^5 + 3x^6$
This shows that sum that occurs most often is 4.

5. (4) Sum that occurs for minimum times is 2 or 6.

6. (3) The number of ways in which different sums can occur is $(3+2+1)(1+2+3) = 36$. The probability of 4 is $14/36 = 7/18$.

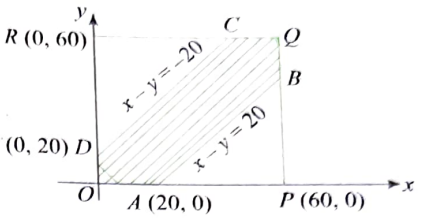
7. (2) **8. (2), 9. (4)**
Clearly, cube at the centre is the only cube which is not painted. On each edge, we have one cube pained on two sides. So, there are twelve cubes pained on two sides.

10. (1) For the favorable cases, the points should lie inside the concentric circle of radius $r/2$. So the desired probability is given by



$$\frac{\text{Area of smaller circle}}{\text{Area of larger circle}} = \frac{\pi\left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

11. (4)



Let A and B arrive at the place of their meeting 'a' minutes and 'b' minutes after 5 pm. Their meeting is possible only if
$|a - b| \leq 20$ (1)
Clearly, $0 \leq a \leq 60$ and $0 \leq b \leq 60$. Therefore, a and b can be selected as an ordered pair (a, b) from the set $[0, 60] \times [0, 60]$.
Alternatively, it is equivalent to select a point (a, b) from the square $OPQR$, where P is $(60, 0)$ and R is $(0, 60)$ in the Cartesian plane. Now,
$|a - b| \leq 20 \Rightarrow -20 \leq a - b \leq 20$
Therefore, points (a, b) satisfy the equation $-20 \leq x - y \leq 20$.
Hence, favorable condition is equivalent to selecting a point from the region bounded by $y \leq x + 20$ and $y \geq x - 20$. Therefore, the required probability is

$$\frac{\text{Area of } OABQCDO}{\text{Area of square } OPQR} = \frac{[\text{Ar}(OPQR) - 2\text{Ar}(\Delta APB)]}{\text{Ar}(OPQR)}$$

$$= \frac{60 \times 60 - \frac{2}{2} \times 40 \times 40}{60 \times 60} = \frac{5}{9}$$

12. **(1)** The total number of ways of painting first column when colors are not alternating is $2^8 - 2$. The total number of ways when no column has alternating colors is $(2^8 - 2)^8/2^{64}$.

13. **(1)** The number of ways the square has equal number of red and black squares is ${}^{64}C_{32}$. Hence, the required probability is ${}^{64}C_{32}/2^{64}$.

14. **(2)** This is possible only when the column are alternating red and black. Hence, the required probability is $2/2^{64} = 1/2^{63}$.

Matrix Match Type

1. **a → r; b → p; c. → q; d → s.**

a. The required event will occur if last digit in all the chosen numbers is 1, 3, 7, or 9. Therefore, the required probability is $(4/10)^n$.

b. The required probability is equal to the probability that the last digit is 2, 4, 6, 8 and is given by
P (last digit is 1, 2, 3, 4, 6, 7, 8, 9)
$-P$ (last digit is 1, 3, 7, 9) $= \dfrac{8^n - 4^n}{10^n}$

c. $P(1, 3, 5, 7, 9) - P(1, 3, 7, 9) = \dfrac{5^n - 4^n}{10^n}$

d. The required probability is

$$P(0, 5) - P(5) = \frac{(10^n - 8^n) - (5^n - 4^n)}{10^n}$$

$$= \frac{10^n - 8^n - 5^n + 4^n}{10^n}$$

2. (3) $n(s) = {}^{100}C_3$

a. $2b = a + c$ = even

This means that a and c are both even or both odd.

$n(E) = {}^{50}C_2 + {}^{50}C_2 = 50 \times 49$

b. Taking $r = 2, 3, \ldots, 10$,

a, b, c can be in GP in 53 ways.

c. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in GP $= a, b, c$ are in GP

d. $P(a + b + c \text{ is even}) = \left(\frac{{}^{50}C_3 + {}^{50}C_1 \times {}^{50}C_2}{{}^{100}C_3}\right) = \frac{1}{2}$

Numerical Value Type

1. (2) Total number of cases $= n(S) = 6!$

Now sum the given digits is $1 + 2 + 3 + 4 + 5 + 6 = 21$, which is divisible by 3.

Now we have to form the number which is divisible by 6, then we have to ensure that the digit in unit place is even.

$\Rightarrow$ Favorable cases $= n(A) = 3 \cdot 5!$

Hence, $P(A) = \frac{3 \cdot 5!}{6!} = \frac{1}{2}$.

2. (6) Total number of cases $n(S) = 6^3 = 216$

Product is prime only when two outcomes are 1 and the third is prime, i.e., 2, 3, 5.

If it is 2, 1, 1, then the number of cases is 3.

Similarly, 3 cases for 3, 1, 1 and 5, 1, 1 each

Hence, favorable cases = 9.

Hence, required probability $p = \frac{1}{24}$

$\Rightarrow \quad \frac{1}{4p} = 6$

3. (4) Let the number of green socks be $x > 0$.

E: Two socks drawn are of the same color

$\Rightarrow \quad P(E) = P(R\,R \text{ or } B\,B \text{ or } W\,W \text{ or } G\,G)$

$$= \frac{3}{{}^{6+x}C_2} + \frac{{}^{x}C_2}{{}^{6+x}C_2}$$

$$= \frac{6}{(x+6)(x+5)} + \frac{x(x-1)}{(x+6)(x+5)} = \frac{1}{5} \text{ (given)}$$

$\Rightarrow \quad 5(x^2 - x + 6) = x^2 + 11x + 30$

or $\quad 4x^2 - 16x = 0$

or $\quad x = 4$

4. (5) $P(n) = Kn^2$

Given $\quad P(1) = K, P(2) = 2^2K, P(3) = 3^2K, P(4) = 4^2K,$

$P(5) = 5^2K, P(6) = 6^2K$

So, total probability $= 91K$

$\Rightarrow \quad 1 = 91K$

$\therefore \quad K = \frac{1}{91}$

Therefore, $P(1) = \frac{1}{91}$; $P(2) = \frac{4}{91}$ and so on.

Let three events A, B and C be defined as

A: $a < b$

B: $a = b$

C: $a > b$

By symmetry, $P(A) = P(C)$

Also, $\quad P(A) + P(B) + P(C) = 1$

Since $P(B) = \sum_{i=1}^{6} [P(i)]^2$

$$= \left[\frac{1 + 16 + 81 + 256 + 625 + 1296}{91 \times 91}\right]$$

$$= \frac{2275}{91 \times 91} = \frac{25}{91}$$

Now, $2P(A) + P(B) = 1$

$\Rightarrow \quad P(A) = \frac{1}{2}[1 - P(B)] = \frac{33}{91}$

5. (6) Given S_2 reaches the semi-finals.

Since all other players $(2^n - 1)$ are equally likely to win the finals with probability p.

$\therefore \quad (2^n - 1)p + \frac{1}{4} = 1$

$\Rightarrow \quad (2^n - 1)p = \frac{3}{4}$

$\Rightarrow \quad p = \frac{3}{4(2^n - 1)}$

If $\quad p = \frac{1}{84}$, then

$\therefore \quad \frac{1}{84} = \frac{3}{4(2^n - 1)}$

$\Rightarrow \quad 2^n - 1 = 63$

$\Rightarrow \quad 2^n = 64$

$\Rightarrow \quad n = 6$

6. (4) Total ways of distribution $= n(S) = 4^5$

Total ways of distribution so that each child get at least one game

$$n(E) = 4^5 - {}^4C_1\,3^5 + {}^4C_2\,2^5 - {}^4C_3$$
$$= 1024 - 4 \times 243 + 6 \times 32 - 4$$
$$= 240$$

Required probability $p = \frac{n(E)}{n(S)} = \frac{240}{4^5} = \frac{15}{64}$

7. (7) Number of ways of drawing 7 balls (second draw) $= {}^{10}C_7$

For each set of 7 balls of the second draw, 3 must be common to the set of 5 balls of the first draw, i.e., 2 other balls can be drawn in ${}^3C_2$ ways.

Thus, for each set of 7 balls of the second draw, there are ${}^7C_3 \times {}^3C_2$ ways of making the first draw so that there are 3 balls common.

$\therefore \quad$ Probability of getting 3 balls in common $= \frac{{}^7C_3 \times {}^3C_2}{{}^{10}C_7}$

$$= \frac{7}{8}$$

Archives

JEE Main

Single Correct Answer Type

1. (4) Total number of ways of selecting four numbers $= {}^{20}C_4$

Let d be common difference of A.P.

If $d = 1$, we have
$(1, 2, 3, 4), (2, 3, 4, 5), \ldots, (17, 18, 19, 20) \rightarrow 17$ A.P.s
If $d = 2$, we have
$(1, 3, 5, 7), (2, 4, 6, 8), \ldots, (14, 16, 18, 20) \rightarrow 14$ A.P.s
If $d = 3$, we have
$(1, 4, 7, 10), (2, 5, 8, 11), \ldots, (11, 14, 17, 20) \rightarrow 11$ A.P.s
If $d = 4$, we have
$(1, 5, 9, 13), (2, 6, 10, 14), \ldots, (8, 12, 16, 20) \rightarrow 8$ A.P.s
If $d = 5$, we have
$(1, 6, 11, 16), (2, 7, 12, 17), \ldots, (5, 10, 15, 20) \rightarrow 5$ A.P.s
If $d = 6$, we have
$(1, 7, 13, 19), (2, 8, 14, 20) \rightarrow 2$ A.P.s
So required probability

$$= \frac{17+14+11+8+5+2}{{}^{20}C_4} = \frac{1}{85}$$

2. (3) $N(S) = {}^9C_3$
$N(E) = {}^3C_1 \times {}^4C_1 \times {}^2C_1$

$$\text{Probability} = \frac{3 \times 4 \times 2}{{}^9C_3}$$

$$= \frac{24 \times 3!}{9!} \times 6! = \frac{24 \times 6}{9 \times 8 \times 7} = \frac{2}{7}$$

3. (2) We have, $A \equiv \{0, 1, 2, 3, 4, \ldots, 10\}$
$n(S) = {}^{11}C_2$
Let E be the given event.
$\therefore \quad E \equiv \{(0, 4), (0, 8), (2, 6), (2, 10), (4, 8), (6, 10)\}$
$\therefore \quad n(E) = 6$

$$\therefore \quad P(E) = \frac{6}{55}$$

4. (3) P(Exactly one of A or B occurs)

$$= P(A) + P(B) - 2P(A \cap B) = \frac{1}{4}$$

P(Exactly one of B or C occurs)

$$= P(B) + P(C) - 2P(B \cap C) = \frac{1}{4}$$

P(Exactly one of C or A occurs)

$$= P(C) + P(A) - 2P(C \cap A) = \frac{1}{4}$$

Adding all, we get

$$P(A) + P(B) + P(A) - P(A \cap B) - P(B \cap C) - P(A \cap C) = \frac{3}{8}$$

$$\therefore \quad \Sigma P(A) - \Sigma P(A \cap B) = \frac{3}{8}$$

Now, $P(A \cup B \cup C)$
$= \Sigma P(A) - \Sigma P(A \cap B) + P(A \cap B \cap C)$

$$= \frac{3}{8} + \frac{1}{16} = \frac{7}{16}$$

EE ADVANCED

Single Correct Answer Type

1. (3) $r_1, r_2, r_3 \in \{1,2,3,4,5,6\}$
r_1, r_2, r_3 are of the form $3k, 3k+1, 3k+2$

$$\text{Required probability} = \frac{3! \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{6 \times 6 \times 6} = \frac{6 \times 8}{216} = \frac{2}{9}$$

2. (1) 3 Boys and 2 Girls ...
(1) B (2) B (3) B (4)
Girl can't occupy 4th position.
Either girls can occupy 2 of 1, 2, 3 position or they can both be at position (1) or (2).
Hence, total number of ways in which girls can be seated
$= {}^3C_2 \times 2! \times 3! + {}^2C_1 \times 2! \times 3! = 36 + 24 = 60.$
Number of ways in which 3 boys and 2 girls can be seated $= 5!$

$$\therefore \quad \text{Required probability} = \frac{60}{5!} = \frac{1}{2}$$

3. (3) $x + y + z = 10$
Total number of non-negative solutions $= {}^{10+3-1}C_{3-1}$
$= {}^{12}C_2 = 66$
Now let $z = 2n$
$\therefore \quad x + y = 10 - 2n; \; n \geq 0$
Number of non-negative solutions of above equation is
$11 + 9 + 7 + 5 + 3 + 1 = 36.$

$$\therefore \quad \text{Required probability} = \frac{36}{66} = \frac{6}{11}$$

Linked Comprehension Type

1. (2) For, $x_1 + x_2 + x_3 =$ odd.
Case I: One odd, two even
(OEE) or (EOE) or (EEO)
Total number of ways $= 2 \times 2 \times 3 + 1 \times 3 \times 3 + 1 \times 2 \times 4 = 29.$
Case II: All three odd
Number of ways $= 2 \times 3 \times 4 = 24$
$\therefore \quad$ Favorable ways $= 53$

$$\therefore \quad \text{Required probability} = \frac{53}{3 \times 5 \times 7} = \frac{53}{105}$$

2. (3) Here, $2x_2 = x_1 + x_3$
$\Rightarrow \quad x_1 + x_3 =$ even
So, either x_1 and x_3 are both odd or both even.
Hence, number of favorable ways $= {}^2C_1 \cdot {}^4C_1 + {}^1C_1 \cdot {}^3C_1 = 11.$
Therefore, required probability is $\frac{11}{105}$.

3. (1) Total number of cases $= 5!$
Since S_1 gets seat R_1 and none of the other gets previously allotted seat, we have derangement of 4 students.
So, required probability

$$= \frac{4!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)}{5!} = \frac{9}{120} = \frac{3}{40}$$

4. (3) Event $T_1 \cap T_2 \cap T_3 \cap T_4$: No two students with consecutive index sit adjacent.
So, we have following possible arrangements:
13524, 14253
24135, 24153, 25314
31425, 31524, 35142, 35241
41352, 42531, 42513
52413, 53142
Thus, favorable cases $= 14$

$$\text{Hence, required probability} = \frac{14}{120} = \frac{7}{60}$$

Chapter 10

Concept Application Exercises

Exercise 10.1

1.

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	F	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

2. **(a)** $(p \vee \sim r) \wedge (q \vee \sim r)$

p	q	r	$\sim r$	$p \vee \sim r$	$q \vee \sim r$	$(p \vee \sim r) \wedge (q \vee \sim r)$
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	F	F
F	T	T	F	F	T	F
T	F	F	T	T	T	T
F	F	T	F	F	F	F
F	F	F	T	T	T	T

(b) $\sim (p \vee \sim q) \wedge (\sim p \vee r)$

p	q	r	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$	$\sim p \vee r$	$\sim (p \vee \sim q) \wedge (\sim p \vee r)$
T	T	T	F	F	T	F	T	F
T	T	F	F	F	T	F	F	F
T	F	T	F	T	T	F	T	F
F	T	T	T	F	F	T	T	T
T	F	F	F	T	T	F	F	F
F	T	F	T	F	F	T	T	T
F	F	T	T	T	T	F	T	F
F	F	F	T	T	T	F	T	F

3. **(a)** $\sim p \rightarrow (q \rightarrow p)$

p	q	$\sim p$	$q \rightarrow p$	$\sim p \rightarrow (q \rightarrow p)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

(b) $(p \rightarrow q) \rightarrow (p \wedge q)$

p	q	$p \rightarrow q$	$p \wedge q$	$(p \rightarrow q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

4. **(a)** Truth values of $(p \leftrightarrow \sim q) \leftrightarrow (q \rightarrow p)$:

p	q	$\sim q$	$p \leftrightarrow \sim q$	$q \rightarrow p$	$(p \leftrightarrow \sim q) \leftrightarrow (q \rightarrow p)$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F

(b) Truth values of $(p \rightarrow q) \vee \sim (p \leftrightarrow \sim q)$:

p	q	$\sim q$	$p \rightarrow q$	$p \leftrightarrow \sim q$	$\sim (p \leftrightarrow \sim q)$	$(p \rightarrow q) \vee \sim (p \leftrightarrow \sim q)$
T	T	F	T	F	T	T
T	F	T	F	T	F	F
F	T	F	T	T	F	T
F	F	T	T	F	T	T

5. **(a)** Truth table for $(p \wedge q) \rightarrow \sim p$

p	q	$p \wedge q$	$\sim p$	$(p \wedge q) \rightarrow \sim p$
T	T	T	F	F
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

(b) Truth table for $(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(c) Truth table for $(p \wedge q) \rightarrow r$

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	F	T
F	F	F	F	T

(d) Truth table for $[p \wedge (\sim r)] \rightarrow (q \vee r)$

p	q	r	$\sim r$	$p \wedge (\sim r)$	$q \vee r$	$(p \wedge (\sim r)) \rightarrow (p \vee r)$
T	T	T	F	F	T	T
T	T	F	T	T	T	T
T	F	T	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T

F	T	F	T	F	T	T
F	F	T	F	F	T	T
F	F	F	T	F	F	T

6. Truth values of $\sim (p \leftrightarrow q)$

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

Truth values of $(p \wedge \sim q) \vee (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \vee (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	T
F	T	T	F	F	T	T
F	F	T	T	F	F	F

Statements $\sim (p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (\sim p \wedge q)$ have identical truth values.

$\therefore \quad \sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

7. Let p be the statement 'Traders do not reduce the prices' and q be the statement 'Government takes action against them'.

The first statement in symbolic form is $p \rightarrow q$ and the second statement is $-(p \wedge -q)$.

In order to check the equivalence of the above statements, let us prepare the following truth table.

p	q	$\sim q$	$p \wedge \sim q$	$\sim (p \wedge \sim q)$	$p \rightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Clearly, $p \rightarrow q$ and $\sim(p \wedge \sim q)$ have same truth values for all the values of p and q. Hence, the two statements are equivalent.

8. **Converse:** If a quadrilateral has two pairs of parallel sides, then it is a rectangle.

Inverse: If a quadrilateral is not a rectangle, then it does not have two pairs of parallel sides.

Contrapositive: If a quadrilateral does not have two pairs of parallel sides, then it is not a rectangle.

Here, both converse and inverse are false. Since the given statement is true, its contrapositive is true.

Exercise 10.2

1.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$(p \wedge q) \vee (\sim p)$	$p \wedge \sim q$	$(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$
T	T	F	F	T	T	F	T
T	F	F	T	F	F	T	T
F	T	T	F	F	T	F	T
F	F	T	T	F	T	F	T

For all possible truth values of p and q, the compound statement $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$ is true.

Thus, $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$ is a tautology.

2. Truth values of $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

For all possible truth values of p and q, the compound statement $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is true.

Thus, $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is a tautology.

3. $\sim ((\sim p) \wedge q \equiv \sim (\sim p) \vee \sim q \equiv p \vee (\sim q)$

4. $(p \wedge q) \wedge \sim (p \vee q) \equiv (p \wedge q) \wedge (\sim p \wedge \sim q)$
$\equiv (p \wedge \sim p) \wedge (q \wedge \sim q)$
$\equiv f \wedge f$
$\equiv f$

Thus, $(p \wedge q) \wedge \sim (p \vee q)$ is fallacy, i.e., contradiction.

5. $\sim (\sim p \rightarrow \sim q) \equiv \sim (\sim (\sim p) \vee \sim q$
$\equiv \sim (p \vee \sim q)$
$\equiv \sim p \wedge q$

Exercises

Single Correct Answer Type

1. (4) Clearly, 'please do me a favour' is not a statement.

2. (1) $\sim p$: Ram is not tall

Thus, $\sim p \vee q$ means "Ram is not tall or he is intelligent".

3. (4) All the statements in (1), (2) and (3) are equivalent and each is the negation of p.

4. (4) If p : A number is a prime

q : It is odd

We have $p \rightarrow q$.

The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

$\sim p \rightarrow \sim q$: If a number is not a prime then it is not odd.

5. (1) Correct result is: $(\sim p \vee \sim q) \rightarrow (r \wedge s)$

or $\quad \sim (p \wedge q) \rightarrow (r \wedge s)$

6. (2) Let p : Paris is in France, q : London is in England

The given statement is $p \wedge q$.

Its negation is $\sim (p \wedge q) = \sim p \vee \sim q$.

So, the required statement is "Paris is not in France or London is not in England".

7. (3) "Mathematics is interesting" is not a logical sentence. It may be interesting for some persons and may not be interesting for others.

Therefore, this is not a proposition.

8. (2) $\sim ((\sim (\sim p)) \wedge q) \equiv \sim (p \wedge q)$
$\equiv \sim p \vee \sim q$

9. (2) $\sim (p \vee (\sim q)) \equiv \sim p \wedge \sim (\sim q) \equiv (\sim p) \wedge q$

10. (4) $\sim(\sim p \to q) \equiv \sim((\sim(\sim p)) \vee q)$
$\equiv \sim(p \vee q)$
$\equiv \sim p \wedge \sim q$

11. (1) $(\sim p \vee q) \wedge \sim r \to p$
$\equiv (F \vee F) \wedge F \to T$
$\equiv F \to T$
$\equiv T$

12. (1) $(p \wedge \sim r) \to (q \vee r)$ is false.
Thus, $p \wedge \sim r$ is true and $q \vee r$ is false.
Hence, p must be true.

13. (2) $(p \wedge q) \wedge (q \wedge r)$ is true which means that $p \wedge q$ and $q \wedge r$ are both true.
Therefore, p, q and r are all true.

14. (1) $\sim[p \vee (\sim p \vee q)]$
$\equiv \sim p \wedge \sim(\sim p \vee q)$
$\equiv \sim p \wedge (\sim(\sim p) \wedge \sim q)$
$\equiv \sim p \wedge (p \wedge \sim q)$

15. (2) $(\sim(p \vee q)) \vee (\sim p \wedge q)$
$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$\equiv \sim p \wedge (\sim q \vee q)$
$\equiv \sim p$

16. (3) Inverse of $p \to q$ is $\sim p \to \sim q$
Therefore, inverse of $(p \wedge \sim q) \to r$ is
$\sim(p \wedge \sim q) \to \sim r$
$\equiv \sim(p \vee q) \to \sim r$

17. (1) $\sim(q \vee \sim(p \wedge r))$
$\equiv \sim(p \wedge (\sim(\sim(p \wedge r))$
$\equiv \sim q \wedge (p \wedge r)$

18. (3) Contrapositive of $p \to q$ is $\sim q \to \sim p$
Therefore, contrapositive of $(p \vee q) \to r$ is
$\sim r \to \sim(p \vee q)$
$\equiv \sim r \to (\sim p \wedge \sim q)$

19. (4) $p \to q$ is false only when p is true and q is false.
Therefore, $p \to (q \vee r)$ is false when p is true and $q \vee r$ is false.
$q \vee r$ is false only when both q and r are false.
Hence, truth values of p, q and r are, respectively, T, F and F.

20. (2) $(p \wedge \sim q) \wedge (\sim p \wedge q)$
$\equiv (p \wedge \sim p) \wedge (\sim q \wedge q)$
$\equiv f \wedge f$
$\equiv f$ (By using associative laws and commutative laws)
Therefore, $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a contradiction.

21. (3)

p	$\sim p$	$p \to \sim p$	$\sim p \to p$	$(p \to \sim p) \wedge (\sim p \to p)$
T	F	F	T	F
F	T	T	F	F

Clearly, $(p \to \sim p) \wedge (\sim p \to p)$ is a contradiction.

22. (2) $p \to q$ is logically equivalent to $\sim q \to \sim p$
Therefore, $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology but not a contradiction.

23. (4) We know that $p \to q \equiv \sim p \vee q$
$\therefore \quad \sim p \to q \equiv p \vee q$
$\therefore \quad \sim(p \to q) \equiv \sim(p \vee q)$
$\equiv \sim p \wedge \sim q$

24. (4) $p \to (\sim p \vee q)$ is false then p is true and $\sim p \vee q$ is false.
So, p is true and both $\sim p$ and q are false.
Therefore, p is true and q is false.

25. (1)

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Therefore, $(p \wedge q) \to p$ is a tautology.

26. (1)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Clearly, $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction.

27. (4)

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$	$p \to \sim(p \wedge \sim q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	F	T	T

Result is neither tautology nor contradiction.

28. (1)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$(p \wedge \sim q) \wedge (\sim p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Clearly, $(p \wedge \sim q) \wedge (\sim p \vee q)$ is a contradiction.

29. (1)

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

30. (4) Since p is true and $p \to \sim q$ is also true, therefore $\sim q$ is true.
Also, since $\sim r \to q$ is true, $\sim r$ is false and so, r is true.
From above $r \to q$ is false.

31. (4) $\sim q \to \sim p$ is contrapositive of $p \to q$ and both convey the same meaning.

32. (4) Since p is true and q is false, $p \vee q$ is true.
$p \wedge (\sim q)$ is true as $\sim q$ is true.
Also, $p \to p$ is true.
Clearly, $p \to q$ is false.

33. (3) $(p \wedge \sim r) \to (\sim p \vee q)$ is false.
Thus, $(p \wedge \sim r)$ is true and $(\sim p \vee q)$ is false.

So, (p is true and $\sim r$ is true) and ($\sim p$ is false and q is false)
Therefore, p is true, r is false and q is false.

14. (2) $(\sim q \to \sim p)$ is contrapositive of $(p \to q)$.
Therefore, $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is tautology.

15. (2) Inverse is $\sim p \to q$
Contrapositive is $\sim p \to q$ is $\sim q \to p$

16. (3) The symbolic meaning of the statement is $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

17. (3) $\sim(p \vee q) \vee (\sim p \wedge q)$
$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$\equiv \sim p \wedge (\sim q \vee q)$
$\equiv \sim p \wedge t$
$\equiv \sim p$

18. (3) Proposition $p \to (q \vee r)$ is false. So, p is true and $(q \vee r)$ is false.
Therefore, p is true and q and r are both false.

Archives

JEE MAIN

Single Correct Answer Type

1. (3)

p	q	$p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$
T	T	T	F	F	T
T	F	F	T	T	F
F	T	F	F	T	F
F	F	T	T	F	T

2. (4) P : There is a rational number $x \in S$ such that $x > 0$.
$\sim P$: Every rational number $x \in S$ satisfies $x \le 0$.

3. (3) Suman is brilliant and dishonest is $P \wedge \sim R$
Suman is brilliant and dishonest iff Suman is rich is
$Q \leftrightarrow (P \wedge \sim R)$
Negative of statement is expressed as
$\sim(Q \leftrightarrow (P \wedge \sim R))$

4. (1) Let p : I become a teacher
q : I will open a school
Negation of $p \to q$ is $\sim(p \to q) = p \wedge \sim q$
i.e. I will become a teacher and I will not open a school.

5. (2)

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

It is fallacy.

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	T	T	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

It is tautology.

6. (1)

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

7. (4) $\sim s \vee (\sim r \wedge s) = (\sim s \vee \sim r) \wedge (\sim s \vee s)$
$= \sim(s \wedge r) \wedge t$
$= \sim(s \wedge r)$
So, negation is $s \wedge r$.

8. (2) $[(p \wedge \sim q) \vee q] \vee (\sim p \wedge q)$
$= (p \vee q) \wedge (\sim q \vee q) \vee (\sim p \wedge q)$
$= (p \vee q) \wedge [t \vee (\sim p \wedge q)]$
$= (p \vee q) \wedge t$
$= p \vee q$

9. (2) $(p \to q) \to \{(\sim p \to q) \to q\}$
$= (p \to q) \to \{(p \vee q) \to q\}$
$= (p \to q) \to \{(\sim p \to \sim q) \vee q\}$
$= (p \to q) \to \{(\sim p \vee q) \wedge (\sim q \vee q)\}$
$= (p \to q) \to (\sim p \vee q)$
$= (p \to q) \to (p \to q)$
$= T$

10. (2) $\sim(p \vee q) \vee (\sim p \wedge q)$
$= (\sim p \wedge \sim q) \vee (\sim p \wedge q)$
$= \sim p \wedge (\sim q \wedge q)$
$= \sim p$

Chapter 11

Concept Application Exercises

Exercise 11.1

1. (2) Total students = 100

Total marks of 70 students is $75 \times 70 = 5250$.

Total marks of girls is $7200 - 5250 = 1950$

So average of girls is

$$\frac{1950}{30} = 65$$

2. (4) Since $n = 9$, the median term is the

$$\left(\frac{9+1}{2}\right)^{\text{th}} = 5^{\text{th}} \text{ term.}$$

Last four observations are increased by 2.

Since the median is the 5^{th} observation, which remains unchanged, there will be no change in the median.

3. (4) Mode + 2 Mean = 3 Median

$\Rightarrow$ Mode $= 3 \times 22 - 2 \times 21 = 66 - 42 = 24$

4. (3) $52x + 42y = 50(x + y)$

$\Rightarrow \quad 2x = 8y$

$\Rightarrow \quad \frac{x}{y} = \frac{4}{1}$ and $\frac{x}{x+y} = \frac{4}{5}$

The percentage of boys is 80.

5. (1)

Marks obtained	No. of students	Cumulative frequency
0-10	2	2
10-20	18	20
20-30	30	50
30-40	45	95
40-50	35	130
50-60	20	150
60-70	6	156
70-80	3	159

$N = \Sigma f = 159$ (Odd number)

Median is $\frac{1}{2}(n+1) = \frac{1}{2}(159+1) = 80$th value, which lies in the class 30–40 (see the row of cumulative frequency 95, which contains 80).

Hence median class is 30–40.

$\therefore$ We have

l = Lower limit of median class = 30

f = Frequency of median class = 45

C = Total of all frequencies preceding median class = 50

i = Width of class interval of median class = 10

$$\therefore \quad \text{Required median} = l + \frac{\frac{N}{2} - C}{f} \times i = 30 + \frac{\frac{159}{2} - 50}{45} \times 10$$

$$= 30 + \frac{295}{45} = 36.55$$

Exercise 11.2

1. (1) $\Sigma x = 170$, $\Sigma x^2 = 2830$

The increase in Σx is 10, then

$\Sigma x' = 170 + 10 = 180$

The increase in Σx^2 is $900 - 400 = 500$, then

$\Sigma x^2 = 2830 + 500 = 3330$

$$\text{Variance} = \frac{1}{n}\Sigma x^2 - \left(\frac{1}{n}\Sigma x'\right)^2$$

$$= \frac{1}{15} \times 3330 - \left(\frac{1}{15} \times 180\right)^2$$

$$= 222 - 144 = 78$$

2. (3) In the $2n$ observations, half of them equal to a and the remaining half equal to $-a$. Then the mean of total $2n$ observations is equal to zero. Therefore,

$$\text{S.D.} = \sqrt{\frac{\Sigma(x-x)^2}{N}}$$

$$2 = \sqrt{\frac{\Sigma x^2}{2n}}$$

$$\Rightarrow \quad 4 = \frac{\Sigma x^2}{2n} \quad \Rightarrow \quad 4 = \frac{2a^2}{2n} \quad \Rightarrow \quad a^2 = 4$$

$$\Rightarrow \quad |a| = 2$$

3. (1) $\sigma_x^2 = \frac{\Sigma d_i^2}{n}$ (Here deviations are taken from the mean.)

Since both A and B have 100 consecutive integers, they have the same standard deviation and hence variance. Therefore,

$$\frac{V_A}{V_B} = 1 \quad (\text{As } \Sigma d_i^2 \text{ is same in both the cases.})$$

4. (4) The mean of a, b, 8, 5, and 10 is 6.

$$\frac{a+b+8+5+10}{5} = 6$$

$$\Rightarrow \quad a + b = 7 \qquad (1)$$

Given that the variance is 6.8.

$$\text{Variance} = \frac{\Sigma(X_i - A)^2}{n}$$

$$= \frac{(a-6)^2 + (b-6)^2 + 4 + 1 + 16}{5} = 6.8$$

$$\Rightarrow \quad a^2 + b^2 = 25$$

$$\Rightarrow \quad a^2 + (7-a)^2 = 25 \qquad [\text{Using (1)}]$$

$$\Rightarrow \quad a^2 - 7a + 12 = 0$$

$$\therefore \quad a = 4, 3 \text{ and } b = 3, 4$$

5. (3) Here, $m = \frac{\Sigma x_i}{5}$, $s = \sqrt{\frac{\Sigma x_i^2}{5} - \left(\frac{\Sigma x_i}{5}\right)^2}$

So, for observations $kx_1, kx_2, kx_3, kx_4, kx_5$,

$$\text{S.D.} = \sqrt{\frac{k^2\Sigma x_i^2}{5} - \left(\frac{k\Sigma x_i}{5}\right)^2}$$

$$= \sqrt{\frac{k^2\Sigma x_i^2}{5} - k^2\left(\frac{\Sigma x_i}{5}\right)^2}$$

$$= k\sqrt{\left(\frac{\Sigma x_i^2}{5}\right) - \left(\frac{\Sigma x_i}{5}\right)^2} = ks$$

(4) Given, $w_i = lx_i + k$, $\bar{x}_i = 48$, S.D. of x_i's $= 12$, $w_i = 55$ and S.D. of w_i's $= 15$

Then, $\bar{w}_i = l\bar{x}_i + k$ [where, $\bar{w}_i$ is mean of w_i's and $\bar{x}_i$ is mean of x_i's]

$\Rightarrow \quad 55 = 48l + k \quad \ldots$ (i)

Now, S.D. of w_i's $= l \times$ (S.D. of x_i's)

$\Rightarrow \quad 15 = l \times 12$

$\Rightarrow \quad l = \dfrac{15}{12} = 1.25$

From Eq. (i), we get

$k = 55 - 1.25 \times 48 = -5$

Exercises

Single Correct Answer Type

1. (1) Here, $(CV)_1 = 50$, $(CV)_2 = 60$, $\bar{x}_1 = 30$ and $\bar{x}_2 = 25$

$\therefore \quad (CV)_1 = \dfrac{\sigma_1}{\bar{x}_1} \times 100$

$\Rightarrow \quad 50 = \dfrac{\sigma_1}{30} \times 100$

$\therefore \quad \sigma_1 = \dfrac{30 \times 50}{100} = 15$

and $\quad CV_2 = \dfrac{\sigma_2}{\bar{x}_2} \times 100$

$\Rightarrow \quad 60 = \dfrac{\sigma_2}{25} \times 100$

$\therefore \quad \sigma_2 = \dfrac{60 \times 25}{100} = 15$

Now, $\sigma_1 - \sigma_2 = 15 - 15 = 0$

2. (4) Let $x_1, x_2, \ldots, x_n$ be n numbers. Then $\bar{X} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$.

If each number is divided by 3, then the new mean $\bar{Y}$ is given by

$$\bar{Y} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i}{3}\right) = \frac{1}{3}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{\bar{X}}{3}$$

3. (2) The required AM is

$$\bar{X} = \frac{1 + 2 + 2^2 + 2^3 + \cdots + 2^n}{n+1} = \frac{\left(2^{n+1} - 1\right)}{(n+1)(2-1)} = \frac{2^{n+1} - 1}{n+1}$$

4. (3) Let $x_1, x_2, \ldots, x_n$ be n items. Then $\bar{X} = \dfrac{1}{n}\Sigma x_i$. Let

$y_1 = x_1 + 1, y_2 = x_2 + 2, y_3 = x_3 + 3, \ldots, y_n = x_n + n.$

Then the mean of the new series is

$$\frac{1}{n}\sum (x_i + i) = \frac{1}{n}\sum x_i + \frac{1}{n}(1 + 2 + 3 + \cdots + n)$$

$$= \bar{X} + \frac{1}{n}\frac{n(n+1)}{2}$$

$$= \bar{X} + \frac{n+1}{2}$$

5. (1) For a moderately skewed distribution, we have

Mode $= 3$ median $- 2$ mean

$\Rightarrow$ Mode $= 3\,(6) - 2\,(5) = 8.$

6. (4) It is obvious.

7. (2) Arranging the data in ascending order of magnitude, we obtain

Height (in cm)	150	152	154	155	156	160	161
Number of students	8	4	3	7	3	12	4
Cumulative frequency	8	12	15	22	25	37	41

Here, the total number of items is 41, i.e., an odd number.

Hence, the median is $\dfrac{41+1}{2}$ th i.e., 21st item.

From cumulative frequency table, we find that the median, i.e. 21st item is 155.

(All items from 16th to 22nd are equal, each 155).

8. (3) Average speed $= \dfrac{120 + 120 + 120}{\frac{120}{30} + \frac{120}{25} + \frac{120}{50}}$

$$= \frac{3}{\frac{1}{30} + \frac{1}{25} + \frac{1}{50}} \text{ km/hour}$$

9. (2) Given, observations are 3, 10, 10, 4. 7, 10 and 5.

$\therefore \quad \bar{x} = \dfrac{3 + 10 + 10 + 4 + 7 + 10 + 5}{7} = \dfrac{49}{7} = 7$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
3	4
10	3
10	3
4	3
7	0
10	3
5	2
Total	$\Sigma d_i = 18$

Now, M.D. $= \dfrac{\Sigma d_i}{N} = \dfrac{18}{7} = 2.57$

10. (1) The lives of 5 bulbs are: 1357, 1090, 1666, 1494, 1623

Mean $(\bar{x}) = \dfrac{1357 + 1090 + 1666 + 1494 + 1623}{5}$

$= \dfrac{7230}{5} = 1446$

x_i	$d_i = \lvert x_i - \bar{x} \rvert$
1357	89
1090	356
1666	220
1494	48
1623	177
Total	$\Sigma d_i = 890$

M.D. $= \dfrac{\Sigma d_i}{N} = \dfrac{890}{5} = 178$

11. (3) Marks obtained by 9 students in Mathematics are: 50, 69, 20, 33, 53, 39, 40, 65, 59

Let us rewrite the given data in ascending order.

20, 33, 39, 40, 50, 53, 59, 65, 69,

Here, $n = 9$ [odd]

Median (M_e) = 5th term = 50

x_i	$d_i = \lvert x_i - M_e \rvert$
20	30
33	17
39	11
40	10
50	0
53	3
59	9
65	15
69	19
N = 2	**$\Sigma d_i = 114$**

$$\text{M.D.} = \frac{114}{9} = 12.67$$

12. (3) $\text{Mean} = \dfrac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$

$$\therefore \quad 2.6 = \frac{1\times 4 + 2\times 5 + 3\times y + 4\times 1 + 5\times 2}{4 + 5 + y + 1 + 2}$$

$$\Rightarrow \quad 31.2 + 2.6y = 28 + 3y$$

$$\Rightarrow \quad 0.4y = 3.2$$

$$\Rightarrow \quad y = 8$$

13. (2) We know that $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$

$$\Rightarrow \quad \sum_{i=1}^{n} x_i = n\bar{x}$$

$$\therefore \quad \frac{\sum_{i=1}^{n}(x_i + 2i)}{n} = \frac{\sum_{i=1}^{n} x_i + 2\sum_{i=1}^{n} i}{n} = \frac{n\bar{x} + 2(1 + 2 + \ldots n)}{n}$$

$$= \frac{n\bar{x} + 2\dfrac{n(n+1)}{2}}{n} = \bar{x} + (n+1)$$

14. (3) H.M. of 4, 8, 16 $= \dfrac{3}{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}} = \dfrac{48}{7} = 6.85$

15. (2) $M = \dfrac{x_1 + x_2 + x_3 \ldots\ldots x_n}{n}$

$$\Rightarrow \quad nM = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x_n$$

$$\Rightarrow \quad nM - x_n = x_1 + x_2 + x_3 + \ldots + x_{n-1}$$

$$\Rightarrow \quad nM - x_n + x' = x_1 + x_2 + x_3 + \ldots + x_{n-1} + x'$$

$$\Rightarrow \quad \frac{nM - x_n + x'}{n} = \frac{x_1 + x_2 + x_3 + \ldots + x_{n-1} + x'}{n}$$

$$\therefore \quad \text{New average} = \frac{nM - x_n + x'}{n}$$

16. (2) Arranging the data in ascending order of magnitude, we obtain

Height (in cm)	150	152	154	155	156	160	161
Number of students	8	4	3	7	3	12	4
Cumulative frequency	8	12	15	22	25	37	41

Here, total number of items is 41, which is an odd number.

Hence, the median is $\dfrac{41+1}{2}$ th = 21st item.

From cumulative frequency table, we find that median i.e., 21st item is 155.

17. (4) We know that

Mode = (3 × Median) − (2 × Mean) = 3(6) − 2(5) = 8

18. (1) Arranging the given data in ascending order, we have

34, 38, 42, 44, 46, 48, 54, 55, 63, 70,

Median, $M = \dfrac{46 + 48}{2} = 47$ ($\because$ $n = 10$, median is the mean of 5th and 6th items)

$$\therefore \quad \text{Mean deviation} = \frac{\Sigma \lvert x_i - M \rvert}{n}$$

$$= \frac{\Sigma \lvert x_i - 47 \rvert}{10}$$

$$= \frac{13 + 9 + 5 + 3 + 1 + 1 + 7 + 8 + 16 + 23}{10}$$

$$= 8.6$$

19. (3) We have,

$$\Sigma f_i(y_i - \mu) = \Sigma f_i y_i - \mu \Sigma f_i = \mu \Sigma f_i - \mu \Sigma f_i = 0$$

$$\left[\because \mu = \frac{\Sigma f_i y_i}{\Sigma f_i}\right]$$

20. (2) Range = $X_{\text{highest}} - X_{\text{lowest}} = 9 - 2 = 7$

21. (2) Let $x_1, x_2, \ldots, x_n$ be the raw data. Then,

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{X}\right)^2$$

If each value is multiplied by h, then the values become hx_1, $hx_2, \ldots, hx_n$. Then AM of the values is

$$\frac{hx_1 + hx_2 + \cdots + hx_n}{n} = h\bar{X}$$

Therefore, the variance of the new set of values is

$$\frac{1}{n}\sum_{i=1}^{n}\left(hx_1 - h\bar{X}_2\right) = h^2\left(\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{X}\right)^2\right) = h^2\sigma^2$$

22. (3) We have

$$\bar{X} = \frac{0^nC_0 + 1^nC_1 + 2^nC_2 + \cdots + n^nC_n}{{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n} = \frac{\sum_{r=0}^{n} r^nC_r}{\sum_{r=0}^{n} {}^nC_r}$$

$$= \frac{1}{2^n}\sum_{r=1}^{n} r\frac{n}{r}\,{}^{n-1}C_{r-1}$$

$$\left[\because \sum_{r=0}^{n} {}^nC_r = 2^n;\ {}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}\right]$$

$$= \frac{n}{2^n}\sum_{r=1}^{n} {}^{n-1}C_{r-1} = \frac{n}{2^n}\,2^{n-1} = \frac{n}{2} \left[\because \sum_{r=1}^{n} {}^{n-1}C_{r-1} = 2^{n-1}\right]$$

and $\frac{1}{N}\sum f_i x_i^2 = \frac{1}{2^n}\sum r^2\,{}^nC_r = \frac{1}{2^n}\sum_{r=0}^{n}\left[r(r-1)+r\right]{}^nC_r$

$$= \frac{1}{2^n}\left\{\sum_{r=0}^{n} r-(r-1)\,{}^nC_r + \sum_{r=0}^{n} r\,{}^nC_r\right\}$$

$$= \frac{1}{2^n}\left\{\sum_{r=2}^{n} r(r-1)\frac{n}{r}\frac{n-1}{r-1}\,{}^{n-2}C_{r-2} + \sum_{r=1}^{n} r\frac{n}{r}\,{}^{n-1}C_{r-1}\right\}$$

$$= \frac{1}{2^n}\{n(n-1)\,2^{n-2} + n2^{n-1}\} = \frac{n(n-1)}{4} + \frac{n}{2}$$

$\therefore$ Var $(X) = \frac{1}{N}\sum f_i x_i^2 - \bar{X}^2 = \frac{n(n-1)}{4} + \frac{n}{2} - \frac{n^2}{4} = \frac{n}{4}$

3. (3) Here, $\bar{x} = \frac{2+4+6+8+10}{5} = 6$

Hence, variance $= \frac{1}{n}\sum (x_i - \bar{x})^2$

$$= \frac{1}{5}\{(2-6)^2 + (4-6)^2 + (6-6) + (8-6)^2 + (10-6)^2\}$$

$$= \frac{1}{5}\{(16+4+0+4+16\} = \frac{1}{5}\{40\} = 8$$

4. (1) It is obvious.

5. (4) Coefficient of variation $= \frac{\text{SD}}{\text{Mean}} \times 100 = \frac{19.76}{35.16} \times 100$

6. (1) Mean $\bar{x} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$

$$\text{SD} = \sigma = \sqrt{\frac{1}{n}\sum{}_i^{\,2} - (\bar{x})^2}$$

$$= \sqrt{\frac{1}{6}(1+4+9+16+25+36) - \frac{49}{4}}$$

$$= \sqrt{\frac{91}{6} - \frac{49}{4}} = \sqrt{\frac{182-147}{12}} = \sqrt{\frac{35}{12}}$$

7. (1) If each item of the data is increased or decreased by the same constant, the standard deviation of the data remains unchanged.

8. (2) Mean deviation is minimum when it is considered about an item equidistant from the beginning and the end, i.e., the median. In this case, the median is $\frac{101+1}{2}$ th, i.e., 51st item, i.e., x_{51}.

9. (2) On arranging the given observations in ascending order, we get

All negative terms $\underbrace{0}_{(n+1)^{\text{th}}\text{ term}}$ All positive terms

Median of given observations $= (n+1)$th term $= 0$

$\therefore$ SD > MD

10. (1) We have $r = \max |x_i - x_j|$

and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Now, $(x_i - \bar{x})^2 = \left(x_i - \frac{x_1 + x_2 + \cdots + x_n}{n}\right)^2$

$$= \frac{1}{n^2}[(x_i - x_1) + (x_i - x_2) + \cdots + (x_i - x_i - 1)$$

$$+ (x_i - x_1 + 1) + \cdots + (x_i - x_n)] \le \frac{1}{n^2}[(n-1)r]^2,$$

$[\because |x_i - x_j| \le r]$

$\Rightarrow (x_i - \bar{x})^2 \le r^2$

$\Rightarrow \sum_{i=1}^{n}(x_i\,\bar{x})^2 \le nr^2$

$\Rightarrow \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 \le \frac{nr^2}{(n-1)}$

$\Rightarrow S^2 \le \frac{nr^2}{(n-1)}$

$\Rightarrow S \le r\sqrt{\frac{n}{n-1}}$

31. (2) Let $y = \frac{\alpha x + b}{c}$, i.e., $y = \frac{a}{c}x + \frac{b}{c}$

i.e, $y = \frac{a}{c}x + \frac{b}{c}$

$\therefore \bar{y} = A\bar{x} + B$

$\therefore y - \bar{y} = A(x - \bar{x})$

$\Rightarrow (y - \bar{y})^2 = A^2(x - \bar{x})^2$

$\Rightarrow \Sigma(y - \bar{y})^2 = A^2\,\Sigma(x - \bar{x})^2$

$\Rightarrow n \cdot \sigma_y^2 = A^2 \cdot n\sigma_x^2$

$\Rightarrow \sigma_y^2 = A^2\sigma_x^2$

$\Rightarrow \sigma_y = |A|\,\sigma_x$

$\Rightarrow \sigma_y = \left|\frac{a}{c}\right|\,\sigma_x$

Thus, new SD $= \left|\frac{a}{c}\right|\,\sigma$

32. (1) Given, numbers are: 6, 5, 9, 13, 12, 8, 10

x_i	x_i^2
6	36
5	25
9	81
13	169
12	144
8	64
10	100
$\Sigma x_i = 63$	$\Sigma x_i^2 = 619$

$\therefore$ S.D. $= \sigma = \sqrt{\frac{\Sigma x_i^2}{N} - \left(\frac{\Sigma x_i}{N}\right)^2} = \sqrt{\frac{619}{7} - \left(\frac{63}{7}\right)^2}$

$$= \sqrt{\frac{7 \times 619 - 3969}{49}} = \sqrt{\frac{4333 - 3969}{49}}$$

$$= \sqrt{\frac{364}{49}} = \sqrt{\frac{52}{7}}$$

33. (3) Given, $\bar{x} = 50$, $n = 100$ and $\sigma = 5$

$$\Rightarrow \quad 50 = \frac{\Sigma x_i}{100}$$

$$\Rightarrow \quad \Sigma x_i = 50 \times 100 = 5000$$

$$\text{Now, } s^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2$$

$$\Rightarrow \quad 25 = \frac{\Sigma x_i^2}{100} - (50)^2$$

$$\Rightarrow \quad 2525 = \frac{\Sigma x_i^2}{100}$$

$$\therefore \quad \Sigma x_i^2 = 252500$$

34. (4) We know that, SD of first n natural numbers $= \sqrt{\frac{n^2 - 1}{12}}$

$$\therefore \quad \text{SD of first 10 numbers} = \sqrt{\frac{10^2 - 1}{12}}$$

$$= \sqrt{\frac{100 - 1}{12}} = \sqrt{\frac{99}{12}} = \sqrt{8.25} = 2.87$$

35. (4) Given numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

If 1 is added to each number, then observations will be 2, 3, 4, 5, 6, 7, 8, 9, 10 and 11.

$$\therefore \quad \Sigma x_i = 2 + 3 + 4 + \ldots + 11$$

$$= \frac{10}{2}\,[2 \times 2 + 9 \times 1] = 5[4 + 9] = 65$$

and $\quad \Sigma x_i^2 = 2^2 + 3^2 + 4^2 + 5^2 + \ldots + 11^2$

$$= (1^2 + 2^2 + 3^2 + \ldots + 11^2) - (1^2)$$

$$= \frac{11 \times 12 \times 23}{6} - 1 = 505$$

$$\therefore \quad s^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 = \frac{505}{10} - \left(\frac{65}{10}\right)^2$$

$$= 50.5 - (6.5)^2$$

$$= 50.5 - 42.25$$

$$= 8.25$$

36. (1) The first 10 positive integers are 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.

On multiplying each number by -1, we get

$-1, -2, -3, -4, -5, -6, -7, -8, -9, -10$

On adding 1 to each number, we get

$0, -1, -2, -3, -4, -5, -6, -7, -8, -9$

$$\therefore \quad \Sigma x_i = -\frac{9 \times 10}{2} = -45$$

and $\quad \Sigma x_i^2 = 0^2 + (-1)^2 + (-2)^2 + \ldots + (-9)^2 = \frac{9 \times 10 \times 19}{6} = 285$

$$\therefore \quad \text{S.D.} = \sqrt{\frac{285}{10} - \left(\frac{-45}{10}\right)^2} = \sqrt{\frac{285}{10} - \frac{2025}{100}}$$

$$= \sqrt{\frac{2850 - 2025}{100}} = \sqrt{8.25}$$

Now, variance $= (\text{S.D.})^2 = (\sqrt{8.25})^2 = 8.25$

37. (4) Variance $= \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$

$$= \frac{18000}{60} - \left(\frac{960}{60}\right)^2 = 300 - 256 = 44$$

38. (1) Given, $\sigma_C = 5$

Relation between °C and °F is given by

$$F = \frac{9C}{5} + 32$$

$$\therefore \quad \sigma_F = \frac{9}{5}\sigma_C = \frac{9}{5} \times 5 = 9$$

Here, $\sigma_F^2 = (9)^2 = 81$

39. (3)

Class	Frequency	y_i	$u_i = \frac{y_i - A}{10}$, $A = 25$	$f_i u_i$	$f_i u_i^2$
0-10	1	5	−2	−2	4
10-20	3	15	−1	−3	3
20-30	4	25	0	0	0
30-40	2	35	1	2	2
Total	10			−3	9

$$\sigma^2 = c^2\left[\frac{\Sigma f_i u_i^2}{\Sigma f_i} - \left(\frac{\Sigma f_i u_i}{\Sigma f_i}\right)^2\right]$$

$$= 10^2\left[\frac{9}{10} - \left(\frac{-3}{10}\right)^2\right] = 90 - 9 = 81$$

$$\Rightarrow \quad \sigma = 9$$

Archives

JEE Main

Single Correct Answer Type

1. (3) Mean $(\bar{x}) = \frac{\text{sum of quantities}}{n}$

$$= \frac{n/2(a+1)}{n} = \frac{1}{2}\,[1 + 1 + 100d]$$

$$= 1 + 50d$$

$$\text{M.D.} = \frac{1}{n}\Sigma\,|x_i - \bar{x}|$$

$$\Rightarrow \quad 25 = \frac{1}{101}\,[50d + 49d + 48d + \cdots + d + 0 + d + \cdots + 50d]$$

$$= \frac{2d}{101}\left[\frac{50 \times 51}{2}\right]$$

$$\Rightarrow \quad d = \frac{255 \times 101}{50 \times 51} = 10.1$$

2. (1) $1 - q^n \geq \frac{9}{10}$

$$\Rightarrow \quad \left(\frac{3}{4}\right)^n \leq \frac{1}{10}$$

$$\Rightarrow \quad n \geq -\log_{3/4} 10$$

$$\Rightarrow \quad n \geq \frac{1}{\log_{10} 4 - \log_{10} 3}$$

3. (3) $x_1^2 = 4$

$\sigma_y^2 = 5$

$\bar{x} = 2$

$\bar{y} = 4$

$$\frac{\Sigma x_i}{5} = 2 \quad \Rightarrow \Sigma x_i = 10, \Sigma y_i = 20$$

$$\sigma_x^2 = \left(\frac{1}{2}\Sigma x_i^2\right) - (\bar{x})^2 = \frac{1}{5}(\Sigma y_i^2) - 16$$

$\Sigma x_i^2 = 40$

$\Sigma y_i^2 = 105$

$$\sigma_z^2 = \frac{1}{10}(\Sigma x_i^2 + \Sigma y_i^2) - \left(\frac{\bar{x} + \bar{y}}{2}\right)^2$$

$$= \frac{1}{10}(40 + 105) - 9 = \frac{145 - 90}{10} = \frac{55}{10} = \frac{11}{2}$$

4. (4) From the given data, median $= \frac{25+26}{2}a = 25.5a$

Required mean deviation about median

$$= \frac{2|0.5+1.5+2.5+\cdots+24.5|}{50}|a| = 50$$

$$\Rightarrow \quad |a| = 4$$

5. (4) A.M. of $2x_1, 2x_2 \ldots 2x_n$ is $\frac{2x_1+2x_2+\cdots+2x_n}{n}$

$$= 2\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) = 2\bar{x}$$

So statement 2 is false.

variance $(2x_i) = 2^2$ variance $(x_i) = 4\sigma^2$

So statement 1 is true.

6. (4)

7. (2) $\sigma^2 = \left(\frac{\sum x_i^2}{n}\right) - \bar{x}^2$

$$\bar{x} = \frac{\sum_{r=1}^{50} 2r}{50} = 51$$

$$\sigma^2 = \frac{\sum_{r=1}^{50} 4r^2}{50} - (51)^2 = 833$$

8. (4) $\frac{x_1+x_2+\ldots+x_{16}}{16} = 16$

If $x_1 = 16$

$$\frac{x_1+x_2+\ldots+x_{10}-16+3+4+5}{18}$$

$$= \frac{16\times 10-16+12}{18} = 14$$

9. (1) Standard deviation of numbers 2, 3, a and 11 is 3.5

$$\therefore \quad (3.5)^2 = \frac{\sum x_i^2}{4} - (\bar{x})^2$$

$$\Rightarrow \quad (3.4)^2 = \frac{4+9+a^2+121}{4} - \left(\frac{2+3+a+11}{4}\right)^2$$

On solving, we get

$$3a^2 - 32a + 84 = 0$$

10. (4) Let $x_i - 5 = y_i$

$$\therefore \quad \sum_{i=1}^{9} y_i = 9 \text{ and } \sum_{i=1}^{9} y_i^2 = 45$$

So, required standard deviation is

$$\sigma = \sqrt{\frac{\sum_{i=1}^{9} y_i^2}{9} - \left(\frac{\sum_{i=1}^{9} y_i}{9}\right)^2} = \sqrt{\frac{45}{9} - \left(\frac{9}{9}\right)^2} = 2$$

Chapter 12

Concept Application Exercises

Exercise 12.1

1. $\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Expanding along C_3, we have

$$\Delta = -\sin\alpha(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha) + \cos\alpha(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta)$$
$$= \sin^2\alpha(\sin^2\beta + \cos^2\beta) + \cos^2\alpha(\cos^2\beta + \sin^2\beta)$$
$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$
$$= 1$$

2. Since the determinant is symmetrical, it is easier to expand by Sarrus rule. Hence,

$$\Rightarrow \begin{vmatrix} \tan A & 1 & 1 \\ 1 & \tan B & 1 \\ 1 & 1 & \tan C \end{vmatrix}$$
$$= \tan A\tan B\tan C + 1 + 1 - \tan A - \tan B - \tan C$$
$$= 2 \text{ [as in a triangle } \tan A + \tan B + \tan C = \tan A\tan B\tan C)$$

3. Expanding by Sarrus rule,

$$\begin{vmatrix} 1 & e^{i\pi/3} & e^{i\pi/4} \\ e^{-i\pi/3} & 1 & e^{i2\pi/3} \\ e^{-i\pi/4} & e^{-i2\pi/3} & 1 \end{vmatrix} = 1 + e^{i\pi/3}\times e^{i2\pi/3}\times e^{-i\pi/4} + e^{-i\pi/3}$$
$$\times e^{-i2\pi/3}\times e^{i\pi/4} - (e^{-i\pi/4}\times e^{i\pi/4} + e^{-i\pi/3}\times e^{i\pi/3} + e^{-i2\pi/3}\times e^{i2\pi/3})$$
$$= 1 + e^{i3\pi/4} + e^{-i3\pi/4} - (1+1+1)$$
$$= -2 + 2\cos(3\pi/4)$$
$$= -2 - \sqrt{2}$$

4. Expanding Using Sarrus' Rule,

$$\Delta = (x-a)(x+b)(x-c) + (x-b)(x+a)(x+c)$$
$$= 2x(x^2 + ac - ab - bc)$$

Now $\Delta = 0$ gives $x = 0$

or $x^2 = b(a+c) - ac$

If $b(a+c) > ac$, we have three roots $0, \pm\sqrt{\{b(a+c)-ac\}}$.

5. $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = (\alpha+\beta+\gamma)\frac{1}{2}[(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2] = 0$

Since $\alpha \neq \beta \neq \gamma$, we have

$$\alpha + \beta + \gamma = 0$$
$\therefore$ $\alpha + \beta - \gamma = -2\gamma$

$$\beta + \gamma - \alpha = -2\alpha$$
$$\gamma + \alpha - \beta = -2\beta$$

Let $y = -2x$

or $x = \frac{-y}{2}$

Therefore, the required equation is

$$a\left(-\frac{y}{2}\right)^3 + b\left(-\frac{y}{2}\right)^2 + c\left(-\frac{y}{2}\right) + d = 0$$

$$\Rightarrow \frac{-a}{8}y^3 + \frac{b}{4}y^2 - \frac{c}{2}y + d = 0$$
$$\Rightarrow ay^3 - 2by^2 + 4cy - 8d = 0$$
or $ax^3 - 2bx^2 + 4cx - 8d = 0$

6. Altitudes of triangle are concurrent at $H(0, 0)$,

Then $A_1H \perp A_2A_3$

$$\Rightarrow \frac{y_1 - 0}{x_1 - 0}\cdot\frac{y_2 - y_3}{x_2 - x_3} = -1$$

or $y_1(y_2 - y_3) + x_1(x_2 - x_3) = 0$

Similarly, $A_2H \perp A_3A_1$ and $A_3H \perp A_1A_2$

$$\Rightarrow y_2(y_3 - y_1) + x_2(x_3 - x_1) = 0$$
and $y_3(y_1 - y_2) + x_3(x_1 - x_2) = 0$

$$\therefore \Delta = \begin{vmatrix} x_2 - x_3 & y_2 - y_3 & y_1(y_2 - y_3) + x_1(x_2 - x_3) \\ x_3 - x_1 & y_3 - y_1 & y_2(y_3 - y_1) + x_2(x_3 - x_1) \\ x_1 - x_2 & y_1 - y_2 & y_3(y_1 - y_2) + x_3(x_1 - x_2) \end{vmatrix}$$
$$= \begin{vmatrix} x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \\ x_1 - x_2 & y_1 - y_2 & 0 \end{vmatrix} = 0$$

7. As $1 + \omega + \omega^2 = 0$

$$D = \begin{vmatrix} \frac{x}{1+\omega} & \frac{y}{\omega+\omega^2} & \frac{z}{\omega^2+1} \\ \frac{y}{\omega+\omega^2} & \frac{z}{\omega^2+1} & \frac{x}{1+\omega} \\ \frac{z}{\omega^2+1} & \frac{x}{1+\omega} & \frac{y}{\omega+\omega^2} \end{vmatrix}$$
$$= \begin{vmatrix} -\frac{x}{\omega^2} & -y & -\frac{z}{\omega} \\ -y & -\frac{z}{\omega} & -\frac{x}{\omega^2} \\ -\frac{z}{\omega} & -\frac{x}{\omega^2} & -y \end{vmatrix}$$
$$= x^3 + y^3 + z^3 - 3xyz$$
$$= \frac{1}{2}(x+y+z)\{(x-y)^2 + (y-z)^2 + (z-x)^2\}$$
$$\Rightarrow x = y = z \qquad (\because x + y + z \neq 0)$$

Exercise 12.2

1. $\Delta = \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

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Applying $R_1 \to \omega R_1$, we get

$$\Delta = \frac{1}{\omega}\begin{vmatrix} \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

$$= \frac{1}{\omega}\begin{vmatrix} \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \qquad (\text{As } \omega^3 = 1)$$

$$= 0 \qquad (\text{As } R_1 \text{ and } R_2 \text{ are identical})$$

2. We have, $\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix}$

Multiplying R_1, R_2 and R_3 by a, b and c, respectively, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad (\text{Taking } abc \text{ common from } C_3)$$

$$= -\begin{vmatrix} 1 & a^3 & a^2 \\ 1 & b^3 & b^2 \\ 1 & c^3 & c^2 \end{vmatrix} \qquad (C_1 \leftrightarrow C_3)$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad (C_2 \leftrightarrow C_3)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} \qquad (\text{Taking transpose})$$

3. $\Delta = \begin{vmatrix} abc & b^2c & c^2b \\ abc & c^2a & ca^2 \\ abc & a^2b & b^2a \end{vmatrix} = 0$

Taking bc, ca and ab common from R_1, R_2 and R_3, respectively, we get

$$\therefore \quad (bc \cdot ca \cdot ab)\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \qquad (\text{As } a, b, c \neq 0)$$

$$\Rightarrow \quad -(a^3 + b^3 + c^3 - 3abc) = 0$$

$$\Rightarrow \quad \frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$$

$$\Rightarrow \quad a + b + c = 0 \qquad (\text{As } a, b, c \text{ are distinct})$$

4. $a_r = (\cos 2r\pi + i \sin 2r\pi)^{\frac{1}{9}} = e^{i\frac{2r\pi}{9}}$ (Using Euler's formula)

$$\therefore \quad \Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix} = \begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{8\pi}{9}} & e^{i\frac{10\pi}{9}} & e^{i\frac{12\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$

Now, taking $e^{i\frac{6\pi}{9}}$ common from R_2, we get

$$\Delta = e^{i\frac{6\pi}{9}}\begin{vmatrix} e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{2\pi}{9}} & e^{i\frac{4\pi}{9}} & e^{i\frac{6\pi}{9}} \\ e^{i\frac{14\pi}{9}} & e^{i\frac{16\pi}{9}} & e^{i\frac{18\pi}{9}} \end{vmatrix}$$

$$= 0 \qquad (\text{As } R_1 \text{ and } R_2 \text{ are identical})$$

5. $B = 2 \times 2\begin{vmatrix} f & d & e \\ n & l & m \\ c & a & b \end{vmatrix}$

$$= 2\begin{vmatrix} 2f & d & e \\ 2n & l & m \\ 2c & a & b \end{vmatrix}$$

$$= 2\begin{vmatrix} 2c & a & b \\ 2f & d & e \\ 2n & l & m \end{vmatrix} \qquad [R_3 \leftrightarrow R_2 \text{ and then } R_2 \leftrightarrow R_1]$$

$$= 2\begin{vmatrix} a & b & 2c \\ d & e & 2f \\ l & m & 2n \end{vmatrix} \qquad [C_1 \leftrightarrow C_2 \text{ and then } C_2 \leftrightarrow C_3]$$

$$= 2A$$

$$\therefore \quad B/A = 2$$

Exercise 12.3

1. (a) Operating $C_2 - lC_1 - mC_3$, we get the value of the determinant is 0.

(b) Operating $C_1 \to C_1 + C_2 + C_3$, we get the value of the determinant is 0.

(c) $\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2 + \log x & \log 2 + \log y & \log 2 + \log z \\ \log 3 + \log x & \log 3 + \log y & \log 3 + \log z \end{vmatrix}$

Operating $R_3 \to R_3 - R_1$ and $R_2 \to R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} \log x & \log y & \log z \\ \log 2 & \log 2 & \log 2 \\ \log 3 & \log 3 & \log 3 \end{vmatrix}$$

$$= 0 \qquad (\because R_2 \text{ and } R_3 \text{ are same})$$

(d) Using $C_1 \to C_1 - C_2 - 4C_3$, we have

$$\begin{vmatrix} 0 & (a^x - a^{-x})^2 & 1 \\ 0 & (b^y - b^{-y})^2 & 1 \\ 0 & (c^z - c^{-z})^2 & 1 \end{vmatrix} = 0$$

(e) Applying $C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} \sin^2\left(x+\frac{3\pi}{2}\right) & \sin^2\left(x+\frac{5\pi}{2}\right) & \sin(2x+5\pi)\sin(2\pi) \\ \sin\left(x+\frac{3\pi}{2}\right) & \sin\left(x+\frac{5\pi}{2}\right) & 2\cos\left(x+\frac{5\pi}{2}\right)\sin(\pi) \\ \sin\left(x-\frac{3\pi}{2}\right) & \sin\left(x-\frac{5\pi}{2}\right) & 2\cos\left(x-\frac{5\pi}{2}\right)\sin(-\pi) \end{vmatrix} = 0$$

[$\because$ all elements of C_3 are zero]

2. Let $D = \begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$

Let us first reduce the value of elements by performing some operation.

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 5R_1$, we get

$$D = \begin{vmatrix} 18 & 40 & 89 \\ 4 & 9 & 20 \\ -1 & -2 & -5 \end{vmatrix}$$

Applying $C_3 \to C_3 - 2C_2$, we get

$$D = \begin{vmatrix} 18 & 40 & 9 \\ 4 & 9 & 2 \\ -1 & -2 & -1 \end{vmatrix}$$

Now apply $C_1 \to C_1 - 2C_3$ to get zeros in C_1

$$D = \begin{vmatrix} 0 & 40 & 9 \\ 0 & 9 & 2 \\ 1 & -2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 40 & 9 \\ 9 & 2 \end{vmatrix} = 80 - 81 = -1$$

3. $\Delta = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$

$= \begin{vmatrix} b+c & -b & a \\ a+c & -c & b \\ a+b & -a & c \end{vmatrix}$ [Applying $C_2 \to C_2 - C_3$]

$= \begin{vmatrix} c & -b & a \\ a & -c & b \\ b & -a & c \end{vmatrix}$ [Applying $C_1 \to C_1 + C_2$]

$= \begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$ [Interchanging C_2 and C_3]

$= 3abc - a^3 - b^3 - c^3$

4. Let $\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix}$

Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 1+3p \\ 3 & 6 & 1+6p \end{vmatrix}$$

$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 3 & 6 & 1 \end{vmatrix}$ [Applying $C_3 \to C_3 - pC_2$]

$= \begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 2 & 5 & 1 \end{vmatrix}$ [Applying $C_1 \to C_1 - C_3, C_2 \to C_2 - C_3$]

$= 5 - 4 = 1$ [Expanding along R_1]

5. Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c-x & c & b \\ a+b+c-x & b-x & a \\ a+b+c-x & a & c-x \end{vmatrix}$$

$$= (a+b+c-x)\begin{vmatrix} 1 & c & b \\ 1 & b-x & a \\ 1 & a & c-x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2, R_2 \to R_2 - R_3$, we get

$$\Delta = (a+b+c-x)\begin{vmatrix} 0 & c-b+x & b-a \\ 0 & b-a-x & a-c+x \\ 1 & a & c-x \end{vmatrix}$$

$= (a+b+c-x)\{(x+c-b)(x+a-c)$
$\quad + (x+a-b)(b-a)\}$
$= (a+b+c-x)\{(x^2 + x(a-b) + (c-b)(a-c)$
$\quad + x(b-a) - (a-b)^2\}$
$= (a+b+c-x)(x^2 + ac - c^2 - ab + bc - a^2 - b^2 + 2ab)$
$= (a+b+c-x)\,(x^2 - a^2 - b^2 - c^2 + ab + bc + ca)$

$$= (a+b+c-x)\left(x^2 - \frac{1}{2}\{(a-b)^2 + (b-c)^2 + (c-a)^2\}\right)$$

Since $\Delta = 0$, we have

$$x = a+b+c, \pm\sqrt{\frac{1}{2}((a-b)^2 + (b-c)^2 + (c-a)^2)}$$

6. Let $\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1, R_3 \to R_3 - R_1$]

$$= (a+b+c)\begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix}$$ [Expanding along C_1]

$= (a+b+c)\,\{(2b+a)(2c+a) - (-b+a)(-c+a)\}$
$= (a+b+c)\,\{(4bc + 2ab + 2ca + a^2)$
$\quad - (bc - ab - ac + a^2)\}$
$= (a+b+c)\,(3bc + 3ab + 3ca)$
$= 3(a+b+c)\,(ab + bc + ca)$

7. $\Delta = -\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$

Applying $R_3 \to R_3 + R_2$, $R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} a+b & a+b & -(a+b) \\ -c & a+b+c & -a \\ -(b+c) & b+c & b+c \end{vmatrix}$$

$$= (a+b)(b+c)\begin{vmatrix} 1 & 1 & -1 \\ -c & a+b+c & -a \\ -1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$

$$\Delta = (a+b)(b+c)\begin{vmatrix} 0 & 2 & 0 \\ -c & a+b+c & -a \\ -1 & 1 & 1 \end{vmatrix}$$

$$= (a+b)(b+c)(-2)\begin{vmatrix} -c & -a \\ -1 & 1 \end{vmatrix}$$

$$= 2(a+b)(b+c)(c+a)$$

8. $\begin{vmatrix} x^2-1 & x^2+2x+1 & 2x^2+3x+1 \\ 2x^2+x-1 & 2x^2+5x-3 & 4x^2+4x-3 \\ 6x^2-x-2 & 6x^2-7x+2 & 12x^2-5x-2 \end{vmatrix} = 0$

or $\begin{vmatrix} (x-1)(x+1) & (x+1)^2 & (x+1)(2x+1) \\ (2x-1)(x+1) & (2x-1)(x+3) & (2x-1)(2x+3) \\ (3x-2)(2x+1) & (3x-2)(2x-1) & (3x-2)(4x+1) \end{vmatrix} = 0$

or $(x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & x+1 & 2x+1 \\ x+1 & x+3 & 2x+3 \\ 2x+1 & 2x-1 & 4x+1 \end{vmatrix} = 0$

$C_2 \to C_2 - C_1$ and $C_3 \to C_3 - 2C_1$

$\Rightarrow$ $(x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & 2 & 3 \\ x+1 & 2 & 1 \\ 2x+1 & -2 & -1 \end{vmatrix} = 0$

Applying $C_2 \to C_2 - 2C_3$, we get

$$(x+1)(2x-1)(3x-2)\begin{vmatrix} x-1 & -4 & 3 \\ x+1 & 0 & 1 \\ 2x+1 & 0 & -1 \end{vmatrix} = 0$$

or $4(x+1)(2x-1)(3x-2)(3x+2) = 0$

or $x = -1, \frac{1}{2}, \frac{2}{3}, -\frac{2}{3}$

9. Operating $C_1 \to C_1 - bC_3$, $C_2 \to C_2 + aC_3$, we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2 \times \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$R_3 \to R_3 - bR_1 + aR_2$ gives

$$\Delta = (1+a^2+b^2)^2 \times \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & 0 & 1+a^2+b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^3 \quad \text{[Expanding along } C_1\text{]}$$

10. The given determinant can be written as the sum of two determinants

$$\begin{vmatrix} x_1 & a_1b_2 & a_1b_3 \\ 0 & x_2+a_2b_2 & a_2b_3 \\ 0 & a_3b_2 & x_3+a_3b_3 \end{vmatrix} + \begin{vmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & x_2+a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & x_3+a_3b_3 \end{vmatrix}$$

Expanding the first determinant along C_1, we get

$$x_1\begin{vmatrix} x_2+a_2b_2 & a_2b_3 \\ a_3b_2 & x_3+a_3b_3 \end{vmatrix}$$

$$= x_1[(x_2+a_2b_2)(x_3+a_3b_3) - a_3b_2a_2b_3]$$

$$= x_1(x_2x_3 + x_3a_2b_2 + x_2a_3b_3 + a_2b_2a_3b_3 - a_3b_2a_2b_3]$$

$$= x_1x_2x_3 + x_1x_3a_2b_2 + x_1x_2a_3b_3$$

In the second determinant, taking b_1 common from C_1, and then applying $C_2 \to C_2 - b_2C_1$ and $C_3 \to C_3 - b_3C_1$, we obtain

$$b_1\begin{vmatrix} a_1 & 0 & 0 \\ a_2 & x_2 & 0 \\ a_3 & 0 & x_3 \end{vmatrix} = a_1b_1x_2x_3$$

Therefore, the given determinant is

$$x_1x_2x_3 + x_1x_3a_2b_2 + x_1x_2a_3b_3 + a_1b_1x_2x_3$$

$$= x_1x_2x_3\left(1 + \frac{a_1b_1}{x_1} + \frac{a_2b_2}{x_2} + \frac{a_3b_3}{x_3}\right)$$

11. $\Delta = \begin{vmatrix} -1+\cos B & \cos C+\cos B & \cos B \\ \cos C+\cos A & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix}$

Applying $C_1 \to C_1 - C_3 : C_2 \to C_2 - C_3$

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} -a & \cos C & \cos B \\ a\cos C & -1 & \cos A \\ a\cos B & \cos A & -1 \end{vmatrix}$$, where a, b and c are sides opposite to angles A, B and C, respectively.

Applying $C_1 \to C_1 + bC_2 + cC_3$

$$= \frac{1}{a}\begin{vmatrix} -a+b\cos C+c\cos B & \cos C & \cos B \\ a\cos C-b+c\cos A & -1 & \cos A \\ a\cos B+b\cos A-c & \cos A & -1 \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix} = 0$$

(Using projection rule in triangle)

12. Let $\Delta = \begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_3$; then taking 4 common from R_2]

$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad [R_3 \to R_3 - R_1 + 2R_2]$$

$$= -4\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= -4(a-b)(b-c)(c-a)$$

Hence $k = -4$.

13. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc+a^2 & ac+b^2 & ab+c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

Now, $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ abc & abc & abc \end{vmatrix} \quad \begin{matrix}[\text{Multiplying } C_1, C_2, C_3 \text{ by} \\ a, b, c, \text{ respectively}]\end{matrix}$$

$$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$\Rightarrow \quad \Delta = 2\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)$$

14. $\Delta = \begin{vmatrix} x & \frac{1}{2}x(x-1) & \frac{1}{6}x(x-1)(x-2) \\ y & \frac{1}{2}y(y-1) & \frac{1}{6}y(y-1)(y-2) \\ z & \frac{1}{2}z(z-1) & \frac{1}{6}z(z-1)(z-2) \end{vmatrix}$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x-1 & x^2-3x+2 \\ 1 & y-1 & y^2-3y+2 \\ 1 & z-1 & z^2-3z+2 \end{vmatrix}$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2-3x+2 \\ 1 & y & y^2-3y+2 \\ 1 & z & z^2-3z+2 \end{vmatrix} \qquad (C_2 \to C_2 + C_1)$$

$$= \frac{1}{12}xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \qquad (C_3 \to C_3 + 3C_2 - 2C_1)$$

$$= \frac{1}{12}xyz(x-y)(y-z)(z-x)$$

15. $\sum_{r=1}^{n} 2^{r-1} = 1 + 2 + 2^2 + \cdots + 2^{n-1} = 1 \times \frac{(2^n - 1)}{2-1} = 2^n - 1$

$$\sum_{r=1}^{n} 2 \times 3^{r-1} = 2(1 + 3 + 3^2 + \cdots + 3^{n-1}) = \frac{2(3^n - 1)}{3-1} = 3^n - 1$$

$$\sum_{r=1}^{n} 4 \times 5^{r-1} = 4(1 + 5 + 5^2 + \cdots + 5^{n-1}) = \frac{4(5^n - 1)}{5-1} = 5^n - 1$$

$$\therefore \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} 2^{n-1} & 3^{n-1} & 5^{n-1} \\ \alpha & \beta & \gamma \\ 2^{n-1} & 3^{n-1} & 5^{n-1} \end{vmatrix} = 0$$

16. Applying

$$R_1 \to \frac{1}{a}R_1, R_2 \to \frac{1}{b}R_2, R_3 \to \frac{1}{c}R_3, R_4 \to \frac{1}{d}R_4,$$

we get

$$\Delta = abcd\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3 + R_4$ and taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$ common, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Now applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_4 \to C_4 - C_1$,

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\times\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding again along R_1, we get

$$\Delta = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

2nd part: $\Delta = abcd + (bcd + acd + abd + abc)$

$$= \frac{t}{p} - \frac{s}{p} = \frac{t-s}{p}$$

17. $\Delta = \begin{vmatrix} 1 & ab & \frac{1}{a}+\frac{1}{b} \\ 1 & bc & \frac{1}{b}+\frac{1}{c} \\ 1 & ca & \frac{1}{c}+\frac{1}{a} \end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} 1 & ab & ac+bc \\ 1 & bc & ab+ca \\ 1 & ca & bc+ab \end{vmatrix}$ (Multiplying C_3 by abc)

$= \frac{1}{abc}\begin{vmatrix} 1 & ab & ac+bc+ab \\ 1 & bc & ab+ca+bc \\ 1 & ca & bc+ab+ca \end{vmatrix}$ (Applying $C_3 \to C_3 + C_2$)

$= \frac{(ab+bc+ca)}{abc}\begin{vmatrix} 1 & ab & 1 \\ 1 & bc & 1 \\ 1 & ca & 1 \end{vmatrix} = 0$

18. $\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$

Applying $R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$, we get

$\Delta = \frac{1}{abc} \times \begin{vmatrix} a(b^2+c^2) & a^2b & a^2c \\ ab^2 & b(c^2+a^2) & cb^2 \\ ac^2 & bc^2 & c(a^2+b^2) \end{vmatrix}$

Taking a, b and c common from C_1, C_2 and C_3, respectively, we get

$\Delta = \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2 - R_3$

$\Delta = \begin{vmatrix} 0 & -2c^2 & -2b^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$

Taking 2 common from R_1 and applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$

$\Delta = 2\begin{vmatrix} 0 & -c^2 & -b^2 \\ b^2 & a^2 & 0 \\ c^2 & 0 & a^2 \end{vmatrix}$

Expanding along R_1, we get

$\Delta = 2[c^2(a^2b^2) - b^2(-a^2c^2)] = 4a^2b^2c^2$

19. Let $\Delta = \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$

$= \frac{1}{a}\begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & b+a & c \end{vmatrix}$ [Multiplying first column by a]

$= \frac{1}{a}\begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & b+a & c \end{vmatrix}$

[Applying $C_1 \to C_1 + bC_2 + cC_3$]

$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$

[Taking $a^2+b^2+c^2$ common from C_1]

$= \frac{1}{a}(a^2+b^2+c^2)\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= \frac{1}{a}(a^2+b^2+c^2)\times 1 \times \begin{vmatrix} c & -a-b \\ a+c & -b \end{vmatrix}$

[Expanding along C_1]

$= \frac{1}{a}(a^2+b^2+c^2)(-bc + a^2 + ac + ba + bc)$

$= (a^2+b^2+c^2)(a+b+c)$

Exercise 12.4

1. Since points are collinear, we have to prove that area is zero. $x_2 = x_1 r$, $x_3 = x_1 r^2$ and $y_2 = y_1 r$, $y_3 = y_1 r^2$

$\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$= \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$

$= x_1y_1\begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix}$

$= 0$

Hence, the points are collinear.

2. For concurrency of three lines

$px + qy + r = 0$; $qx + ry + p = 0$; $rx + py + q = 0$

We must have $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = 0$

or $3pqr - p^3 - q^3 - r^3 = 0$

or $(p+q+r)(p^2+q^2+r^2-pq-pr-rq) = 0$

or $p+q+r = 0$ ($\because$ $p \neq q \neq r$)

3. We have $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} = a$

$\sqrt{(x_2-x_3)^2+(y_2-y_3)^2} = b$

$\sqrt{(x_3-x_1)^2+(y_3-y_1)^2} = c$

Consider triangle having vertices $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$.

So, $AB = a$, $BC = b$ and $AC = c$. Also, $2s = a + b + c$, where s is semi-perimetre.

Area of triangle ABC,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Also, $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\therefore \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \sqrt{s(s-a)(s-b)(s-c)}$$

Squaring and simplifying, we get

$$4\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2$$

$$= (a+b+c)(b+c-a)(c+a-b)(a+b-c)$$

4. Let the equation of pair of asymptotes be

$$x^2 + 4xy + 3y^2 - 4x + 2y + 1 + k = 0.$$

This is the equation of pair of straight lines.

$$\therefore \quad \begin{vmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & 1+k \end{vmatrix} = 0$$

$$\Rightarrow \quad 1(3 + 3k - 1) - 2(2 + 2k + 2) - 2(2 + 6) = 0$$

$$\Rightarrow \quad 3k + 2 - 4k - 8 - 16 = 0$$

$$\Rightarrow \quad k = -22$$

Exercise 12.5

1. $$\begin{vmatrix} (b+x)(c+x) & (c+x)(a+x) & (a+x)(b+x) \\ (b+y)(c+y) & (c+y)(a+y) & (a+y)(b+y) \\ (b+z)(c+z) & (c+z)(a+z) & (a+z)(b+z) \end{vmatrix}$$

$$= \begin{vmatrix} bc+(b+c)x+x^2 & ac+(a+c)x+x^2 & ab+(a+b)x+x^2 \\ bc+(b+c)y+y^2 & ac+(a+c)y+y^2 & ab+(a+b)y+y^2 \\ bc+(b+c)z+z^2 & ac+(a+c)z+z^2 & ab+(a+b)z+z^2 \end{vmatrix}$$

$$= \begin{vmatrix} bc & b+c & 1 \\ ac & a+c & 1 \\ ab & a+b & 1 \end{vmatrix} \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (b-c)(c-a)(a-b)(y-z)(z-x)(x-y)$$

2. $$\Delta = \begin{vmatrix} 1+a^2+a^4 & 1+ab+a^2b^2 & 1+ac+a^2c^2 \\ 1+ab+a^2b^2 & 1+b^2+b^4 & 1+bc+b^2c^2 \\ 1+ac+a^2c^2 & 1+bc+b^2c^2 & 1+c^2+c^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)^2(b-c)^2(c-a)^2$$

3. Consider the product

$$\begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \gamma\delta+\alpha\beta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \Delta$$

Hence, $\Delta = 0$ [being the product of two determinants, each equal to 0]

4. $$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A\cos P+\sin A\sin P & \cos A\cos Q+\sin A\sin Q & \cos A\cos R+\sin A\sin R \\ \cos B\cos P+\sin B\sin P & \cos B\cos Q+\sin B\sin Q & \cos B\cos R+\sin B\sin R \\ \cos C\cos P+\sin C\sin P & \cos C\cos Q+\sin C\sin Q & \cos C\cos R+\sin C\sin R \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$$= 0 \times 0 = 0$$

5. $$\Delta = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix} \begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

$$= (2abc)^2 = 4a^2b^2c^2$$

6. The given determinant is

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = \Delta \quad \text{[row by row multiplication]}$$

Therefore,

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc-a^2) + b(ac-b^2) + c(ab-c^2)]^2$$

$$= [3abc - a^3 - b^3 - c^3]^2$$

Exercise 12.6

1. $f'(x) = \Delta(R_1', R_2, R_3) + \Delta(R_1, R_2', R_3) + \Delta(R_1, R_2, R_3')$

For $x = 0$, $\Delta(R_1', R_2, R_3) = 0$ and $\Delta(R_1, R_2', R_3) = 0$, as in both all the elements in R_3 are zero. Hence,

$$f'(0) = \Delta(R_1, R_2, R_3')_{x=0}$$

$$= \begin{vmatrix} \cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2) \\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2) \\ 2\cos 2x & 0 & 4x\cos(2x^2) \end{vmatrix}_{x=0}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 2$$

Hence, $f'(0) = 2$.

2. Let $a(x) = a_1x^2 + a_2x + a_3$, $g(x) = b_1x^2 + b_2x + b_3$ and $h(x) = c_1x^2 + c_2x + c_3$. Then,

$f'(x) = 2a_1x + a_2$, $g(x) = 2b_1x + b_2$

and $h'(x) = 2c_1x + c_2$

$f''(x) = 2a_1$, $g''(x) = 2b_1$, $h''(x) = 2c_1$

$f'''(x) = g'''(x) = h'''(x) = 0$

In order to prove that $\phi(x)$ is a constant polynomial, it is sufficient to show that $\phi'(x) = 0$ for all x. Now,

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$$

$$\Rightarrow \quad \phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$$= 0 + 0 + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0 + 0 + 0 = 0 \text{ for all } x$$

$\Rightarrow$ $\phi(x)$ = constant for all

Hence, $\phi(x)$ is a constant polynomial.

3. By partial fractions, we have

$$g(x) = \frac{f(a)}{(x-a)(a-b)(a-c)} + \frac{f(b)}{(b-a)(x-b)(b-c)} + \frac{f(c)}{(c-a)(c-b)(x-c)}$$

$$= \frac{1}{(a-b)(b-c)(c-a)} \times \left[\frac{f(a)(c-b)}{(x-a)} + \frac{f(b)(a-c)}{(x-b)} + \frac{f(c)(b-a)}{(x-c)}\right]$$

$$= \begin{vmatrix} 1 & a & f(a)/(x-a) \\ 1 & b & f(b)/(x-b) \\ 1 & c & f(c)/(x-c) \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\Rightarrow \quad \frac{dg(x)}{dx} = \begin{vmatrix} 1 & a & -f(a)(x-a)^{-2} \\ 1 & b & -f(b)(x-b)^{-2} \\ 1 & c & -f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & f(a)(x-a)^{-2} \\ 1 & b & f(b)(x-b)^{-2} \\ 1 & c & f(c)(x-c)^{-2} \end{vmatrix} \div \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix}$$

4. We have $f'(x) = 0$

So, $f(x)$ is constant.

$\therefore$ $|f(1)| + |f(2)| + \ldots + |f(30)| = 60$

5. $f'(x) = \begin{vmatrix} -\sin x & 1 & 0 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\cos x & 2x & 2 \\ \tan x & x & 1 \end{vmatrix} + \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \sec^2 x & 1 & 0 \end{vmatrix}$

$$\Rightarrow \quad f'(0) = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

Now, $\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} f'(x)$ (as $f(0) = 0$)

$$= f'(0) = 0$$

Exercise 12.7

1. Since the equations are consistent, we have

$$\Delta = 0$$

$$\Rightarrow \quad \begin{vmatrix} (a+1)^3 & (a+2)^3 & -(a+3)^3 \\ (a+1) & (a+2) & -(a+3) \\ 1 & 1 & -1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} (a+1)^3 & (a+2)^3 & (a+3)^3 \\ (a+1) & (a+2) & (a+3) \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or $$[(a+1)-(a+2)][(a+2)-(a+3)][(a+3)-(a+1)] \times [a+1+a+2+a+3] = 0$$

or $$a = -2$$

2. Here, we use Cramer's rule to solve this system.

$$\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$

$$= abc(a-b)(b-c)(c-a)$$

If a, b, c are distinct and nonzero, then the system has a unique solution. Now,

$$\Delta_x = \begin{vmatrix} d & b & c \\ d^2 & b^2 & c^2 \\ d^3 & b^3 & c^3 \end{vmatrix} = dbc(d-b)(b-c)(c-d)$$

$$\therefore \quad x = \frac{\Delta_x}{\Delta} = \frac{d(d-b)(c-d)}{a(a-b)(c-a)}$$

By symmetry, we have

$$y = \frac{\Delta_y}{\Delta} = \frac{d(a-d)(d-c)}{b(a-b)(b-c)}$$

$$z = \frac{\Delta_z}{\Delta} = \frac{d(b-d)(d-a)}{c(b-c)(c-a)}$$

3. $\Delta = \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 7\lambda + 35$

If $7\lambda + 35 \neq 0$ i.e., $\lambda \neq -5$, then system has a unique solution (As $\Delta \neq 0$ gives unique solution).

But if $\lambda = -5$, then we have $\Delta = 0$. Solution exists in this case if $\Delta_x = \Delta_y = \Delta_z = 0$.

For $\lambda = -5$,

$$\Delta_x = \begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & -5 \end{vmatrix} = 0,$$

$$\Delta_y = \begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & -5 \end{vmatrix} = 0,$$

and $$\Delta_z = \begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix} = 0$$

Thus, $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ and hence, there exists infinite number of solutions.

Now, eliminating x from the equations, we get

$7y - 13z = -9$

Let us put $z = k \in R$. $\therefore$ $y = (13k - 9)/7$ and so $x = (4 - 5k)/7$, where k is any real number.

Thus, $\dfrac{7x-4}{-5} = \dfrac{7y+9}{13} = z$

4. $2x + 3y + 1 = 0,\ 3x + y - 2 = 0$

$\Rightarrow$ $x = 1, y = -1$

If the equations are consistent, then

$a(1) + 2(-1) - b = 0$

or $a - b = 2$

5. Since x, y, and z are not all zero, the determinant of the coefficient of the given set of equations must satisfy

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 + \lambda q_1 & p_2 + \lambda q_2 & p_3 + \lambda q_3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ p_1 & p_2 & p_3 \end{vmatrix} \div \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ q_1 & q_2 & q_3 \end{vmatrix}$$

Exercises

Single Correct Answer Type

1. (2) Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 1 & \sin\theta & 0 \\ 1 & 0 & \cos\theta \end{vmatrix} = \sin\theta\cos\theta = (\sin 2\theta)/2$$

So, maximum value of Δ is 1/2.

2. (1) $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$

$= pqr(a^3 + b^3 + c^3 - 3abc) - abc(p^3 + q^3 + r^3 - 3pqr)$

$= pqr(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$\quad - abc(p + q + r)(p^2 + q^2 + r^2 - pq - qr - pr)$

$= 0$

3. (3) Operation $C_1 \to C_1 + C_2 + C_3$ gives

$$(\alpha\beta + \beta\gamma + \gamma\alpha) \begin{vmatrix} 1 & \beta\gamma & \gamma\alpha \\ 1 & \gamma\alpha & \alpha\beta \\ 1 & \alpha\beta & \beta\gamma \end{vmatrix}$$

From the given equation, $\alpha\beta + \beta\gamma + \gamma\alpha = 0$. So, the value of determinant is 0.

4. (4) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$= -(a + b + c)(a + b\omega^2 + c\omega)(a + b\omega + c\omega^2)$

(where ω is cube roots of unity)

$= -f(\alpha) f(\beta) f(\gamma)$ $\quad [\because \alpha = 1, \beta = \omega, \gamma = \omega^2]$

5. (3) $\because -1 \le x < 0$ $\quad \therefore [x] = -1$

$0 \le y < 1$ $\quad \therefore [y] = 0$

$1 \le z < 2$ $\quad \therefore [z] = 1$

Hence, the given determinant is

$$\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

6. (1) $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -\dfrac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] = 0$

$\Rightarrow$ $a + b + c = 0$ or $a = b = c$

If $a + b + c = 0$, we have

$\cos\theta + \cos 2\theta + \cos 3\theta = 0$ and $\sin\theta - \sin 2\theta + \sin 3\theta = 0$

or $\cos 2\theta(2\cos\theta + 1) = 0$ and $\sin 2\theta(1 - 2\cos\theta) = 0$ (1)

which is not possible as $\cos 2\theta = 0$ gives $\sin 2\theta \ne 0$, $\cos\theta \ne 1/2$. And $\cos\theta = -1/2$ gives $\sin 2\theta \ne 0$, $\cos\theta \ne 1/2$. Therefore, Eq. (1) does not hold simultaneously. Therefore,

$a + b + c \ne 0$

$\Rightarrow$ $a = b = c$

$\therefore$ $e^{i\theta} = e^{-2i\theta} = e^{3i\theta}$

which is satisfied only by $e^{i\theta} = 1$, i.e., $\cos\theta = 1$, $\sin\theta = 0$;

so $\theta = 2k\pi$, $k \in Z$.

7. (2) The degree of each term of the determinant is $n + (n + 2) + (n + 3) = 3n + 5$ and the degree of each term of the expression on R.H.S. is 2.

$\therefore$ $3n + 5 = 2$ or $n = -1$

8. (3) $f(x) = \begin{vmatrix} 1 - 2\sin^2 x & \sin^2 x & 1 - 8\sin^2 x(1 - \sin^2 x) \\ \sin^2 x & 1 - 2\sin^2 x & 1 - \sin^2 x \\ 1 - 8\sin^2 x(1 - \sin^2 x) & 1 - \sin^2 x & 1 - 2\sin^2 x \end{vmatrix}$

The required constant term is

$$f(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1(0 - 1) = -1$$

9. (2) Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Applying $C_2 \to C_2 - \dfrac{a_{12}}{a_{11}} C_1$, $C_3 \to C_3 - \dfrac{a_{13}}{a_{11}} C_1$, we get

$$\Delta = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & \left(a_{22} - \frac{a_{12}}{a_{11}} \times a_{21}\right) & \left(a_{23} - \frac{a_{13}}{a_{11}} a_{21}\right) \\ a_{31} & \left(a_{32} - \frac{a_{12}}{a_{11}} \times a_{31}\right) & \left(a_{32} - \frac{a_{13}}{a_{11}} \times a_{31}\right) \end{vmatrix}$$

which has minimum value of -4.

10. (4) Since $A + B + C = \pi$ and $e^{i\pi} = \cos\pi + i\sin\pi = -1$,

$e^{i(B+C)} = e^{i(\pi - A)} = -e^{-iA}$ and $e^{-i(B+C)} = -e^{iA}$

By taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2, and R_3, respectively, we have

$$\Delta = -\begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= -\begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

By taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2, and C_3, respectively, we have

$$\Delta = \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

11. (4) Since for $x = 0$, the determinant reduces to the determinant of a skew-symmetric matrix of odd order which is always zero, hence $x = 0$ is the solution of the given equation.

12. (2) We have,

$$\Delta = \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = abc - (a + b + c) + 2$$

$\therefore \quad \Delta > 0 \Rightarrow abc + 2 > a + b + c$

$\Rightarrow \quad abc + 2 > 3(abc)^{1/3}$

$$\left[\because \text{A.M.} > \text{G.M.} \Rightarrow \frac{a+b+c}{3} > (abc)^{1/3}\right]$$

$\Rightarrow \quad x^3 + 2 > 3x$, where $x = (abc)^{1/3}$

$\Rightarrow \quad x^3 - 3x + 2 > 0 \Rightarrow (x-1)^2 (x+2) > 0$

$\Rightarrow \quad x + 2 > 0 \Rightarrow x > -2 \Rightarrow (abc)^{1/3} > -2 \Rightarrow abc > -8$

13. (2) $B_2 = a_1c_3 - a_3c_1$, $C_2 = -(a_1b_3 - a_3b_1)$

$B_3 = -(a_1c_2 - a_2c_1)$, $C_3 = a_1b_2 - a_2b_1$

$$\therefore \quad \begin{vmatrix} B_2 & C_2 \\ B_3 & C_2 \end{vmatrix} = \begin{vmatrix} a_1c_3 - a_3c_1 & -a_1b_3 + a_3b_1 \\ -a_1c_2 + a_2c_1 & a_1b_2 - a_2b_1 \end{vmatrix}$$

$$= \begin{vmatrix} a_1c_3 & -a_1b_3 \\ -a_1c_2 & a_1b_2 \end{vmatrix} + \begin{vmatrix} a_1c_3 & a_3b_1 \\ -a_1c_2 & -a_2b_1 \end{vmatrix} + \begin{vmatrix} -a_3c_1 & -a_1b_3 \\ a_2c_1 & a_1b_2 \end{vmatrix} + \begin{vmatrix} -a_3c_1 & a_3b_1 \\ a_2c_1 & -a_2b_1 \end{vmatrix}$$

$$= a_1^2 \begin{vmatrix} c_3 & -b_3 \\ -c_2 & b_2 \end{vmatrix} + a_1b_1 \begin{vmatrix} c_3 & a_3 \\ -c_2 & -a_2 \end{vmatrix} + a_1c_1 \begin{vmatrix} -a_3 & -b_3 \\ a_2 & b_2 \end{vmatrix} + b_1c_1 \begin{vmatrix} -a_3 & a_3 \\ a_2 & -a_2 \end{vmatrix}$$

$$= a_1\{a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)\}$$

$$= a_1 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\Delta$$

14. (4) Since a, b, c, d, e, f are in G. P. and if r is the common ratio of the G.P., then

$b = ar$

$c = ar^2$

$d = ar^3$

$e = ar^4$

$f = ar^5$

Therefore, given determinant is

$$\begin{vmatrix} a^2 & a^2r^6 & x \\ a^2r^2 & a^2r^8 & y \\ a^2r^4 & a^2r^{10} & z \end{vmatrix} = a^2a^2r^6 \begin{vmatrix} 1 & 1 & x \\ r^2 & r^2 & y \\ r^4 & r^4 & z \end{vmatrix}$$

$$= a^4r^6\,(0) = 0$$

$[\because C_1, C_2$ are identical$]$

15. (3) Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ reduce the determinant to

$$\begin{vmatrix} x^2 - 2x + 1 & x - 1 & 0 \\ 2x - 2 & x - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix} = (x-1)^3 - 2(x-1)^2$$

$$= (x-1)^2\,(x - 1 - 2)$$

$$= (x-1)^2(x-3),$$

which is clearly negative for $x < 1$.

16. (2) Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} -4 - 2\sqrt{2} & -2\sqrt{2} & 0 \\ 4\sqrt{2} & 4\sqrt{2} & 0 \\ 3 - 2\sqrt{2} & 2 - 2\sqrt{2} & 1 \end{vmatrix}$$

$$= 1(-(4 + 2\sqrt{2}))\,4\sqrt{2} + 2\sqrt{2} \times 4\sqrt{2})$$

$$= -16\sqrt{2}$$

17. (2) The total number of third-order determinants is 9!. The number of determinants is even and of these there are 9!/2 pairs of determinants which are obtained by changing two consecutive rows, so $\sum_{i=1}^{n} D_i = 0$.

18. (1) $$\Delta = \begin{vmatrix} a_1 + b_1w & a_1w^2 + b_1 & c_1 + b_1\bar{w} \\ a_2 + b_2w & a_2w^2 + b_2 & c_2 + b_2\bar{w} \\ a_3 + b_3w & a_3w^2 + b_3 & c_3 + b_3\bar{w} \end{vmatrix}$$

Operating $C_2 \to wC_2$, we have

$$\Delta = \frac{1}{w}\begin{vmatrix} a_1 + b_1w & a_1w^3 + b_1w & c_1 + b_1\bar{w} \\ a_2 + b_2w & a_2w^3 + b_2w & c_2 + b_2\bar{w} \\ a_3 + b_3w & a_3w^3 + b_3w & c_3 + b_3\bar{w} \end{vmatrix}$$

$$= \frac{1}{w}\begin{vmatrix} a_1 + b_1w & a_1 + b_1w & c_1 + b_1\bar{w} \\ a_2 + b_2w & a_2 + b_2w & c_2 + b_2\bar{w} \\ a_3 + b_3w & a_3 + b_3w & c_3 + b_3\bar{w} \end{vmatrix} \quad (\because \omega^3 = 1)$$

$$= 0$$

19. (4) Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$(a+b+c-x)\begin{vmatrix}1 & c & b\\ 1 & b-x & a\\ 1 & a & c-x\end{vmatrix}=0$$

$\therefore \quad x = a+b+c = 0$

20. (2) Since x, y, z are in A.P., therefore, $x + z - 2y = 0$. Now,

$$\begin{vmatrix}a+2 & a+3 & a+2x\\ a+3 & a+4 & a+2y\\ a+4 & a+5 & a+2z\end{vmatrix}=\begin{vmatrix}0 & 0 & 2(x+z-2y)\\ a+3 & a+4 & a+2y\\ a+4 & a+5 & a+2z\end{vmatrix}$$

[Applying $R_1 \to R_1 + R_3 - 2R_2$]

$$=\begin{vmatrix}0 & 0 & 0\\ a+3 & a+4 & a+2y\\ a+4 & a+5 & a+2z\end{vmatrix} \qquad [\because x+z-2y=0]$$

$= 0$

21. (1) We have,

$(a_{n+1})^2 = a_n a_{n+2}$

$\Rightarrow \quad 2\log a_{n+1} = \log a_n + \log a_{n+2}$

Similarly,

$2\log a_{n+4} = \log a_{n+3} + \log a_{n+5}$

$2\log a_{n+7} = \log a_{n+6} + \log a_{n+8}$

Substituting these values in second column of determinant, we get

$$\Delta = \frac{1}{2}\begin{vmatrix}\log a_n & \log a_n + \log a_{n+2} & \log a_{n+2}\\ \log a_{n+3} & \log a_{n+3} + \log a_{n+5} & \log a_{n+5}\\ \log a_{n+6} & \log_{n+6} + \log a_{n+8} & \log a_{n+8}\end{vmatrix}$$

$$= \frac{1}{2}(0) = 0 \qquad [\text{Using } C_2 \to C_2 - C_1 - C_3]$$

22. (4) Applying $R_1 \to R_1 - (R_2 + R_3)$, we get

$$D=\begin{vmatrix}0 & -2y & -2x\\ x & y+z & x\\ y & y & z+x\end{vmatrix}$$

$$=2\begin{vmatrix}0 & -y & -x\\ x & y+z & x\\ y & y & z+x\end{vmatrix}$$

$$=2\begin{vmatrix}0 & -y & -x\\ x & z & 0\\ y & 0 & z\end{vmatrix} \qquad (R_2 \to R_2 + R_1 \text{ and } R_3 \to R_3 + R_1)$$

$= 4xyz$

23. (2) Operation $R_1 \to R_1 - R_2$, gives

$$\Delta=\begin{vmatrix}x-2 & 3(x-2) & -(x-2)\\ 2 & -3x & x-3\\ -3 & 2x & x+2\end{vmatrix}$$

$$=(x-2)\begin{vmatrix}1 & 3 & -1\\ 2 & -3x & x-3\\ -3 & 2x & x+2\end{vmatrix}$$

$$=(x-2)\begin{vmatrix}1 & 3 & -1\\ 0 & -3(x+2) & x-1\\ 0 & 2x+9 & x-1\end{vmatrix} \qquad [R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + 3R_1]$$

$= (x-2)\{-(3x+6)(x-1) - (x-1)(2x+9)\}$

$= -(x-2)(x-1)(5x+15)$

Therefore, $\Delta = 0$ gives $x = 2, 1, -3$.

24. (3) We have,

$$\begin{vmatrix}ka & k^2+a^2 & 1\\ kb & k^2+b^2 & 1\\ kc & k^2+c^2 & 1\end{vmatrix}=\begin{vmatrix}ka & k^2 & 1\\ kb & k^2 & 1\\ kc & k^2 & 1\end{vmatrix}+\begin{vmatrix}ka & a^2 & 1\\ kb & b^2 & 1\\ kc & c^2 & 1\end{vmatrix}$$

$$=0+k\begin{vmatrix}a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}$$

$= k(a-b)(b-c)(c-a)$

25. (2) We have

$$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c) \qquad (1)$$

Also, $\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}$

$$=abc\begin{vmatrix}\frac{1}{a} & \frac{1}{b} & \frac{1}{c}\\ 1 & 1 & 1\\ a^2 & b^2 & c^2\end{vmatrix} \text{ (taking } a,b,c \text{ common from } R_1, R_2, R_3)$$

$$=\begin{vmatrix}bc & ac & ab\\ 1 & 1 & 1\\ a^2 & b^2 & c^2\end{vmatrix} \text{ (Multiplying } R_1 \text{ by } abc)$$

$$=\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ bc & ac & ab\end{vmatrix}$$

Then, $D=\begin{vmatrix}1 & 1 & 1\\ (x-a)^2 & (x-b)^2 & (x-c)^2\\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b)\end{vmatrix}$

$= (a-b)(b-c)(c-a)(3x-a-b-c)$

Now, given that a, b and c are all different. So, $D = 0$ when

$$x = \frac{1}{3}(a+b+c)$$

26. (2) $R_3 \to R_3 - 2R_2$, hence two identical rows $\Rightarrow f(x)$ = constant.

27. (2) In each determinant applying $R_1 \to R_1 + R_2 + R_3$ and then taking out $(x + 9)$ common, we get

$x + 9 = 0$ or $x = -9$

28. (3) Taking x^5 common from last row, we get

$$x^5\begin{vmatrix}x^n & x^{n+2} & x^{2n}\\ 1 & x^a & a\\ x^n & x^{a+1} & x^{2n}\end{vmatrix}=0, \forall x \in R$$

$\Rightarrow \quad a + 1 = n + 2$ or $a = n + 1$

(as it will make first and third row is identical)

29. (3) $\Delta=(1+x+x^2)\begin{vmatrix}1 & 1 & 1\\ x^2 & 1 & x\\ x & x^2 & 1\end{vmatrix}=(1-x^3)^2$

Therefore, $\Delta = 0$ has roots $1, 1, \omega, \omega, \omega^2, \omega^2$.

30. (3) Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & 1+b^2x & (1+c^2)x \\ 1+2x+(a^2+b^2+c^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix} \qquad [\because a^2+b^2+c^2 = -2]$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 0 & 1-x & 0 \\ 0 & 0 & 1-x \end{vmatrix} \quad \begin{matrix}[\text{Operating } R_2 \to R_2 - R_1 \\ \text{and } R_3 \to R_3 - R_1]\end{matrix}$$

$= (1)\,[(1-x)^2 - 0]$

$= (1-x)^2$

which is a polynomial of degree 2.

31. (1) $\begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+2}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+2}C_2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 1 & 1 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 + {}^{m+1}C_1 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 + {}^{m+1}C_2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^{m+1}C_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^{m+1}C_2 & {}^{m+1}C_1 \end{vmatrix} \qquad [\text{Applying } C_3 \to C_3 - C_2]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ {}^mC_1 & {}^mC_0 + {}^mC_1 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 + {}^mC_2 & {}^{m+1}C_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ {}^mC_1 & {}^mC_0 & {}^{m+1}C_0 \\ {}^mC_2 & {}^mC_1 & {}^{m+1}C_1 \end{vmatrix} \qquad [\text{Applying } C_2 \to C_2 - C_1]$$

$= {}^mC_0\,{}^{m+1}C_1 - {}^{m+1}C_0\,{}^mC_1$

$= m + 1 - m$

$= 1$

32. (2) $\Delta = \begin{vmatrix} {}^nC_{r-1} & {}^nC_r & (r+1)\,{}^{n+2}C_{r+1} \\ {}^nC_r & {}^nC_{r+1} & (r+2)\,{}^{n+2}C_{r+2} \\ {}^nC_{r+1} & {}^nC_{r+2} & (r+3)\,{}^{n+2}C_{r+3} \end{vmatrix}$

Applying $C_1 \to C_1 + C_2$ and using ${}^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1}$ in C_3, we get

$$\Delta = \begin{vmatrix} {}^{n+1}C_r & {}^nC_r & (n+2)\,{}^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & (n+2)\,{}^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & (n+2)\,{}^{n+1}C_{r+2} \end{vmatrix}$$

$$= (n+2)\begin{vmatrix} {}^{n+1}C_r & {}^nC_r & {}^{n+1}C_r \\ {}^{n+1}C_{r+1} & {}^nC_{r+1} & {}^{n+1}C_{r+1} \\ {}^{n+1}C_{r+2} & {}^nC_{r+2} & {}^{n+1}C_{r+2} \end{vmatrix}$$

$= 0$ (as C_1 and C_3 are identical)

33. (2) Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x+2a & a & a \\ x+2a & x & a \\ x+2a & a & x \end{vmatrix} = (x+2a)\begin{vmatrix} 1 & a & a \\ 1 & x & a \\ 1 & a & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = (x+2a)\begin{vmatrix} 0 & a-x & 0 \\ 0 & x-a & a-x \\ 1 & a & x \end{vmatrix} = (x-a)^2\,(x+2a)$$

34. (3) Since each element of C_1 is the sum of two elements, putting the determinant as sum of two determinants, we get

$$\Delta = \begin{vmatrix} x^3 & x^2 & x \\ y^3 & y^2 & y \\ z^3 & z^2 & z \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= xyz\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} + \begin{vmatrix} 1 & x^2 & x \\ 1 & y^2 & y \\ 1 & z^2 & z \end{vmatrix}$$

$$= -(xyz+1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$= -(xyz+1)\,(x-y)\,(y-z)\,(z-x)(x+y+z)$

Since $\Delta = 0$, x, y, z all are distinct, we have $xyz + 1 = 0$

or $xyz = -1$.

35. (3) $\begin{vmatrix} 1+x & 1 & 1 \\ 1+y & 1+2y & 1 \\ 1+z & 1+z & 3+3z \end{vmatrix}$

$$= xyz\begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 1 & 1 \\ 1+\frac{1}{y} & 2+\frac{1}{y} & \frac{1}{y} \\ 1+\frac{1}{z} & 1+\frac{1}{z} & 3+\frac{1}{z} \end{vmatrix}$$

$$= xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 0 & 0 \\ 1+\frac{1}{y} & 1 & -1 \\ 1+\frac{1}{z} & 0 & 2 \end{vmatrix}$$

$$= 2xyz\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

Hence, the given equation gives $x^{-1} + y^{-1} + z^{-1} = -3$.

36. (1) As $a_1b_1c_1$, $a_2b_2c_2$ and $a_3b_3c_3$ are even natural numbers, each of c_1, c_2, c_3 is divisible by 2. Let $c_i = 2k_i$ for $i = 1, 2, 3$. Thus,

$$\Delta = 2\begin{vmatrix} k_1 & a_1 & b_1 \\ k_2 & a_2 & b_2 \\ k_3 & a_3 & b_3 \end{vmatrix} = 2m$$

where m is some natural number. Thus, Δ is divisible by 2. That Δ may not be divisible by 4 can be seen by taking the three numbers as 112, 122, and 134. Note that

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 4 & 1 & 3 \end{vmatrix} = 2$$

which is divisible by 2 but not by 4.

37. (2) We have

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

or $$\begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + (C_2 + C_3)$ on L.H.S.]

or $$2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -a & -b \\ a+b+c & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$ on L.H.S.]

or $$\begin{vmatrix} a & -b & -c \\ c & -a & -b \\ b & -c & -a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_2 + C_3$ on L.H.S.]

or $$2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = k\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

$\therefore \quad k = 2$

38. (4) $$D' = \begin{vmatrix} a_1 + pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 + pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 + pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 + qc_1 & c_1 + ra_1 \\ a_2 & b_2 + qc_2 & c_2 + ra_2 \\ a_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 + qc_1 & c_1 + ra_1 \\ pb_2 & b_2 + qc_2 & c_2 + ra_2 \\ pb_3 & b_3 + qc_3 & c_3 + ra_3 \end{vmatrix}$$

In the first determinant, apply $C_3 \to C_3 - rC_1$ and then $C_2 \to C_2 - qC_3$.

In the second determinant, take p common from C_1 and then apply

$C_2 \to C_2 - C_1$. Then take q common from C_2 and apply

$C_3 \to C_3 - C_2$. Finally taking r common from C_3, we have finally $D' = (1 + pqr)D$.

39. (3) $$\Delta = \begin{vmatrix} \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) \\ \log_{a^2}\left(\frac{y}{z}\right) & \log_{a^2}\left(\frac{z}{x}\right) & \log_{a^2}\left(\frac{x}{y}\right) \\ \log_{a^3}\left(\frac{z}{x}\right) & \log_{a^3}\left(\frac{x}{y}\right) & \log_{a^3}\left(\frac{y}{z}\right) \end{vmatrix}$$

$$= \frac{1}{2}\times\frac{1}{3}\begin{vmatrix} \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) \\ \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) & \log_a\left(\frac{x}{y}\right) \\ \log_a\left(\frac{z}{x}\right) & \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) \end{vmatrix}$$

Now, $\log_a\left(\frac{x}{y}\right) + \log_a\left(\frac{y}{z}\right) + \log_a\left(\frac{z}{x}\right)$

$$= \log_a\left(\frac{x}{y}\right)\left(\frac{y}{z}\right)\left(\frac{z}{x}\right) = \log_a 1 = 0$$

So, applying operation $R_1 \to R_1 + R_2 + R_3$ on Δ, we get

$$\Delta = \frac{1}{2}\times\frac{1}{3}\begin{vmatrix} 0 & 0 & 0 \\ \log_a\left(\frac{y}{z}\right) & \log_a\left(\frac{z}{x}\right) & \log_a\left(\frac{x}{y}\right) \\ \log_a\left(\frac{z}{x}\right) & \log_a\left(\frac{x}{y}\right) & \log_a\left(\frac{y}{z}\right) \end{vmatrix} = 0$$

40. (1) Let the first term of G.P. be A and common ratio be R. Then,

$a = AR^{p-1}$

$\Rightarrow \quad \log a = \log A + (p-1)\log R$, etc.

$$\Rightarrow \quad \begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix} \quad [C_1 \to C_1 - (\log A)C_3]$$

$$= \log R\begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

$$= \log R\begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} = 0 \qquad (C_1 \to C_1 + C_3)$$

41. (3) Here $a > 0$ and $4b^2 - 4ac < 0$, i.e., $ac - b^2 > 0$.

$\therefore \quad ax^2 + 2bx + c > 0, \forall\, x \in R$

Now,

$$\Delta = \begin{vmatrix} a & b & ax + b \\ b & c & bx + c \\ 0 & 0 & -(ax^2 + 2bx + c) \end{vmatrix}$$

[Operating $R_3 \to R_3 - xR_1 - R_2$]

$= -(ax^2 + 2bx + c)(ac - b^2)$

$= -(+\text{ve})(+\text{ve}) = -\text{ve}$

42. (3) Using $C_1 \to C_1 + C_2 + C_3$,

$$\Delta = \begin{vmatrix} \sin x + 2\cos x & \cos x & \cos x \\ \sin x + 2\cos x & \sin x & \cos x \\ \sin x + 2\cos x & \cos x & \sin x \end{vmatrix}$$

$$= (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}$$

$$= (\sin x + 2\cos x)(\sin x - \cos x)^2$$

Thus, $\Delta = 0 \Rightarrow \tan x = -2$ or $\tan x = 1$

As $-\pi/4 \le x \le \pi/4$, we get $-1 \le \tan x \le 1$

$\therefore$ $\tan x = 1 \Rightarrow x = \pi/4$

43. (4) $\sum_{k=1}^{n} D_k = 56$

$$\Rightarrow \begin{vmatrix} \sum_{k=1}^{n} 1 & n & n \\ \sum_{k=1}^{n} 2k & n^2+n+1 & n^2+n \\ \sum_{k=1}^{n} (2k-1) & n^2 & n^2+n+1 \end{vmatrix} = 56$$

or $$\begin{vmatrix} n & n & n \\ n(n+1) & n^2+n+1 & n^2+n \\ n^2 & n^2 & n^2+n+1 \end{vmatrix} = 56$$

Applying $C_3 \to C_3 - C_1$ and $C_2 \to C_2 - C_1$, we get

$$\begin{vmatrix} n & 0 & 0 \\ n(n+1) & 1 & 0 \\ n^2 & 0 & n+1 \end{vmatrix} = 56$$

or $n(n+1) = 56$ or $n = 7$

44. (1) Applying $C_1 \to C_1 + 2C_2 + C_3$, we get

$$S = \sum_{r=2}^{n} (-2)^r \begin{vmatrix} {}^nC_r & {}^{n-2}C_{r-1} & {}^{n-2}C_r \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$= \sum_{r=2}^{n} (-2)^r {}^nC_r = \sum_{r=0}^{n} (-2)^r {}^nC_r - ({}^nC_0 - 2\,{}^nC_1)$$

$$= (1-2)^n - (1-2n) = 2n - 1 + (-1)^n$$

45. (1) Applying $R_1 \to R_1 + R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & x+z-2y \\ 4 & 5 & 6 & y \\ 5 & 6 & 7 & z \\ x & y & z & 0 \end{vmatrix}$$

$$= -(x+z-2y)\begin{vmatrix} 4 & 5 & 6 \\ 5 & 6 & 7 \\ x & y & z \end{vmatrix}$$ [Expanding along R_1]

$$= -(x+z-2y)\begin{vmatrix} 0 & -1 & 6 \\ 0 & -1 & 7 \\ x-2y+z & y-z & z \end{vmatrix}$$

[Applying $C_1 \to C_1 + C_3 - 2C_2$ and $C_2 \to C_2 - C_3$]

$$= -(x+z-2y)^2 \begin{vmatrix} -1 & 6 \\ -1 & 7 \end{vmatrix}$$

$$= (x-2y+z)^2$$

Hence, $\Delta = 0 \Rightarrow x, y, z$ are in A.P.

46. (1) $$\begin{vmatrix} x & m & n & 1 \\ a & x & n & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix}$$ $[R_1 \to R_1 - R_2, R_2 \to R_2 - R_3, R_3 \to R_3 - R_4]$

or $$\begin{vmatrix} x-a & m-x & 0 & 0 \\ 0 & x-b & n-x & 0 \\ 0 & 0 & x-c & 0 \\ a & b & c & 1 \end{vmatrix} = 0$$

or $$\begin{vmatrix} x-a & m-x & 0 \\ 0 & x-b & n-x \\ 0 & 0 & x-c \end{vmatrix} = 0$$

or $$(x-a)\begin{vmatrix} (x-b) & n-x \\ 0 & (x-c) \end{vmatrix} = 0$$

or $(x-a)(x-b)(x-c) = 0$

Thus, roots are independent of m and n.

47. (4) Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0$$

Expanding along C_3, we get

$$(c-z)\begin{vmatrix} -x & y \\ -x & 0 \end{vmatrix} + z\begin{vmatrix} a & b-y \\ -x & y \end{vmatrix} = 0$$

or $(c-z)(xy) + z(ay + bx - xy) = 0$

or $cxy - xyz + ayz + bxz - xyz = 0$

or $ayz + bzx + cxy = 2xyz$

or $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

48. (2) $$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$$

Multiplying R_1 by x, R_2 by y and R_3 by z, we get

$$\frac{1}{xyz}\begin{vmatrix} x^4+x & x^3y & x^3z \\ xy^3 & y^4+y & y^3z \\ xz^3 & yz^3 & z^4+z \end{vmatrix} = 11$$

Taking x, y, z common from C_1, C_2, C_3, respectively, we get

$$\begin{vmatrix} x^3+1 & x^3 & x^3 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$$

Using $R_1 \to R_1 + R_2 + R_3$, we have

$$(x^3+y^3+z^3+1)\begin{vmatrix} 1 & 1 & 1 \\ y^3 & y^3+1 & y^3 \\ z^3 & z^3 & z^3+1 \end{vmatrix} = 11$$

Using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$(x^3+y^3+z^3+1)\begin{vmatrix} 1 & 0 & 0 \\ y^3 & 1 & 0 \\ z^3 & 0 & 1 \end{vmatrix} = 11$$

Hence, $x^3+y^3+z^3 = 10$

Therefore, the ordered triplets are (2, 1, 1), (1, 2, 1), (1, 1, 2).

49. (2) Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \tan\frac{B}{2} - \tan\frac{A}{2} & \tan\frac{C}{2} - \tan\frac{B}{2} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ \cot\frac{A}{2} - \cot\frac{B}{2} & \cot\frac{B}{2} - \cot\frac{C}{2} & \cot\frac{C}{2} \\ \dfrac{\cot\frac{A}{2} - \cot\frac{B}{2}}{\cot\frac{A}{2}\cot\frac{B}{2}} & \dfrac{\cot\frac{B}{2} - \cot\frac{C}{2}}{\cot\frac{B}{2}\cot\frac{C}{2}} & \tan\frac{A}{2} + \tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)$$

$$\times \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & \cot\frac{C}{2} \\ \tan\frac{A}{2}\tan\frac{B}{2} & \tan\frac{B}{2}\tan\frac{C}{2} & \tan\frac{A}{2}\tan\frac{B}{2} \end{vmatrix}$$

$$= \left(\cot\frac{A}{2} - \cot\frac{B}{2}\right)\left(\cot\frac{B}{2} - \cot\frac{C}{2}\right)\left(\tan\frac{C}{2} - \tan\frac{A}{2}\right)\tan\frac{B}{2}$$

Since $\Delta = 0$, we have

$$\therefore \quad \cot\frac{A}{2} = \cot\frac{B}{2} \text{ or } \cot\frac{B}{2} = \cot\frac{C}{2} \text{ or } \tan\frac{A}{2} = \tan\frac{C}{2}$$

Hence, the triangle is definitely isosceles.

50. (2) Applying $C_1 \to aC_1$ and then $C_1 \to C_1 + bC_2 + cC_3$, and taking $(a^2+b^2+c^2)$ common from C_1, we get

$$\Delta = \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

$$= \frac{(a^2+b^2+c^2)}{a}\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= \frac{(a^2+b^2+c^2)}{a}(-bc + a^2 + ab + ac + bc)$$

(expanding along C_1)

$$= (a^2+b^2+c^2)(a+b+c)$$

Hence, $\Delta = 0 \Rightarrow a+b+c = 0$

Therefore, line $ax + by + c = 0$ passes through the fixed point (1, 1).

51. (4) Let $\Delta = \begin{vmatrix} y^2 & -xy & x^2 \\ a & b & c \\ a' & b' & c' \end{vmatrix}$

Then,

$$\Delta = \frac{1}{xy}\begin{vmatrix} xy^2 & -xy & x^2y \\ ax & b & cy \\ a'x & b' & c'y \end{vmatrix}$$ [Applying $C_1 \to xC_1$, $C_3 \to yC_3$]

$$= \frac{1}{xy}\begin{vmatrix} 0 & -xy & 0 \\ ax+by & b & bx+cy \\ a'x+b'y & b' & b'x+c'y \end{vmatrix}$$

[Applying $C_1 \to C_1 + yC_2$, $C_3 \to C_3 + xC_2$]

$$= \frac{1}{xy}xy\begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$ [Expanding along R_1]

$$= \begin{vmatrix} ax+by & bx+cy \\ a'x+b'y & b'x+c'y \end{vmatrix}$$

52. (2) $\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\therefore \quad \Delta^2 = \begin{vmatrix} x_1 & y_1 & z_2 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1^2+y_1^2+z_1^2 & x_1x_2+y_1y_2+z_1z_2 & x_1x_3+y_1y_3+z_1z_3 \\ x_1x_2+y_2y_1+z_2z_1 & x_2^2+y_2^2+z_2^2 & x_2x_3+y_2y_3+z_2z_3 \\ x_3x_1+y_3y_1+z_3z_1 & x_2x_3+y_2y_3+z_2z_3 & x_3^2+y_3^2+z_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1$$

53. (3) The given determinant Δ_1 is obtained by corresponding cofactors of determinant Δ_2 and hence, $\Delta_1 = \Delta_2^2$.

Now, $\Delta_1\Delta_2 = \Delta_2^2\Delta_2 = \Delta_2^3$.

54. (4) The given determinant, on simplification, gives

$$\Delta_1 = \begin{vmatrix} a_1^2 & -2a_1 & 1 & 0 \\ a_2^2 & -2a_2 & 1 & 0 \\ a_3^2 & -2a_3 & 1 & 0 \\ a_4^2 & -2a_4 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 & 0 \\ 1 & b_2 & b_2^2 & 0 \\ 1 & b_3 & b_3^2 & 0 \\ 1 & b_4 & b_4^2 & 0 \end{vmatrix}$$

$$= 0 \times 0 = 0$$

55. (1) $\dfrac{\Delta}{h^2} = \begin{vmatrix} \tan x & \dfrac{\tan(x+h) - \tan x}{h} & \dfrac{\tan(x+2h) - \tan x}{h} \\ \tan(x+2h) & \dfrac{\tan x - \tan(x+2h)}{h} & \dfrac{\tan(x+h) - \tan(x+2h)}{h} \\ \tan(x+h) & \dfrac{\tan(x+2h) - \tan(x+h)}{h} & \dfrac{\tan x - \tan(x+h)}{h} \end{vmatrix}$

$$\Rightarrow \lim_{h\to 0}\frac{\Delta}{h^2} = \begin{vmatrix} \tan x & \sec^2 x & 2\sec^2 x \\ \tan x & -2\sec^2 x & -\sec^2 x \\ \tan x & \sec^2 x & -\sec^2 x \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 3\sec^2 x \\ 0 & -3\sec^2 x & 0 \\ \tan x & \sec^2 x & -\sec^2 x \end{vmatrix}$$

$$= 9 \tan x \sec^4 x$$

$$\Rightarrow \lim_{h\to 0} \frac{\Delta(\pi/3)}{\sqrt{3}h^2} = 144$$

56. (4) $\begin{vmatrix} 1+x_1 & 1+x_1x & 1+x_1x^2 \\ 1+x_2 & 1+x_2x & 1+x_2x^2 \\ 1+x_3 & 1+x_3x & 1+x_3x^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & x_1 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & 0 \end{vmatrix} \begin{vmatrix} 1 & 1 & 0 \\ 1 & x & 0 \\ 1 & x^2 & 0 \end{vmatrix}$$

$$= 0$$

57. (3) We observe that the elements in the pre-factor are the cofactors of the corresponding elements of the post-factor. Hence,

$$\begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix}^3 = [\lambda(\lambda^2 + a^2 + b^2 + c^2)]^3 = (1 + a^2 + b^2 + c^2)^3$$

$$\Rightarrow \lambda = 1$$

Alternate solution:

Writing $a = 0, b = 0, c = 0$ on both sides, we get

$\lambda^6\lambda^3 = 1$ or $\lambda = 1$

58. (4) Let the given determinant be equal to $\Delta(x)$. Then,

$5A + 4B + 3C + 2D + E = \Delta(1) + \Delta'(1)$

Now, $\Delta(1) = 0$ as R_2 and R_3 are identical.

$$\Delta'(x) = \begin{vmatrix} 1 & 0 & 1 \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ 2x & 1 & 0 \\ x & x & 6 \end{vmatrix} + \begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ 1 & 1 & 0 \end{vmatrix}$$

$$\Delta'(1) = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 6 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= -17 + (12 + 1 - 1 - 6) = -11$$

59. (2) $\Delta_1 = x(x^2 - ab) - b(ax - ab) + b(a^2 - ax)$

$$= x^3 - 3abx + ab^2 + a^2 b$$

$$\frac{d}{dx}(\Delta_1) = 3x^2 - 3ab = 3(x^2 - ab) = 3\Delta_2$$

60. (4) We have $y = \sin mx$, therefore $y_1 = m \cos mx$, $y_2 = -m^2 \sin mx$, etc.

$$\therefore \quad \Delta = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$$

$$= \begin{vmatrix} \sin mx & m\cos mx & -m^2 \sin mx \\ -m^3 \cos mx & m^4 \sin mx & m^5 \cos mx \\ -m^6 \sin mx & -m^7 \cos mx & m^8 \sin mx \end{vmatrix}$$

$$= m^{12} \begin{vmatrix} \sin mx & \cos mx & -\sin mx \\ -\cos mx & \sin mx & \cos mx \\ -\sin mx & -\cos mx & \sin mx \end{vmatrix} = 0$$

61. (1) Applying $C_1 \to C_1 - 2\sin x C_3$ and $C_2 \to C_2 + 2\cos x\, C_3$, we get

$$f(x) = \begin{vmatrix} 2 & 0 & -\sin x \\ 0 & 2 & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$

$$= 2\cos^2 x + 2\sin^2 x = 2$$

$$\therefore \quad f'(x) = 0$$

$$\therefore \quad \int_0^{\pi/2} [f(x) + f'(x)] = dx = \int_0^{\pi/2} 2dx = \pi$$

62. (2) Since the system has non-trivial solution, we have

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} a-1 & 1-b & 0 \\ 0 & b-1 & 1-c \\ 1 & 1 & c \end{vmatrix} = 0$$

or $\quad c(1-a)(1-b) + (1-b)(1-c) - (1-c)(a-1) = 0$

Dividing throughout by $(1-a)(1-b)(1-c)$, we get

$$\frac{c}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 0$$

or $$-1 + \frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 0$$

or $$\frac{1}{1-c} + \frac{1}{1-a} + \frac{1}{1-b} = 1$$

63. (1) The given system of linear equations has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & \lambda \end{vmatrix} \neq 0$$

i.e., if $\lambda - 8 \neq 0$ or $\lambda \neq 8$.

64. (2) Let $\Delta = \begin{vmatrix} \cos(\alpha-\beta) & \cos(\beta-\gamma) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \cos(\beta+\gamma) & \cos(\gamma+\alpha) \\ \sin(\alpha+\beta) & \sin(\beta+\gamma) & \sin(\gamma+\alpha) \end{vmatrix}$

It is clear that either $\alpha = \beta$ or $\beta = \gamma$ or $\gamma = \alpha$ is sufficient to make $\Delta = 0$. It is not necessary that triangle is equilateral.

Also, isosceles triangle can be obtuse one.

65. (3) $a = x/(y-z) \Rightarrow x - ay + az = 0$ (1)

$b = y/(z-x) \Rightarrow bx + y - bz = 0$ (2)

$c = z/(x-y) \Rightarrow -cx + cy + z = 0$ (3)

Since x, y, z are not all zero, the above system has a non-trivial solution. So,

$$\Delta = \begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ -c & c & 1 \end{vmatrix} = 0$$

or $\quad 1 + ab + bc + ca = 0$

66. (1) $\Delta = \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{vmatrix} p+a & b & c \\ -p & q & 0 \\ -p & 0 & r \end{vmatrix} = 0$$

or $pqc + [q(p+a) + bp]r = 0$

Dividing by pqr, we obtain

$$\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1$$

67. (2) For no solution or infinitely many solutions

$$\begin{vmatrix} \alpha & -1 & -1 \\ 1 & -\alpha & -1 \\ 1 & -1 & -\alpha \end{vmatrix} = 0$$

or $\alpha(\alpha^2 - 1) - 1(\alpha - 1) + 1(1 - \alpha) = 0$

or $\alpha(\alpha^2 - 1) - 2\alpha + 2 = 0$

or $\alpha(\alpha - 1)(\alpha + 1) - 2(\alpha - 1) = 0$

or $(\alpha - 1)(\alpha^2 + \alpha - 2) = 0$

or $(\alpha - 1)(\alpha + 2)(\alpha - 1) = 0$

or $(\alpha - 1)^2(\alpha + 2) = 0$

or $\alpha = 1, 1, -2$

But for $\alpha = 1$, there are infinite solutions. When $\alpha = -2$, we have

$-2x - y - z = -3$

$x + 2y - z = -3$

$x - y + 2z = -3$

Adding, we get $0 = -9$, which is not true. Hence there is no solution.

68. (1) $D = \cos\theta - \cos^2\theta + 6 > 0$. Since $D > 0$ only trivial solution is possible.

69. (3) The given system is consistent. Therefore,

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -c \\ -b & 3b & -c \end{vmatrix} = 0$$

or $c + bc - 6b + b + 2c + 3bc = 0$

or $3c + 4bc - 5b = 0$

or $c = \dfrac{5b}{4b+3}$

Now, $c < 1$

$\Rightarrow \dfrac{5b}{4b+3} < 1$

or $\dfrac{5b}{4b+3} - 1 < 0$

or $\dfrac{b-3}{4b+3} < 0$

$\Rightarrow b \in \left(-\dfrac{3}{4}, 3\right)$

70. (4) As a, b, c are in G.P. with common ratio r_1 and α, β, γ are in G.P. having common ratio r_2, $a \neq 0$, $\alpha \neq 0$, $b = ar_1$, $c = ar_1^2$, $\beta = ar_2$, $\gamma = ar_2^2$.

Also the system of equations has only zero (trivial) solution.

$$\Delta = \begin{vmatrix} a & \alpha & 1 \\ b & \beta & 1 \\ c & \gamma & 1 \end{vmatrix} \neq 0$$

$$\Rightarrow a\alpha \begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & 1 \\ r_1^2 & r_2^2 & 1 \end{vmatrix} \neq 0$$

$\Rightarrow a\alpha(r_1 - 1)(r_2 - 1)(r_1 - r_2) \neq 0$

$\Rightarrow r_1 \neq 1, r_2 \neq 1$, and $r_1 \neq r_2$

71. (1) The given system of equations will have a non-trivial solution if the determinant of coefficients is 0.

$$\Delta = \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & b-t \end{vmatrix} = 0 \qquad \ldots(1)$$

$\Delta = 0$ is a cubic equation in t, so it has 3 solutions say t_1, t_2 and t_3.

Let $\Delta = p_0t^3 + p_1t^2 + p_2t + p_3$,

Clearly, p_0 = coeff. of t^3 which is equal to -1, so

$$t_1t_2t_3 = -\frac{p_3}{(-1)} = p_3$$

= Constant term in the expansion of Δ i.e. $\Delta_{(t=0)}$.

Hence, $t_1t_2t_3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

72. (4) The given system has a non-trivial solution if

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

By expanding the determinant along first column, we get

$\lambda = \sin 2\alpha + \cos 2\alpha$

We know that

$-\sqrt{2} \le \sin 2\alpha + \cos 2\alpha \le \sqrt{2}$

$\therefore \quad -\sqrt{2} \le \lambda \le \sqrt{2}$

Hence, integral values of λ are -1, 0 and 1.

Multiple Correct Answers Type

1. (1), (2), (3)

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix} \qquad [R_3 \to R_3 - R_2 \text{ and } R_2 \to R_2 - R_1]$$

$= 0$

$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ac \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix}$$

$[R_1 \to aR_1, R_2 \to bR_2, R_3 \to cR_3]$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

[taking abc common from C_3]

$= 0$

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix} = \begin{vmatrix} a+b & 2a+b & 3a+b \\ a & a & a \\ 2a & 2a & 2a \end{vmatrix}$$

$[R_3 \to R_3 - R_2, R_2 \to R_2 - R_1]$

$= 0$

$$\begin{vmatrix} 2 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix} \qquad [C_2 \to C_2 - 7C_3]$$

$$= \begin{vmatrix} 1 & 1 & 6 \\ 0 & 7 & 4 \\ 0 & 3 & 2 \end{vmatrix} \qquad [C_1 \to C_1 - C_2]$$

$= 2$

2. (1), (3)

$$\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ 0 & 0 & 1+\sin\beta-\cos\beta \end{vmatrix}$$

[Applying $R_3 \to R_3 - R_1(\cos\beta) + R_2(\sin\beta)$]

$= (1 + \sin\beta - \cos\beta)(\cos^2\alpha + \sin^2\alpha)$

$= 1 + \sin\beta - \cos\beta$, which is independent of α.

3. (1), (3)

$f(\theta) = \sin^3\theta + \cos^3\theta - \cos\theta\sin\theta(\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)^3 - 4\sin\theta\cos\theta(\sin\theta + \cos\theta)$

$= (\sin\theta + \cos\theta)[1 - \sin 2\theta]$

Now, $f(\theta) = 0$

$\Rightarrow$ $\tan\theta = -1$ or $\sin 2\theta = 1$

$\Rightarrow$ $f(\theta) = 0$ has 2 real solutions in $[0, \pi]$

Also, $\dfrac{f(\theta)}{1-\sin 2\theta} = \sin\theta + \cos\theta \in [-\sqrt{2}, \sqrt{2}]$

4. (1), (2), (3)

Applying $R_3 \to R_3 - xR_2$ and $R_2 \to R_2 - xR_1$, we get

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & 0 & a+x \end{vmatrix} = a(a+x)^2$$

Hence,

$f(2x) - f(x) = a[(a+2x)^2 - (a+x)^2]$

$= a(a+2x-a-x)(a+2x+a+x)$

$= ax(2a+3x)$

5. (1), (2), (3)

Operating $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & ac & bc \\ 1 & ad & bd \\ 1 & ae & be \end{vmatrix} = ab \begin{vmatrix} 1 & c & c \\ 1 & d & d \\ 1 & e & e \end{vmatrix} = ab(0) = 0$$

6. (3), (4)

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a+b-x & a & b \\ a+b-x & -x & a \\ a+b-x & b & -x \end{vmatrix} = (a+b-x)\begin{vmatrix} 1 & a & b \\ 1 & -x & a \\ 1 & b & -x \end{vmatrix}$$

$$= (a+b-x)\begin{vmatrix} 1 & a & b \\ 0 & -x-a & a-b \\ 0 & b-a & -x-b \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$]

$= (a+b-x)[(x+a)(x+b) + (a-b)^2]$

[expanding along C_1]

$= (a+b-x)[x^2 + (a+b)x + a^2 + b^2 - ab]$

7. (1), (2)

$$\Delta = \frac{1}{a}\begin{vmatrix} a^3+ax & ab & ac \\ a^2b & b^2+x & bc \\ a^2c & bc & c^2+x \end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & ab & ac \\ b & b^2+x & bc \\ c & bc & c^2+x \end{vmatrix}$$

Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta = \frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & 0 & x \end{vmatrix}$$

$$= \frac{1}{a}(a^2+b^2+c^2+x)(ax^2)$$

$= x^2(a^2+b^2+c^2+x)$

Thus Δ is divisible by x and x^2.

8. (1), (2)

$$\Delta = \begin{vmatrix} a & a^2 & 0 \\ 1 & 2a+b & (a+b) \\ 0 & 1 & 2a+2b \end{vmatrix}$$

$$= a\begin{vmatrix} 1 & a & 0 \\ 1 & (2a+b) & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix}$$

$$= a(a+b)\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & (a+b) \\ 0 & 1 & 2a+3b \end{vmatrix} \qquad (R_2 \to R_2 - R_1)$$

$= a(a+b)(2a+3b-a-b)$

$= a(a+b)(a+2b)$

9. (1), (3)

$$\begin{vmatrix} {}^{x}C_r & {}^{n-1}C_r & {}^{n}C_r \\ {}^{x+1}C_r & {}^{n}C_r & {}^{n+1}C_r \\ {}^{x+2}C_r & {}^{n+1}C_r & {}^{n+2}C_r \end{vmatrix} = 0 \qquad (1)$$

$$\Rightarrow \begin{vmatrix} \dfrac{x!}{r!(x-r)!} & \dfrac{(n-1)!}{r!(n-r-1)!} & \dfrac{n!}{r!(n-r)!} \\ \dfrac{(x+1)!}{r!(x+1-r)!} & \dfrac{n!}{r!(n-r)!} & \dfrac{(n+1)!}{r!(n-r+1)!} \\ \dfrac{(x+2)!}{r!(x+2-r)!} & \dfrac{(n+1)!}{r!(n+1-r)!} & \dfrac{(n+2)!}{r!(n-r+2)!} \end{vmatrix} = 0$$

Taking $\dfrac{x!}{r!(x-r)!}$ common from C_1, we have quadratic equation in x.

Now in (1), if we put $x = n-1$, C_1 and C_2 are the same; hence, $x = n-1$ is one root of the equation.

If we put $x = n$, then C_1 and C_3 are same. Hence, $x = n$ is the other root.

10. (1), (2)

Applying $C_3 \to C_3 - xC_2$, $C_2 \to C_2 - xC_1$, we obtain

$$\Delta(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2+2a^2 & 4a^2x & 6a^2x^2+2a^4 \end{vmatrix}$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & 2x \\ 3x^2+2a^2 & 2x & 3x^2+a^2 \end{vmatrix}$$

Applying $C_3 \to C_3 - xC_2$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2+2a^2 & 2x & x^2+2a^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - 3C_3$, we get

$$\Delta(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -4a^2 & 2x & x^2+2a^2 \end{vmatrix} = 16a^6$$

11. (1), (3)

$$f(n) = \begin{vmatrix} n & n+1 & n+2 \\ n! & (n+1)! & (n+2)! \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} n & 1 & 1 \\ n! & nn! & (n+1)(n+1)! \\ 1 & 0 & 0 \end{vmatrix}$$

[Applying $C_3 \to C_3 - C_2$ and $C_2 \to C_2 - C_1$]

$= (n+1)(n+1)! - nn! = n![(n+1)^2 - n]$

$= n!(n^2+n+1)$

Thus, $f(n)$ is divisible by $n!$ and n^2+n+1.

12. (2), (4)

Given that $\begin{vmatrix} a & b & a\alpha+b \\ b & c & b\alpha+c \\ a\alpha+b & b\alpha+c & 0 \end{vmatrix} = 0$

Operating $C_3 \to C_3 - C_1\alpha - C_2$, we get

$$\begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & -(a\alpha^2+b\alpha+b\alpha+c) \end{vmatrix} = 0$$

$$\Rightarrow \quad (a\alpha^2+2b\alpha+c) \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha+b & b\alpha+c & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (ac-b^2)(a\alpha^2+2b\alpha+c) = 0$$

So, either $ac - b^2 = 0$ or $a\alpha^2 + 2b\alpha + c = 0$.

This means that either a, b, c are in G.P. or $(x-\alpha)$ is a factor of $ax^2 + 2bx + c$.

13. (1), (2), (3), (4)

$$\begin{vmatrix} \sin x & \sin y & \sin z \\ \cos x & \cos y & \cos z \\ \cos^3 x & \cos^3 y & \cos^3 z \end{vmatrix}$$

Taking cos x, cos y and cos z common from C_1, C_2 and C_3, respectively.

$$\Delta = \cos x \cos y \cos z \begin{vmatrix} \tan x & \tan y & \tan z \\ 1 & 1 & 1 \\ \cos^2 x & \cos^2 y & \cos^2 z \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$, we get

$$\Delta = \cos x \cos y \cos z \begin{vmatrix} \tan x - \tan y & \tan y - \tan z & \tan z \\ 0 & 0 & 1 \\ \cos^2 x - \cos^2 y & \cos^2 y - \cos^2 z & \cos^2 z \end{vmatrix}$$

$$= -\cos x \cos y \cos z \begin{vmatrix} \tan x - \tan y & \tan y - \tan z \\ \cos^2 x - \cos^2 y & \cos^2 y - \cos^2 z \end{vmatrix}$$

$$= -\cos x \cos y \cos z \times \begin{vmatrix} \frac{\sin(x-y)}{\cos x \cos y} & \frac{\sin(y-z)}{\cos y \cos z} \\ -\sin(x-y)\sin(x+y) & -\sin(y-z)\sin(y+z) \end{vmatrix}$$

$$= \sin(x-y)\sin(y-z) \begin{vmatrix} \cos z & \cos x \\ \sin(x+y) & \sin(y+z) \end{vmatrix}$$

$= \sin(x-y)\sin(y-z)[\cos z \sin(y+z) - \cos x \sin(x+y)]$

$= \sin(x-y)\sin(y-z)[\sin y \cos^2 z + \sin z \cos y \cos z - \sin x \cos y \cos x - \sin y \cos^2 x]$

$= \sin(x-y)\sin(y-z)[\sin y(\cos^2 z - \cos^2 x) + \cos y(\sin z \cos z - \sin x \cos x)]$

$= \sin(x-y)\sin(y-z)[\sin y \sin(x+z)\sin(x-z) + \cos y(\sin 2z - \sin 2x)/2]$

$= \sin(x-y)\sin(y-z)[\sin y \sin(x+z)\sin(x-z) + \cos y \sin(z-x)\cos(z+x)]$

$= \sin(x-y)\sin(y-z)\sin(z-x)[-\sin y \sin(x+z) + \cos y \cos(z+x)]$

$= \sin(x-y)\sin(y-z)\sin(z-x)\cos(x+y+z)$

For $\Delta = 0$, $x = y$ or $y = z$ or $z = x$ or $x + y + z = \pi/2$.

14. (1), (4)

Let $\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 & x+1 & x-2 \\ 0 & x-2 & x+1 \\ 0 & x-2 & x+1 \end{vmatrix} + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

$$= 0 + \begin{vmatrix} x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_2$, we get

$$\Delta = \begin{vmatrix} x & x+1 & x-2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} x & x & x \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix} x + \begin{vmatrix} 0 & 1 & -2 \\ -1 & -3 & 3 \\ 4 & 0 & 0 \end{vmatrix}$$

15. (1), (2)

We have

$$(a-b)(b-c)(c-a) = (a-b)(b-c)(c-a)(a+b+c) \quad (a \neq b \neq c)$$

$$\Rightarrow \quad a+b+c = 1$$

As a, b and c are positive.

Using AM > GM, we get

$$\frac{a+b+c}{3} > (abc)^{\frac{1}{3}}$$

$$\therefore \quad abc < \frac{1}{27}$$

16. (1), (2), (3), (4)

$$\Delta = \begin{vmatrix} {}^{x}C_r & {}^{x}C_{r+1} & {}^{x}C_{r+2} \\ {}^{y}C_r & {}^{y}C_{r+1} & {}^{y}C_{r+2} \\ {}^{z}C_r & {}^{z}C_{r+1} & {}^{z}C_{r+2} \end{vmatrix} \quad \ldots(1)$$

Applying $C_2 \to C_2 + C_1$, we get

$$\Delta = \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z}C_{r+2} \end{vmatrix}.$$

So, (4) is correct option.

In (1), applying $C_3 \to C_3 + C_2$, we get

$$\Delta = \begin{vmatrix} {}^{x}C_r & {}^{x}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^{y}C_r & {}^{y}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^{z}C_r & {}^{z}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}.$$

So, (3) is correct option.

In above applying $C_2 \to C_2 + C_1$, we get

$$\Delta = \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}.$$

So, (1) is correct option.

In above applying $C_3 \to C_3 + C_2$, we get

$$\Delta = \begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}.$$

So, (2) is correct option.

17. (2), (4)

Applying $C_1 \to C_1 - (\cot\phi)\, C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta/\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= -\frac{\sin\theta}{\sin\phi}\,[-\sin\phi\sin^2\theta - \cos^2\theta\sin\phi] \quad \text{[expanding along } C_1\text{]}$$

$= \sin\theta$, which is independent of ϕ.

Also, $\dfrac{d\Delta}{d\theta} = \cos\theta \Rightarrow \left.\dfrac{d\Delta}{d\theta}\right]_{\theta=\pi/2} = \cos(\pi/2) = 0$

18. (4)

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin(B+A)\sin(B-A) & \dfrac{\sin(A-B)}{\sin A \sin B} & 0 \\ \sin(C+A)\sin(C-A) & \dfrac{\sin(A-C)}{\sin A \sin C} & 0 \end{vmatrix}$$

$$\left[\because \cot\alpha - \cot\beta = \frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta}\right]$$

Expanding along C_3, we get

$$\Delta = \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(B+A)}{\sin C} + \frac{\sin(C+A)}{\sin B}\right]$$

$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin(\pi-C)}{\sin C} + \frac{\sin(\pi-B)}{\sin B}\right]$$

$$= \frac{\sin(A-B)\sin(A-C)}{\sin A}\left[-\frac{\sin C}{\sin C} + \frac{\sin B}{\sin B}\right] = 0$$

19. (1), (2)

Applying $R_1 \to R_1 + \sin\phi\,(R_2) + \cos\phi\,(R_3)$,

$$f(x) = \Delta = \begin{vmatrix} 0 & 0 & \cos 2\phi + 1 \\ \sin\theta & \cos\theta & \sin\phi \\ -\cos\phi & \sin\theta & \cos\phi \end{vmatrix}$$

$$= (\cos 2\phi + 1)(\sin^2\theta + \cos^2\theta)$$

$$= (1 + \cos 2\phi)$$

Hence, Δ is independent of θ.

20. (1), (3)

$$g(x) = \begin{vmatrix} a^{-x} & e^{\log_e a^x} & x^2 \\ a^{-3x} & e^{\log_e a^{3x}} & x^4 \\ a^{-5x} & e^{\log_e a^{5x}} & 1 \end{vmatrix} = \begin{vmatrix} a^{-x} & a^{x} & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix} \left(e^{\log a^x = a^x}\right)$$

$$\Rightarrow \quad g(-x) = \begin{vmatrix} a^{x} & a^{-x} & x^2 \\ a^{3x} & a^{-3x} & x^4 \\ a^{5x} & a^{-5x} & 1 \end{vmatrix} = -\begin{vmatrix} a^{-x} & a^{x} & x^2 \\ a^{-3x} & a^{3x} & x^4 \\ a^{-5x} & a^{5x} & 1 \end{vmatrix}$$

[interchanging 1st and 2nd columns]

$$= -g(x)$$

$$\Rightarrow \quad g(x) + g(-x) = 0$$

$$\Rightarrow \quad g(x) \text{ is an odd function}$$

Hence, the graph is symmetrical about origin. Also, $g_4(x)$ is an odd function [where $g_4(x)$ is fourth derivative of $g(x)$]. Hence,

$$g_4(x) = -g_4(-x)$$

$$\Rightarrow \quad g_4(0) = -g_4(0)$$

$$\Rightarrow \quad g_4(0) = 0$$

21. **(2), (3)**

$$\Delta'(x)=\begin{vmatrix}2x+4 & 2x+4 & 13\\ 4x+5 & 4x+5 & 26\\ 16x-6 & 16x-6 & 104\end{vmatrix}+\begin{vmatrix}x^2+4x-3 & 2 & 13\\ 2x^2+5x-9 & 4 & 26\\ 8x^2-6x+1 & 16 & 104\end{vmatrix}$$

$= 0 + 2 \times 13 \times (0) = 0$

$\Rightarrow \quad \Delta(x) = \text{constant} \quad \Rightarrow a = 0, b = 0, c = 0$

22. **(2), (3)**

In the left-hand determinant, each element is the cofactor of the elements of the determinant

$$\begin{vmatrix}x & y & z\\ y & z & x\\ z & x & y\end{vmatrix} = \Delta \text{ (says)}$$

Hence,

$$\Delta^2=\begin{vmatrix}x & y & z\\ y & z & x\\ z & x & y\end{vmatrix}\begin{vmatrix}x & y & z\\ y & z & x\\ z & x & y\end{vmatrix}$$

$$=\begin{vmatrix}x^2+y^2+z^2 & xy+yz+zx & xz+yx+zy\\ \Sigma xy & \Sigma x^2 & \Sigma xy\\ \Sigma xy & \Sigma xy & \Sigma x^2\end{vmatrix}$$

$$=\begin{vmatrix}r^2 & u^2 & u^2\\ u^2 & r^2 & u^2\\ u^2 & u^2 & r^2\end{vmatrix}, \qquad \text{where } x^2+y^2+z^2=r^2,\ xy+yz+zx=u^2$$

23. **(1), (3), (4)**

$$\begin{vmatrix}a^2 & 1 & a+c\\ 0 & b^2+1 & b+c\\ 0 & b+c & c^2+1\end{vmatrix}=\begin{vmatrix}a & 0 & 1\\ 0 & b & 1\\ 0 & 1 & c\end{vmatrix}\begin{vmatrix}a & 0 & 1\\ 0 & b & 1\\ 0 & 1 & c\end{vmatrix}$$

$$=\begin{vmatrix}a & 0 & 1\\ 0 & b & 1\\ 0 & 1 & c\end{vmatrix}^2$$

$$= (abc - a)^2 = a^2(bc - 1)^2$$

$$=\begin{vmatrix}bc-1 & 0 & 0\\ 1 & ac & -a\\ -b & -a & ab\end{vmatrix}$$

(Determinant formed by cofactors)

24. **(1), (2)**

$$\begin{vmatrix}1 & k & 3\\ k & 2 & 2\\ 2 & 3 & 4\end{vmatrix}=0$$

or $8 + 4k + 9k - 12 - 4k^2 - 6 = 0$

or $4k^2 - 13k + 10 = 0$

or $4k^2 - 8k - 5k + 10 = 0$

or $(2k - 5)(k - 2) = 0$

or $k = 5/2, 2$

25. **(1), (3)**

$$\Delta=\begin{vmatrix}-2 & 1 & 1\\ 1 & -2 & 1\\ 1 & 1 & -2\end{vmatrix}=0$$

$$\Delta_x=\begin{vmatrix}a & 1 & 1\\ b & -2 & 1\\ c & 1 & -2\end{vmatrix}=3a+3b+3c$$

$$\Delta_y=\begin{vmatrix}-2 & a & 1\\ 1 & b & 1\\ 1 & c & -2\end{vmatrix}=3a+3b+3c$$

$$\Delta_z=\begin{vmatrix}-2 & 1 & a\\ 1 & -2 & b\\ 1 & 1 & c\end{vmatrix}=3a+3b+3c$$

26. **(1), (2), (3)**

Given system of equations has non-trivial solutions.

$$\therefore \quad \begin{vmatrix}\alpha & \beta & \gamma\\ \beta & \gamma & \alpha\\ \gamma & \alpha & \beta\end{vmatrix}=0$$

$\Rightarrow \quad \alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$

$\Rightarrow \quad \alpha=\beta=\gamma \qquad (\because \alpha+\beta+\gamma\neq 0)$

Now, $\alpha+\beta+\gamma=-a,\ \alpha\beta+\beta\gamma+\alpha\gamma=b,\ \alpha\beta\gamma=-c$

$\Rightarrow \quad 3\alpha=-a,\ 3\alpha^2=b,\ \alpha^3=-c$

$\Rightarrow \quad \alpha=-\frac{a}{3},\ \alpha^2=\frac{b}{3},\ \alpha^3=-c$

$\Rightarrow \quad a^2=3b,\ a^3=27c,\ b^3=27c^2$

Linked Comprehension Type

1. **(4)** **2.** **(4),** **3.** **(3)**

$$\Delta=\frac{1}{a}\begin{vmatrix}a^3+ax & ab & ac\\ a^2b & b^2+x & bc\\ a^2c & bc & c^2+x\end{vmatrix}$$

Applying $C_1 \to C_1 + bC_2 + cC_3$ and taking $a^2 + b^2 + c^2 + x$ common, we get

$$\Delta=\frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix}a & ab & ac\\ b & b^2+x & bc\\ c & bc & c^2+x\end{vmatrix}$$

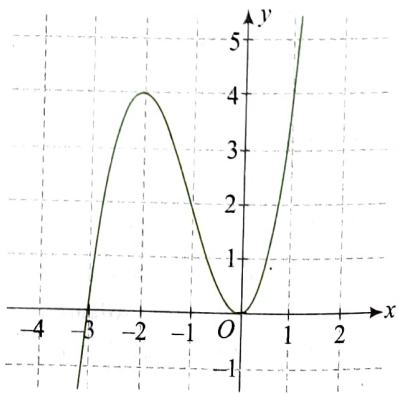
Applying $C_2 \to C_2 - bC_1$ and $C_3 \to C_3 - cC_1$, we get

$$\Delta=\frac{1}{a}(a^2+b^2+c^2+x)\begin{vmatrix}a & 0 & 0\\ b & x & 0\\ c & 0 & x\end{vmatrix}$$

$$=\frac{1}{a}(a^2+b^2+c^2+x)(ax^2)$$

$$= x^2(a^2+b^2+c^2+x)$$

Thus, Δ is divisible by x and x^2. Also, graph of $f(x)$ is



4. (2) 5. (2), 6. (3)

Multiplying R_1, R_2, R_3 by a, b, c, respectively, and then taking a, b, c common from C_1, C_2, and C_3, we get

$$\Delta = \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Now, using $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, and then taking $(ab + bc + ca)$ common from C_2 and C_3, we get

$$\Delta = \begin{vmatrix} -bc & 1 & 1 \\ ab+bc & -1 & 0 \\ ac+bc & 0 & -1 \end{vmatrix} \times (ab+bc+ca)^2$$

Now, applying $R_2 \to R_2 + R_1$, we get

$$\Delta = \begin{vmatrix} -bc & 1 & 1 \\ ab & 0 & 1 \\ ac+bc & 0 & -1 \end{vmatrix} (ab+bc+ca)^2$$

Expanding along C_2, we get

$$\Delta = (ab + bc + ca)^2 [ac + bc + ab]$$
$$= (ab + bc + ca)^3$$
$$= (r/p)^3 = r^3/p^3$$

Now given a, b, c are all positive, then

A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{ab+bc+ac}{3} \geq (ab \times bc \times ac)^{1/3}$$

or $\quad (ab + bc + ac)^3 \geq 27a^2b^2c^2$

or $\quad (ab + bc + ac)^3 \geq 27(s^2/p^2)$

If $\Delta = 27$, then $ab + bc + ca = 3$, and given that $a^2 + b^2 + c^2 = 3$, from $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we have $a + b + c = \pm 3$

$$\Rightarrow \quad a + b + c = 3 \qquad (\because \ a, b, c \in R^+)$$

$$\Rightarrow \quad 3p + q = 0$$

7. (3) 8. (4), 9. (2)

$$\begin{vmatrix} x & n & r \\ m & y & r \\ m & n & z \end{vmatrix} = 0$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\begin{vmatrix} x-m & n-y & 0 \\ 0 & y-n & r-z \\ m & n & z \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-m)(y-n)z + (n-y)(r-z)m - n(r-z)(x-m) = 0$$

Dividing by $(x-m)(y-n)(z-r)$, we have

$$\frac{z}{z-r} + \frac{m}{x-m} + \frac{n}{y-n} = 0$$

or $\quad \frac{z}{z-r} + \frac{m}{x-m} + 1 + \frac{n}{y-n} + 1 = 2$

or $\quad \frac{z}{z-r} + \frac{x}{x-m} + \frac{y}{y-n} = 2$

or $\quad \frac{z}{z-r} - 1 + \frac{x}{x-m} - 1 + \frac{y}{y-n} - 1 = -1$

or $\quad \frac{m}{x-m} + \frac{n}{y-n} + \frac{r}{z-r} = -1$

Now, A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{\frac{z}{z-r} + \frac{x}{x-m} + \frac{y}{y-n}}{3} \geq \left(\frac{z}{(z-r)} \frac{x}{(x-m)} \frac{y}{(y-n)} \right)^{1/3}$$

or $\quad \frac{z}{z-r} \frac{x}{x-m} \frac{y}{y-n} \leq \frac{8}{27}$

10. (3) 11. (4), 12. (2)

In given determinant, applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} x+c_1 & a-c_1 & 0 \\ x+b & c_2-b & a-c_2 \\ x+b & 0 & c_3-b \end{vmatrix}$$

$$= x \begin{vmatrix} 1 & a-c_1 & 0 \\ 1 & c_2-b & a-c_2 \\ 1 & 0 & c_3-b \end{vmatrix} + \begin{vmatrix} c_1 & a-c_1 & 0 \\ b & c_2-b & a-c_2 \\ b & 0 & c_3-b \end{vmatrix}$$

So, $f(x)$ is linear. Let $f(x) = Px + Q$. Then

$f(-a) = -aP + Q, f(-b) = -bP + Q$

Then, $f(0) = 0 \times P + Q$

$$\Rightarrow \quad Q = \frac{bf(-a) - af(-b)}{(b-a)} = f(0) \qquad (1)$$

$$\text{Also, } f(-a) = \begin{vmatrix} c_1-a & 0 & 0 \\ b-a & c_2-a & 0 \\ b-a & b-a & c_3-a \end{vmatrix}$$

$$= (c_1 - a)(c_1 - a)(c_3 - a)$$

Similarly,

$$f(-b) = (c_1 - b)(c_2 - b)(c_3 - b)$$

Now, $g(x) = (c_1 - x)(c_2 - x)(c_3 - x)$

$\Rightarrow \quad g(a) = f(-a)$ and $g(b) = f(-b)$

Now from (1), we get

$$f(0) = \frac{bg(a) - ag(b)}{(b-a)}$$

13. (2) 14. (4), 15. (1)

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$

$$f'(x) = \begin{vmatrix} -(b+1) & -(b+2) & 2ax+b+1 \\ (b+1) & (b+2) & -1 \\ b & b+1 & 2ax+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - R_2$, we get

$$f'(x) = \begin{vmatrix} 0 & 0 & 2ax+b \\ b+1 & b+2 & -1 \\ -1 & -1 & 2ax+b+1 \end{vmatrix}$$

$$= (2ax + b)[-b - 1 + b + 2]$$

$\therefore \quad f'(x) = 2ax + b$

$\therefore \quad f(x) = ax^2 + bx + c$

$f(0) = 2 \Rightarrow c = 2$

$f(1) = 1 \Rightarrow a + b + 2 = 1 \Rightarrow a + b = -1$

$f'(5/2) = 0 \Rightarrow 5a + b = 0$

$\Rightarrow \quad a = 1/4, b = -5/4$

Hence, $f(x)=\frac{1}{4}x^2-\frac{5}{4}x+2$

Clearly, discriminant (D) of the equation $f(x)=0$ is less than 0. Hence, $f(x)=0$ has imaginary roots. Also, $f(2)=1/2$. And minimum value of $f(x)$ is

$$-\frac{\frac{25}{16}-4\frac{1}{4}(2)}{4\frac{1}{4}}=\frac{7}{16}$$

Hence, range of the $f(x)$ is $\left[\frac{7}{16},\infty\right)$.

16. (4) 17. (3), 18. (3)

Let,

$$\begin{vmatrix}(1+x)^a & (1+2x)^b & 1\\ 1 & (1+x)^a & (1+2x)^b\\ (1+2x)^b & 1 & (1+x)^a\end{vmatrix}=A+Bx+Cx^2+\cdots$$

Putting $x=0$, we get

$$A=\begin{vmatrix}1&1&1\\1&1&1\\1&1&1\end{vmatrix}=0$$

Now differentiating both sides with respect to x and putting $x=0$, we get

$$B=\begin{vmatrix}a&2b&0\\1&1&1\\1&1&1\end{vmatrix}+\begin{vmatrix}1&1&1\\0&a&2b\\1&1&1\end{vmatrix}+\begin{vmatrix}1&1&1\\1&1&1\\2b&0&a\end{vmatrix}=0$$

Hence, coefficient of x is 0. Since $f(x)=0$ and $f'(0)=0$, $x=0$ is a repeating root of the equation $f(x)=0$.

19. (3) 20. (1), 21. (4)

The system of equations

$$-x+cy+bz=0 \quad (1)$$

$$cx-y+az=0 \quad (2)$$

$$bx+ay-z=0 \quad (3)$$

has a nonzero solution if

$$\Delta=\begin{vmatrix}-1&c&b\\c&-1&a\\b&a&-1\end{vmatrix}=0$$

$$\Rightarrow \quad a^2+b^2+c^2+2abc-1=0$$

or $\quad a^2+b^2+c^2+2abc=1 \quad (4)$

Then clearly the system has infinitely many solutions. From (1) and (2), we have

$$\frac{x}{ac+b}=\frac{y}{bc+a}=\frac{z}{1-c^2}$$

$$\therefore \quad \frac{x^2}{(ac+b)^2}=\frac{y^2}{(bc+a)^2}=\frac{z^2}{(1-c^2)^2}$$

or $$\frac{x^2}{(1-a^2)(1-c^2)}=\frac{y^2}{(1-b^2)(1-c^2)}=\frac{z^2}{(1-c^2)^2} \quad \text{[from (4)]}$$

or $$\frac{x^2}{1-a^2}=\frac{y^2}{1-b^2}=\frac{z^2}{1-c^2} \quad (5)$$

From (5), we see that $1-a^2$, $1-b^2$, $1-c^2$ are all positive or all negative. Given that one of a, b, c is proper fraction, so

$1-a^2>0$, $1-b^2>0$, $1-c^2>0$, which gives

$a^2+b^2+c^2<3 \quad (6)$

Using (4) and (6), we get

$1<3+2abc$

or $\quad abc>-1 \quad (7)$

22. (1) 23. (2), 24. (3)

$$\Delta=\begin{vmatrix}1&1&1\\1&2&3\\1&2&\lambda\end{vmatrix}=2\lambda+3+2-2-\lambda-6=\lambda-3$$

$$\Delta_1=\begin{vmatrix}6&1&1\\10&2&3\\\mu&2&\lambda\end{vmatrix}=12\lambda+3\mu+20-2\mu-10\lambda-36$$

$$=2\lambda+\mu-16$$

$$\Delta_2=\begin{vmatrix}1&6&1\\1&10&3\\1&\mu&\lambda\end{vmatrix}=10\lambda+18+\mu-10-3\mu-6\lambda$$

$$=4\lambda-2\mu+8$$

$$\Delta_3=\begin{vmatrix}1&1&6\\1&2&10\\1&2&\mu\end{vmatrix}=2\mu+10+12-12-\mu-20$$

$$=\mu-10$$

Thus, the system has unique solutions if $\Delta\neq 0$ or $\lambda\neq 3$ and the system has infinite solutions if $\Delta=\Delta_1=\Delta_2=\Delta_3=0$ or $\lambda=3$ and $\mu=10$. System has no solution if $\Delta=0$ and at least one of Δ_1, Δ_2, Δ_3 is nonzero or $\lambda=3$ and $\mu\neq 10$.

Matrix Match Type

1. a→s; b→p; c→q; d→s.

a. Coefficient of x in $f(x)$ is coefficient of x in $\begin{vmatrix}x&1&1\\1&x&2\\x^2&1&0\end{vmatrix}$

Therefore, coefficient of x is -2.

b. Let $D=\begin{vmatrix}1&3\cos\theta&1\\\sin\theta&1&3\cos\theta\\1&\sin\theta&1\end{vmatrix}$

$=(3\cos\theta-\sin\theta)^2$

$\Delta_{max}=10$

c. $f'(x)=0$

$\Rightarrow \quad f'(0)=0$

d. $a_0=\begin{vmatrix}0&2&0\\1&0&6\\0&0&1\end{vmatrix}=-2(1)=-2$

2. a → p, q, r, s; b → r; c → p, r, s; d → p, q, r, s.

a. We have

$$\Delta=\begin{vmatrix}\frac{1}{c}&\frac{1}{c}&-\frac{(a+b)}{c^2}\\-\frac{(b+c)}{a^2}&\frac{1}{a}&\frac{1}{a}\\-\frac{b(b+c)}{a^2c}&\frac{(a+2b+c)}{ac}&-\frac{b(a+b)}{ac^2}\end{vmatrix}$$

Multiplying C_1 by a, C_2 by b and C_3 by c, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} \frac{a}{c} & \frac{b}{c} & -\frac{a+b}{c} \\ -\frac{b+c}{a} & \frac{b}{a} & \frac{c}{a} \\ -\frac{b(b+c)}{ac} & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} 0 & \frac{b}{c} & -\frac{a+b}{c} \\ 0 & \frac{b}{a} & \frac{c}{a} \\ 0 & \frac{b(a+2b+c)}{ac} & -\frac{b(a+b)}{ac} \end{vmatrix} = 0$$

This shows that Δ is independent of a, b and c.

b. We have

$$\Delta = \begin{vmatrix} \sin a\cos b & \sin a\sin b & \cos a \\ \cos a\cos b & \cos a\sin b & -\sin a \\ -\sin a\sin b & \sin a\cos b & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - (\cot b)\, C_2$, we get

$$\Delta = \begin{vmatrix} 0 & \sin a\sin b & \cos a \\ 0 & \cos a\sin b & -\sin a \\ -\sin a / \sin b & \sin a\cos b & 0 \end{vmatrix}$$

$$= -\frac{\sin a}{\sin b}\,[-\sin b \sin^2 a - \cos^2 a \sin b] \quad \text{[Expanding along } C_1\text{]}$$

$$= \sin a$$

This shows that Δ is independent of b.

c. We have

$$\Delta = \begin{vmatrix} \frac{1}{\sin a\cos b} & \frac{1}{\sin a\sin b} & \frac{1}{\cos a} \\ \frac{-\cos a}{\sin^2 a\cos b} & \frac{-\cos a}{\sin^2 a\sin b} & \frac{\sin a}{\cos^2 a} \\ \frac{\sin b}{\sin a\cos^2 b} & \frac{-\cos b}{\sin a\sin^2 b} & 0 \end{vmatrix}$$

Multiplying C_1 by $\sin a \cos b$, C_2 by $\sin a \sin b$ and C_3 by $\cos a$, we get

$$\Delta = \frac{1}{\sin^2 a\ \cos a\ \sin b\ \cos b}\begin{vmatrix} 1 & 1 & 1 \\ -\cot a & -\cot a & \tan a \\ \tan b & -\cot b & 0 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, we get

$$\Delta = \frac{1}{\sin^2 a\ \cos a\ \sin b\ \cos b}\begin{vmatrix} 0 & 1 & 1 \\ 0 & -\cot a & \tan a \\ \frac{1}{\sin b\cos b} & -\cot b & 0 \end{vmatrix}$$

$$= \frac{1}{\sin^2 a\ \cos a\ \sin^2 b\ \cos^2 b}[\tan a + \cot a]$$

$$= \frac{1}{\sin^3 a\ \cos^2 a\ \sin^2 b\ \cos^2 b}$$

This shows that Δ is dependent on a and b.

d. We have

$$\Delta = \begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & a\sin B & a\sin C \\ a\sin B & 1 & \cos A \\ a\sin C & \cos A & 1 \end{vmatrix} \text{ (Using Sine Rule)}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - (\sin B)\, C_1$ and $C_3 \to C_3 - (\sin C)\, C_1$,

$$= a^2\begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1-\sin^2 B & \cos A - \sin B\sin C \\ \sin C & \cos A - \sin B\sin C & 1-\sin^2 C \end{vmatrix}$$

$$= a^2\,[\cos^2 B\cos^2 C - (\cos A - \sin B\sin C)^2]$$

$$= a^2\,[\cos^2 B\cos^2 C - (\cos(B+C) + \sin B\sin C)^2]$$

$$= a^2\,[\cos^2 B\cos^2 C - \cos^2 B\cos^2 C]$$

$$= 0$$

This shows that Δ is independent of a, b and c.

3. a → s; b → q; c → r; d → p.

$x^3 - 3x^2 + 3x - 1 = -8$

or $x^3 - 3x^2 + 3x + 7 = 0$

or $(x-1)^3 = (-2)^3$

$\therefore$ $\alpha - 1 = -2$

$\beta - 1 = -2\omega$

$\gamma - 1 = -2\omega^2$

$\alpha\beta\gamma = -7$

$(\alpha-1)(\beta-1)(\gamma-1) = (-2)(-2\omega)(-2\omega^2)$

$= -8\omega^3$

$= -8$

$$\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\alpha-1} + \frac{\gamma-1}{\alpha-1} = \frac{-2}{-2\omega} + \frac{-2\omega}{-2\omega^2} + \frac{-2\omega^2}{-2} = 3\omega^2$$

$$\begin{vmatrix} \alpha-1 & \beta-1 & \gamma-1 \\ \beta-1 & \gamma-1 & \alpha-1 \\ \gamma-1 & \alpha-1 & \beta-1 \end{vmatrix} = \begin{vmatrix} -2 & -2\omega & -2\omega^2 \\ -2\omega & -2\omega^2 & -2 \\ -2\omega^2 & -2 & -2\omega \end{vmatrix}$$

$$= -8\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$$

(Applying $R_1 \to R_1 + R_2 + R_3$)

4. (4) $\Delta = (\lambda-1)^2(\lambda+2)$

Now, $\Delta = 0$ if $\lambda = 1$ or $\lambda = -2$

If $\lambda \neq 1$, $\lambda \neq -2$, then the system has a unique solution.

If $\lambda = 1$, the system becomes $x + y + z = 1$ which has infinite solutions. If $\lambda = -2$, the system is inconsistent.

5. (2) Let $\Delta = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

a. $\Delta_1 = |(-1)^{i+j} a_{ij}| = \begin{vmatrix} a & -b & c \\ -p & q & -r \\ x & -y & z \end{vmatrix}$

$= \begin{vmatrix} -a & -b & -c \\ p & q & r \\ -x & -y & -z \end{vmatrix}$ (Multiplying C_1 and C_3 by (-1))

$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \Delta = 2$

b. $\Delta_2 = |2^{i+j} a_{ij}| = \begin{vmatrix} 2^2 a & 2^3 b & 2^4 c \\ 2^3 p & 2^4 q & 2^5 r \\ 2^4 x & 2^5 y & 2^6 z \end{vmatrix}$

$= 2^2 \times 2^3 \times 2^4 \begin{vmatrix} a & 2b & 2^2 c \\ p & 2q & 2^2 r \\ x & 2y & 2^2 z \end{vmatrix}$

$= 2^9 \times 2 \times 2^2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$

$= 2^{12} \times 2 = 2^{13}$

c. $\Delta_3 = |C_{ij}| = \Delta^2 = 4$

d. $\Delta_4 = (\Delta_3)^2 = 4^2 = 2^4$

Numerical Value Type

1. (2) We have $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ 5 & 4 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$

Since $a_n = \frac{20}{n}$; $d = \frac{1}{20}$

Hence, $D = \begin{vmatrix} 20 & \frac{20}{2} & \frac{20}{3} \\ \frac{20}{4} & \frac{20}{5} & \frac{20}{6} \\ \frac{20}{7} & \frac{20}{8} & \frac{20}{9} \end{vmatrix} = \frac{(20)^3}{4 \times 7} \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{4}{5} & \frac{2}{3} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix}$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$= \frac{(20)^3}{4 \times 7} \begin{vmatrix} 0 & \frac{-3}{10} & \frac{-1}{3} \\ 0 & \frac{-3}{40} & \frac{-1}{9} \\ 1 & \frac{7}{8} & \frac{7}{9} \end{vmatrix} = \frac{50}{21}$

$\Rightarrow \quad [D] = 2$

2. (3) $x + y + z = 1$ (1)

$x + 2y + 4z = p$ (2)

$x + 4y + 10z = p^2$ (3)

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix}$

$R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$= \begin{vmatrix} 0 & -1 & -3 \\ 0 & -2 & -6 \\ 1 & 4 & 10 \end{vmatrix} = 0$

Since $\Delta = 0$, solution is not unique.

The system will have infinite solutions if $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0$

$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ p & 2 & 4 \\ p^2 & 4 & 10 \end{vmatrix} = 0$

$C_3 \to C_3 - C_2$

$\Delta_1 = \begin{vmatrix} 1 & 1 & 0 \\ p & 2 & 2 \\ p^2 & 4 & 6 \end{vmatrix} = 0$

or $1(12 - 8) - 1(6p - 2p^2) = 0$

or $4 - 6p + 2p^2 = 0$

or $2(p^2 - 3p + 2) = 0$

or $p^2 - 3p + 2 = 0$

$\Rightarrow \quad p = 1$ or 2

Also for these values of p, $\Delta_2, \Delta_3 = 0$

3. (–4) $\Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix} = 0.$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$

$\Delta = \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ x+1 & x+1 & x+1 \\ x+2 & 2(x+2) & 6(x+2) \end{vmatrix} = 0$

$\therefore \quad \Delta = (x+1)(x+2) \begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 1 & 1 & 1 \\ 1 & 2 & 6 \end{vmatrix} = 0$

$\therefore \quad \Delta = (x+1)(x+2)[(x+2).4 - (2x+3).5 + (3x+4).1]$
$= 0$

$\Delta = (x+1)(x+2)(-3x-3) = 0$

or $(x+1)^2(x+2) = 0$

$\therefore \quad x = -1, -1, -2$

4. (64) Let $D = \begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 & 1 \\ (\gamma+\alpha-\beta-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 & 1 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$

Applying $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$

$= \begin{vmatrix} (\beta+\gamma-\alpha-\delta)^4 - (\alpha+\beta-\gamma-\delta)^4 & (\beta+\gamma-\alpha-\delta)^2 - (\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\gamma+\delta-\beta-\delta)^4 - (\alpha+\beta-\gamma-\delta)^4 & (\gamma+\alpha-\beta-\delta)^2 - (\alpha+\beta-\gamma-\delta)^2 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$

$= 4(\beta-\delta)(\gamma-\alpha).4(\alpha-\delta)(\gamma-\beta)$

$\times \begin{vmatrix} (\beta+\gamma-\alpha-\delta)^2 + (\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\gamma+\alpha-\beta-\delta)^2 + (\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$

Apply $R_1 \to R_1 - R_2$

$= 16(\beta-\delta)(\gamma-\alpha)(\alpha-\delta).4(\gamma-\delta)(\beta-\alpha)$

$$\times\begin{vmatrix} 1 & 0 & 0 \\ (\gamma+\alpha-\beta-\delta)^2+(\alpha+\beta-\gamma-\delta)^2 & 1 & 0 \\ (\alpha+\beta-\gamma-\delta)^4 & (\alpha+\beta-\gamma-\delta)^2 & 1 \end{vmatrix}$$

$= -64(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)$

5. (0) Taking x common from R_2 and $x(x-1)$ common from R_3, we get

$$f(x) = x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & x-1 & x+1 \\ 3 & x-2 & x+1 \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, we get

$$f(x) = x^2(x-1)\begin{vmatrix} 1 & x & 1 \\ 2 & x-1 & 2 \\ 3 & x-2 & 3 \end{vmatrix} = 0$$

Thus, $f(500) = 0$.

6. (4) $$\Delta = x\begin{vmatrix} 1 & x+y & x+y+z \\ 2 & 3x+2y & 4x+3y+2z \\ 3 & 6x+3y & 10x+6y+3z \end{vmatrix}$$

$$= x^2\begin{vmatrix} 1 & 1 & x+y \\ 2 & 3 & 4x+3y \\ 3 & 6 & 10x+6y \end{vmatrix} [C_3 \to C_3 - zC_1, C_2 \to C_2 - yC_1]$$

$$= x^3\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} [C_3 \to C_3 - yC_2]$$

$= x^3(6-8+3) = 64$

$x^3 = 64$ or $x = 4$

7. (2) Using $C_3 \to C_3 - (C_1 + C_2)$ in D_1 and D_2, we have

$$\frac{D_1}{D_2} = \frac{-2b(ad-bc)}{b(ad-bc)} = -2$$

8. (1) $$\Delta_1 = \begin{vmatrix} a_1^2+4a_1d & a_1 & d \\ a_2^2+4a_2d & a_2 & d \\ a_3^2+4a_3d & a_3 & d \end{vmatrix}, [C_3 \to C_3 - C_2]$$

where d is the common difference of A.P.

$$= d\begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} + 4d\begin{vmatrix} a_1 & a_1 & d \\ a_2 & a_2 & d \\ a_3 & a_3 & d \end{vmatrix}$$

$= d(a_1-a_2)(a_2-a_3)(a_3-a_1) = -2d^4$

Similarly, $\Delta_2 = -2d^4$.

9. (8) Putting $x = 0$, $a_0 = 1$

$(1+ax+bx^2)^4 = (1+ax+bx^2)(1+ax+bx^2)$
$(1+ax+bx^2)(1+ax+bx^2)$

Clearly $a_0 = 1$, a_1 = coefficient of $x = a + a + a + a = 4a$

a_2 = coefficient of $x^2 = 4b + 6a^2$

Now $\Delta = -(a_0^3 + a_1^3 + a_2^3 - 3a_0a_1a_2)$

$\because \quad a_0 + a_1 + a_2 \neq 0$

$\therefore \quad a_0 = a_1 = a_2$

$1 = 4a = 6a^2 + 4b$

$\Rightarrow \quad a = \frac{1}{4}, b = \frac{5}{32}$

10. (0) $$\Delta = \begin{vmatrix} 5^{\sqrt{\log_5 3}} & 5^{\sqrt{\log_5 3}} & 3^{\sqrt{\log_3 5}} \\ 3^{\log_3 4} & (0.1)^{\log_{0.1} 4} & 7^{\log_7 3} \\ 7 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 5^{\sqrt{\log_5 3}} & 5^{\sqrt{\log_5 3}} & 3^{\sqrt{\log_3 5}} \\ 3^{\log_3 4} & (0.1)^{\log_{0.1} 2} & 7^{\log_7 3} \\ 7 & 3 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 5^{\sqrt{\log_5 3}} & 5^{\sqrt{\log_5 3}} & 5^{\sqrt{\log_5 3}} \\ 4 & 2 & 3 \\ 7 & 3 & 5 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} 5^{\sqrt{\log_5 3}} & 0 & 0 \\ 4 & -2 & -1 \\ 7 & -4 & -2 \end{vmatrix} = 0$$

11. (6) Given that

$$\begin{vmatrix} s+c & a & b \\ c & s+a & b \\ c & a & s+b \end{vmatrix} = 532$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$(a+b+c+s)\begin{vmatrix} 1 & a & b \\ 1 & s+a & b \\ 1 & a & s+b \end{vmatrix} = 532$$

$$\Rightarrow \quad 2s\begin{vmatrix} 0 & -s & 0 \\ 0 & s & -s \\ 1 & a & s+b \end{vmatrix} = 532$$

$\Rightarrow \quad 2s^3 = 532$

$\therefore \quad s^3 = 216$

$s = 6$

12. (25) $$\Delta_2 = \underbrace{\begin{vmatrix} a & b+3c & c+4a \\ b & c+3a & a+4b \\ c & a+3b & b+4c \end{vmatrix}}_{\Delta'} + 2\underbrace{\begin{vmatrix} b & b+3c & c+4a \\ c & c+3a & a+4b \\ a & a+3b & b+4c \end{vmatrix}}_{\Delta''}$$

Applying $C_3 \to C_3 - 4C_1$ in Δ' and $C_2 \to C_2 - C_1$ in Δ'', we get

$$\Delta_2 = \begin{vmatrix} a & b+3c & c \\ b & c+3a & a \\ c & a+3b & b \end{vmatrix} + 2\times 3\begin{vmatrix} b & c & c+4a \\ c & a & a+4b \\ a & b & b+4c \end{vmatrix}$$

Applying $C_2 \to C_2 - 3C_3$ in Δ' and $C_3 \to C_3 - C_2$ in Δ'', we get

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + 6\begin{vmatrix} b & c & 4a \\ c & a & 4b \\ a & b & 4c \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} + 24\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= 25\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 25\Delta_1$$

$$\Rightarrow \quad \frac{\Delta_2}{\Delta_1} = 25$$

13. (0) $\begin{vmatrix} 3u^2 & 2u^3 & 1 \\ 3v^2 & 2v^3 & 1 \\ 3w^2 & 2w^3 & 1 \end{vmatrix} = 0$

$R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

or $\begin{vmatrix} u^2 - v^2 & u^3 - v^3 & 0 \\ v^2 - w^2 & v^3 - w^3 & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$

or $\begin{vmatrix} u+v & u^2+v^2+vu & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$

$R_1 \to R_1 - R_2$

or $\begin{vmatrix} u-w & (u^2-w^2)+v(u-w) & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$

or $\begin{vmatrix} 1 & u+w+v & 0 \\ v+w & v^2+w^2+vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$

or $(v^2 + w^2 + vw) - (v + w)[(v + w) + u] = 0$

or $v^2 + w^2 + vw - (v + w)^2 - u(v + w) = 0$

or $uv + vw + wu = 0$

14. (2) $\Delta_r = \begin{vmatrix} r & 612 & 915 \\ 101r^2 & 2r & 3r \\ r & \frac{1}{r} & \frac{1}{r^2} \end{vmatrix}$

$= r^2\begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ 1 & \frac{1}{r} & \frac{1}{r^2} \end{vmatrix}$ (Taking r common from C_1 and R_2)

$= \begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ r^2 & r & 1 \end{vmatrix}$

$\therefore \quad \lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} \begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ r^2 & r & 1 \end{vmatrix}$

$= \lim_{n\to\infty} \frac{1}{n^3} \begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ \frac{n(n+1)(2n+1)}{6} & \frac{n(n+1)}{2} & n \end{vmatrix}$

$= \lim_{n\to\infty} \begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ \frac{(n+1)(2n+1)}{6n^2} & \frac{n+1}{2n^2} & \frac{1}{n^2} \end{vmatrix}$

$= \begin{vmatrix} 1 & 612 & 915 \\ 101 & 2 & 3 \\ \frac{1}{3} & 0 & 0 \end{vmatrix}$

$= \frac{1}{3}(3 \times 612 - 2 \times 915)$

$= 2$

15. (8) $\begin{vmatrix} (x^2+1)^2 & (xy+1)^2 & (xz+1)^2 \\ (xy+1)^2 & (y^2+1)^2 & (yz+1)^2 \\ (xz+1)^2 & (yz+1)^2 & (z^2+1)^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= 2(x - y)^2 (y - z)^2 (z - x)^2$

16. (3) Equation $x^3 + ax^2 + bx + c = 0$ has roots α, β, γ. Therefore,

$\alpha + \beta + \gamma = -a$

$\alpha\beta + \beta\gamma + \gamma\alpha = b$

Since the given system of equations has non-trivial solutions, we have

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$$

or $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$

or $(\alpha + \beta + \gamma)[\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha] = 0$

or $(\alpha + \beta + \gamma)[(\alpha + \beta + \gamma)^2 - 3(\alpha\beta + \beta\gamma + \gamma\alpha)] = 0$

$\Rightarrow \quad -a[a^2 - 3b] = 0$

or $a^2/b = 3$

17. (2) System of equations

$\alpha x + y + z = \alpha - 1$ (1)

$x + \alpha y + z = \alpha - 1$ (2)

$x + y + \alpha z = \alpha - 1$ (3)

Since system has no solution.

Therefore, (1) $\Delta = 0$ and (2) $\alpha - 1 \neq 0$

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0, \alpha \neq 1$$

$R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$\begin{vmatrix} \alpha-1 & 0 & 1-\alpha \\ 0 & \alpha-1 & 1-\alpha \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

or $(\alpha-1)[\alpha(\alpha-1)-(1-\alpha)]+(1-\alpha)[-(\alpha-1)]=0$

or $(\alpha-1)[\alpha(\alpha-1)+(\alpha-1)]+(\alpha-1)^2=0$

or $(\alpha-1)^2[(\alpha+1)+1]=0$

or $\alpha=1, 1, -2$ or $\alpha=1, -2$

Since system has no solution, $\alpha \neq 1$.

$\therefore \quad \alpha=-2$

18. (201) The system has a non-trivial solution if

$$\Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad (28-21)\sin 3\theta - (-7-7)\cos 2\theta + 2(-3-4) = 0$

$\Rightarrow \quad \sin 3\theta + 2\cos 2\theta - 2 = 0$

$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta + 2 - 4\sin^2\theta - 2 = 0$

$\Rightarrow \quad 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \quad \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$

$\Rightarrow \quad \sin\theta(2\sin\theta + 3)(2\sin\theta - 1) = 0$

So, $\sin\theta = 0; 1/2$, since $\sin\theta \neq -3/2$

Now, $\sin\theta = 0$ is for $0, \pi, 2\pi, \ldots, 100\pi$, total 101 values

$\sin\theta = 1/2$ for $\theta = \pi/3$ and $2\pi/3$.

So, there will be 100 values lying in $[0, 100\pi]$

Therefore, total number of values of θ is 201.

Archives

JEE MAIN

Single Correct Answer Type

1. (3) $\begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix}$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix}$

$= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+2} \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$

This is equal to zero only if $n+2$ is odd, i.e., n is an odd integer.

2. (1) $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 0, \Delta_1 = \begin{vmatrix} 3 & 2 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & 1 \end{vmatrix} \neq 0$

Hence, the given system does not have any solution.

3 (3) For non-trivial solution of the given system of linear equations

$$\begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 8 + k(2-k) + 2(2k-8) = 0$

$\Rightarrow \quad -k^2 + 6k - 8 = 0$

$\Rightarrow \quad k^2 - 6k + 8 = 0$

$\Rightarrow \quad k = 2, 4$

Clearly there exists two values of k.

4. (2) $\Delta = \begin{vmatrix} k+1 & 8 \\ k & k+3 \end{vmatrix} = k^2 + 4k + 3 - 8k$

$= k^2 - 4k + 3$

$= (k-3)(k-1)$

$\Delta_1 = \begin{vmatrix} 4k & 8 \\ 3k-1 & k+3 \end{vmatrix} = 4k^2 + 12k - 24k + 8$

$= 4k^2 - 12k + 8$

$= 4(k^2 - 3k + 2)$

$= 4(k-2)(k-1)$

$\Delta_2 = \begin{vmatrix} k+1 & 4k \\ k & 3k-1 \end{vmatrix} = 3k^2 + 2k - 1 - 4k^2$

$= -k^2 + 2k - 1$

$= -(k-1)^2$

As given no solution,

$\Rightarrow \quad k = 3$, for which $\Delta = 0$ and $\Delta_1, \Delta_2, \neq 0$

5. (3) $\begin{vmatrix} 1+1+1 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$

$= (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

$\Rightarrow \quad k = 1$

6. (3) System has non-initial solution

$\therefore \quad \begin{vmatrix} \lambda-2 & 2 & -1 \\ 2 & -3-\lambda & 2 \\ -1 & 2 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad (\lambda-2)(3\lambda+\lambda^2-4) - 2(-2\lambda+2) - 1(4-3-\lambda) = 0$

$\Rightarrow \quad (\lambda-2)(\lambda-1)(\lambda+4) + 4(\lambda-1) + (\lambda-1) = 0$

$\Rightarrow \quad (\lambda-1)(\lambda^2+2\lambda-8+5) = 0$

$\Rightarrow \quad (\lambda-1)(\lambda^2+2\lambda-3) = 0$

$\Rightarrow \quad (\lambda-1)^2(\lambda+3) = 0$

7. (3) System has non-trivial solution

$\therefore \quad \begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \quad 1(\lambda+1) - \lambda(-\lambda^2+1) - 1(\lambda+1) = 0$

$\Rightarrow \quad \lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0$

$\Rightarrow \quad \lambda^3 - \lambda = 0$

$\Rightarrow \quad \lambda = 0, \pm 1$

8. (1) Since system of equation has no solution,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad a = 1$

Also, at $a = 1$, $\Delta_1 = \Delta_2 = \Delta_3 = 0$.

But when $a = 1$ and $b = 1$, system of equations becomes

$x + y + z = 1$

and $x + y + z = 0$

This system has no solution.

$\therefore \quad b = \{1\}$

9. (2) Here, ω is complex cube root of unity.

Applying $R_1 \to R_1 + R_2 + R_3$, the given matrix reduces to

$$\begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3(-1 - \omega - \omega) = -3z$$

$\Rightarrow \quad k = -z$

10. (3) Given system of equations

$x + ky + 3z = 0$ ...(i)

$3x + ky - 2z = 0$ ...(ii)

$2x + 4y - 3z = 0$ ...(iii)

Eliminating y from first two equations, we get

$2x - 5z = 0$ or $2x = 5z$

$\therefore \quad x = 5\lambda, z = 2\lambda$

From equation (iii), we have

$10\lambda + 4y - 6\lambda = 0$ or $y = -\lambda$

$$\therefore \quad \frac{xz}{y^2} = \frac{(5\lambda)(2\lambda)}{(-\lambda)^2} = 10$$

11. (4) $\Delta = \begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} 0 & x+4 & 0 \\ 0 & -x-4 & x+4 \\ 5x-4 & 2x & x-4 \end{vmatrix}$$

Expanding C_1, we get

$\Delta = (5x - 4)(x + 4)^2$

$\therefore \quad A = -4, B = 5$

JEE ADVANCED

Multiple Correct Answers Type

1. (2), (3)

We have $\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$

Applying $R_3 \to R_3 - R_2, R_2 \to R_2 - R_1$

$$\Rightarrow \begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$

$$\Rightarrow \begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_3 \to C_3 - C_2, C_2 \to C_2 - C_1$

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ 3+2\alpha & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad 2\alpha^2(2 + 3\alpha) - 2\alpha^2(2 + 5\alpha) = -324\alpha$

$\Rightarrow \quad -4\alpha^3 = -324\alpha$

$\Rightarrow \quad \alpha = 0, \pm 9$

2. (2), (3), (4)

$\alpha x + 2y = \lambda$

$3x - 2y = \mu$

$$\Delta = \begin{vmatrix} \alpha & 2 \\ 3 & -2 \end{vmatrix} = -2\alpha - 6$$

$\Delta = 0, \therefore \alpha = -3$

$$\therefore \quad \Delta_1 = \begin{vmatrix} \lambda & 2 \\ \mu & -2 \end{vmatrix} = -2\lambda - 2\mu = -2(\lambda + \mu)$$

$$\Delta_2 = \begin{vmatrix} -3 & \lambda \\ 3 & \mu \end{vmatrix} = -3\mu - 3\lambda = -3(\lambda + \mu)$$

Now check options.

Numerical Value Type

1. (1) $\omega = e^{i2\pi/3}$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$z \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

(Applying $C_1 \to C_1 + C_2 + C_3$ and using $1 + \omega + \omega^2 = 0$)

or $z\left[(z+\omega^2)(z+\omega) - 1 - \omega(z+\omega-1) + \omega^2(1-z-\omega^2)\right] = 0$

or $z^3 = 0$

or $z = 0$ is only solution.

2. (2) $\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$

$$\Rightarrow \quad x^3 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

$$\Rightarrow \quad x^3 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 3 & 9 & 1 \end{vmatrix} + x^6 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 8 \\ 3 & 9 & 27 \end{vmatrix} = 10$$

$$\Rightarrow \quad x^3\begin{vmatrix}1 & 0 & 0\\ 2 & 2 & -1\\ 3 & 6 & -2\end{vmatrix} + x^6\begin{vmatrix}1 & 0 & 0\\ 2 & 2 & 6\\ 3 & 6 & 24\end{vmatrix} = 10$$

$$\Rightarrow \quad 6x^6 + x^3 - 5 = 0$$

$$\Rightarrow \quad (6x^3 - 5)(x^3 + 1) = 0$$

$$\Rightarrow \quad x^3 = 5/6 \text{ or } x^3 = -1$$

Two real distinct values of x.

3. (1) Since the system of equations hasss infinitely many solutions,

$$\begin{vmatrix}1 & \alpha & \alpha^2\\ \alpha & 1 & \alpha\\ \alpha^2 & \alpha & 1\end{vmatrix} = 0$$

$$\Rightarrow \quad 1(1-\alpha^2) - \alpha(\alpha - \alpha^3) + \alpha^2(\alpha^2 - \alpha^2) = 0$$

$$\Rightarrow \quad (1-\alpha^2) - \alpha^2 + \alpha^4 = 0$$

$$\Rightarrow \quad (\alpha^2 - 1)^2 = 0$$

$$\Rightarrow \quad \alpha = \pm 1$$

For $\alpha = 1$, we get $x + y + z = 1$ and $x + y + z = -1$.

Hence, system has no solution.

For $\alpha = -1$, all three equations become $x - y + z = 1$, which represents coincident planes.

$$\therefore \quad 1 + \alpha + \alpha^2 = 1$$

4. (4) $\det(P) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$

$$= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3)$$

$$= x - y$$

where $x = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$ and

$$y = a_3b_2c_1 + a_1b_3c_2 + a_2b_1c_3$$

Now, $x \le 3$ and $y \ge -3$.

If we consider $\det.(P) = 6$, then

$$a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1$$

and $a_3b_2c_1 = a_1b_3c_2 = a_2b_1c_3 = -1$, which is contradiction.

So, next possibility is 5 for which one of the terms must be zero but that will make two terms zero.

Hence, greatest value of determinant is 4.

Concept Application Exercises

Exercise 13.1

1. A 3×4 matrix is given by $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$a_{ij} = \frac{1}{2}|-3i+j|$, $i = 1, 2, 3$ and $j = 1, 2, 3, 4$

$\therefore \quad a_{11} = \frac{1}{2}|-3\times1+1| = \frac{1}{2}|-3+1| = \frac{1}{2}|-2| = \frac{2}{2} = 1$

$a_{21} = \frac{1}{2}|-3\times2+1| = \frac{1}{2}|-6+1| = \frac{1}{2}|-5| = \frac{5}{2}$

$a_{31} = \frac{1}{2}|-3\times3+1| = \frac{1}{2}|-9+1| = \frac{1}{2}|-8| = \frac{8}{2} = 4$

$a_{12} = \frac{1}{2}|-3\times1+2| = \frac{1}{2}|-3+2| = \frac{1}{2}|-1| = \frac{1}{2}$

$a_{22} = \frac{1}{2}|-3\times2+2| = \frac{1}{2}|-6+2| = \frac{1}{2}|-4| = \frac{4}{2} = 2$

$a_{32} = \frac{1}{2}|-3\times3+2| = \frac{1}{2}|-9+2| = \frac{1}{2}|-7| = \frac{7}{2}$

$a_{13} = \frac{1}{2}|-3\times1+3| = \frac{1}{2}|-3+3| = 0$

$a_{23} = \frac{1}{2}|-3\times2+3| = \frac{1}{2}|-6+3| = \frac{1}{2}|-3| = \frac{3}{2}$

$a_{33} = \frac{1}{2}|-3\times3+3| = \frac{1}{2}|-9+3| = \frac{1}{2}|-6| = \frac{6}{2} = 3$

$a_{14} = \frac{1}{2}|-3\times1+4| = \frac{1}{2}|-3+4| = \frac{1}{2}|1| = \frac{1}{2}$

$a_{24} = \frac{1}{2}|-3\times2+4| = \frac{1}{2}|-6+4| = \frac{1}{2}|-2| = \frac{2}{2} = 1$

$a_{34} = \frac{1}{2}|-3\times3+4| = \frac{1}{2}|-9+4| = \frac{1}{2}|-5| = \frac{5}{2}$

Therefore, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}$

2. $\begin{bmatrix} a-b & 2a+c \\ 2a-b & 3c+d \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$

As the two matrices are equal, their corresponding elements are also equal.

Comparing the corresponding elements, we get

$a - b = -1$ (1)

$2a - b = 0$ (2)

$2a + c = 5$ (3)

$3c + d = 13$ (4)

From (2), we have

$b = 2a$

Then, from (1), we have

$a - 2a = -1$

or $a = 1$

$\Rightarrow \quad b = 2$

Now, from (3), we have

$2 \times 1 + c = 5$

or $c = 3$

From (4), we have

$3 \times 3 + d = 13$

or $9 + d = 13$ or $d = 4$

$\therefore \quad a = 1, b = 2, c = 3$, and $d = 4$

3. The given matrix of the order 3×3 has 9 elements and each of these elements can be either 0 or 1.

Now, each of the 9 elements can be chosen in two possible ways.

Therefore, by the multiplication principle, the required number of possible matrices is $2^9 = 512$.

Each of the diagonal and upper triangle elements can be chosen in two ways.

So, number of symmetric matrices is 2^6, as lower triangle elements are same as corresponding upper triangle elements.

4. $A = \begin{bmatrix} 2/x & -1 & 2 \\ 1 & x & 2x^2 \\ 1 & 1/x & 2 \end{bmatrix}$

We have

$$|A| = \left(\frac{2}{x}\right)\begin{vmatrix} x & 2x^2 \\ 1/x & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2x^2 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & x \\ 1 & 1/x \end{vmatrix}$$

$$= \frac{2}{x}(0) + 2 - 2x^2 + 2\left(\frac{1}{x} - x\right)$$

$$= \frac{2x(1-x^2) + 2(1-x^2)}{x}$$

$$= \frac{2(x+1)^2(1-x)}{x}$$

Now, $|A| = 0 \quad \Rightarrow \quad x = \pm 1$

5. Since A is skew-symmetric matrix, tr. $A = 0$.

Also, order of A is odd, then det. $A = 0$.

Thus, tr. A = det. A

Exercise 13.2

1. We have, $x\begin{bmatrix} 2 \\ 1 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = O$

$$\Rightarrow \quad \begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} 3y \\ 5y \end{bmatrix} + \begin{bmatrix} -8 \\ -11 \end{bmatrix} = O$$

$$\therefore \quad \begin{bmatrix} 2x+3y-8 \\ x+5y-11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2x + 3y - 8 = 0$ (1)

and $x + 5y - 11 = 0$ (2)

Solving equations (1) and (2), we get

$x = 1$ and $y = 2$

2. We have, $A=\begin{bmatrix}1 & 5\\ 7 & 12\end{bmatrix}$ and $B=\begin{bmatrix}9 & 1\\ 7 & 8\end{bmatrix}$

Let $C=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$\therefore \quad 3A+5B+2C=O$

$\Rightarrow \quad \begin{bmatrix}3 & 15\\ 21 & 36\end{bmatrix}+\begin{bmatrix}45 & 5\\ 35 & 40\end{bmatrix}+\begin{bmatrix}2a & 2b\\ 2c & 2d\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow \quad \begin{bmatrix}48+2a & 20+2b\\ 56+2c & 76+2d\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

So, we have

$2a+48=0 \Rightarrow a=-24;$

$20+2b=0 \Rightarrow b=-10;$

$56+2c=0 \Rightarrow c=-28;$

and $76+2d=0 \Rightarrow d=-38.$

$\therefore \quad C=\begin{bmatrix}-24 & -10\\ -28 & -38\end{bmatrix}$

3. Given,

$$2X-Y=\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$$

Multiplying both sides by 2, we have

$$4X-2Y=2\begin{bmatrix}3 & -3 & 0\\ 3 & 3 & 2\end{bmatrix}$$

$$4X-2Y=\begin{bmatrix}6 & -6 & 0\\ 6 & 6 & 4\end{bmatrix} \qquad (1)$$

Also given,

$$X+2Y=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix} \qquad (2)$$

Adding (1) and (2), we have

$$5X=\begin{bmatrix}6 & -6 & 0\\ 6 & 6 & 4\end{bmatrix}+\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$$

$$=\begin{bmatrix}6+4 & -6+1 & 0+5\\ 6-1 & 6+4 & 4-4\end{bmatrix}$$

$$=\begin{bmatrix}10 & -5 & 5\\ 5 & 10 & 0\end{bmatrix}$$

$\therefore \quad X=\frac{1}{5}\begin{bmatrix}10 & -5 & 5\\ 5 & 10 & 0\end{bmatrix}=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}$

Putting the value of X in (2), we have

$$\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}+2Y=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}$$

or $$2Y=\begin{bmatrix}4 & 1 & 5\\ -1 & 4 & -4\end{bmatrix}-\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}4-2 & 1+1 & 5-1\\ -1-1 & 4-2 & -4-0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 2 & 4\\ -2 & 2 & -4\end{bmatrix}$$

$\therefore \quad Y=\begin{bmatrix}1 & 1 & 2\\ -1 & 1 & -2\end{bmatrix}$

Hence,

$$X=\begin{bmatrix}2 & -1 & 1\\ 1 & 2 & 0\end{bmatrix} \text{ and } Y=\begin{bmatrix}1 & 1 & 2\\ -1 & 1 & -2\end{bmatrix}$$

4. $\text{tr.}(A+B^T+3C)=\text{tr.}(A)+\text{tr.}(B^T)+\text{tr.}(3C)$

$=(1+1+3)+(1-1+3)+3(-1+1+3)$

$=17$

5. We have

$$(A-\lambda I)=\begin{bmatrix}3 & -2\\ 4 & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3-\lambda & -2\\ 4 & -1-\lambda\end{bmatrix}$$

$(A-\lambda I)$ is singular.

$\therefore \quad \det.(A-\lambda I)=0$

$\Rightarrow \quad (3-\lambda)(-1-\lambda)-(-2)\,4=0$

$\Rightarrow \quad \lambda^2-2\lambda+5=0.$

$\Rightarrow \quad \lambda=1\pm 2i$

6. Here matrix A is expressed as the sum of symmetric and skew-symmetric matrix. Then

$$B=\frac{1}{2}(A+A^T) \text{ and } C=\frac{1}{2}(A-A^T)$$

Now $A=\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}$

$\Rightarrow \quad A^T=\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}$

$\Rightarrow \quad B=\frac{1}{2}\left(\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}+\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}\right)$

$$=\frac{1}{2}\begin{bmatrix}0 & 5 & 2\\ 5 & -6 & 1\\ 2 & 1 & 8\end{bmatrix}$$

and $C=\frac{1}{2}\left(\begin{bmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{bmatrix}-\begin{bmatrix}0 & 4 & 3\\ 1 & -3 & -3\\ -1 & 4 & 4\end{bmatrix}\right)$

$$=\frac{1}{2}\begin{bmatrix}0 & -3 & -4\\ 3 & 0 & 7\\ 4 & -7 & 0\end{bmatrix}$$

Exercise 13.3

1. (b) $A\to 3\times 3, B\to 3\times 2, C\to 3\times 1$

$AB\to 3\times 2 \Rightarrow (AB)^T=2\times 3 \Rightarrow (AB)^TC$ is defined

$C^T\to 1\times 3, \Rightarrow C^TC\to 1\times 1$

Hence $C^TC(AB)^T$ is not defined. Now, C^TAB is also defined.

$A^T\to 3\times 3, B^T\to 2\times 3$

$A^TA\to 3\times 3$

$BB^T\to 3\times 3$

$\Rightarrow \quad A^TABB^T\to 3\times 3$

$\Rightarrow \quad A^TABB^TC$ is defined

2. $A = BB^T + CC^T$

$$= \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix} + \begin{bmatrix}\sin\theta \\ -\cos\theta\end{bmatrix}\begin{bmatrix}\sin\theta & -\cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta & \cos\theta\cdot\sin\theta \\ \sin\theta\cdot\cos\theta & \sin^2\theta\end{bmatrix} + \begin{bmatrix}\sin^2\theta & -\sin\theta\cdot\cos\theta \\ -\sin\theta\cdot\cos\theta & \cos^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}$$

3. $R(s)R(t) = \begin{bmatrix}\cos s & \sin s \\ -\sin s & \cos s\end{bmatrix}\begin{bmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{bmatrix}$

$$= \begin{bmatrix}\cos s\cos t - \sin s\sin t & \cos s\sin t + \sin s\cos t \\ -\sin s\cos t - \cos s\sin t & \cos s\cos t - \sin s\sin t\end{bmatrix}$$

$$= \begin{bmatrix}\cos(s+t) & \sin(s+t) \\ -\sin(s+t) & \cos(s+t)\end{bmatrix}$$

$$= R(s+t)$$

4. $A^2 = \begin{bmatrix}i & 0 \\ 0 & i\end{bmatrix}\begin{bmatrix}i & 0 \\ 0 & i\end{bmatrix} = \begin{bmatrix}i^2 & 0 \\ 0 & i^2\end{bmatrix} = \begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix}$

$$A^4 = \begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0 \\ 0 & -1\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} = I$$

$\Rightarrow \quad A^{4n} = I^n = I$

5. We have,

$$A^2 = \begin{bmatrix}3 & -4 \\ 1 & -1\end{bmatrix}\begin{bmatrix}3 & -4 \\ 1 & -1\end{bmatrix} = \begin{bmatrix}5 & -8 \\ 2 & -3\end{bmatrix} = \begin{bmatrix}1+2\times2 & -4\times2 \\ 2 & 1-2\times2\end{bmatrix}$$

and

$$A^3 = \begin{bmatrix}5 & -8 \\ 2 & -3\end{bmatrix}\begin{bmatrix}3 & -4 \\ 1 & -1\end{bmatrix} = \begin{bmatrix}7 & -12 \\ 3 & -5\end{bmatrix} = \begin{bmatrix}1+2\times3 & -4\times3 \\ 3 & 1-2\times3\end{bmatrix}$$

Thus, it is true for indices 2 and 3. Now, assume

$$A^k = \begin{bmatrix}1+2k & -4k \\ k & 1-2k\end{bmatrix} \text{ to be true.}$$

Then,

$$A^{k+1} = \begin{bmatrix}1+2k & -4k \\ k & 1-2k\end{bmatrix}\begin{bmatrix}3 & -4 \\ 1 & -1\end{bmatrix}$$

$$= \begin{bmatrix}3+2k & -4(k+1) \\ k+1 & -1-2k\end{bmatrix}$$

$$= \begin{bmatrix}1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1)\end{bmatrix}$$

Thus, if the law is true for A^k, it is also true for A^{k+1}. But it is true for $k = 2, 3$, etc. Hence, by induction, the required result follows.

6. Let $X = \begin{bmatrix}a & b \\ c & d\end{bmatrix}$

Then $\begin{bmatrix}1 & 2 \\ 3 & -5\end{bmatrix} = \begin{bmatrix}1 & 0 \\ 0 & 2\end{bmatrix}\begin{bmatrix}a & b \\ c & d\end{bmatrix} = \begin{bmatrix}a & b \\ 2c & 2d\end{bmatrix}$

$\therefore \quad a = 1, b = 2, 2c = 3, 2d = -5$

$\therefore \quad X = \begin{bmatrix}1 & 2 \\ 3/2 & -5/2\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 4 \\ 3 & -5\end{bmatrix}$

7. We have

$$\begin{bmatrix}1 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2\end{bmatrix}\begin{bmatrix}0 \\ 2 \\ x\end{bmatrix} = 0$$

or $\quad \begin{bmatrix}1+4+1 & 2+0+0 & 0+2+2\end{bmatrix}\begin{bmatrix}0 \\ 2 \\ x\end{bmatrix} = 0$

or $\quad [6(0) + 2(2) + 4(x)] = 0$

or $\quad [4 + 4x] = [0]$

or $\quad 4 + 4x = 0$

or $\quad x = -1$

8. It is given that

$$X\begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9 \\ 2 & 4 & 6\end{bmatrix}$$

The matrix given on the R.H.S. of the equation is a 2×3 matrix and the one given on the L.H.S. is a 2×3 matrix. Therefore, X has to be a 2×2 matrix.

Now, let $X = \begin{bmatrix}a & c \\ b & d\end{bmatrix}$

Therefore, we have

$$\begin{bmatrix}a & c \\ b & d\end{bmatrix}\begin{bmatrix}1 & 2 & 3 \\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9 \\ 2 & 4 & 6\end{bmatrix}$$

or $\quad \begin{bmatrix}a+4c & 2a+5c & 3a+6c \\ b+4d & 2b+5d & 3b+6d\end{bmatrix} = \begin{bmatrix}-7 & -8 & -9 \\ 2 & 4 & 6\end{bmatrix}$

Equating the corresponding elements of the two matrices, we have

$a + 4c = -7, \quad 2a + 5c = -8, \quad 3a + 6c = -9$

$b + 4d = 2, \quad 2b + 5d = 4, \quad 3b + 6d = 6$

Now, $a + 4c = -7 \Rightarrow a = -7 - 4c$

$\therefore \quad 2a + 5c = -8 \Rightarrow -14 - 8c + 5c = -8$

or $\quad -3c = 6$

or $\quad c = -2$

$\therefore \quad a = -7 - 4(-2) = -7 + 8 = 1$

Now, $b + 4d = 2$ or $b = 2 - 4d$

$\therefore \quad 2b + 5d = 4 \Rightarrow 4 - 8d + 5d = 4$

$\Rightarrow \quad -3d = 0$

$d = 0$

$\therefore \quad b = 2 - 4(0) = 2$

Thus, $a = 1, b = 2, c = -2, d = 0$

Hence, the required matrix X is $\begin{bmatrix}1 & -2 \\ 2 & 0\end{bmatrix}$

9. $A = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$

$$A^2 = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 3\theta & \sin 3\theta \\ -\sin 3\theta & \cos 3\theta\end{bmatrix}$$

Hence, $A^n = \begin{bmatrix}\cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta\end{bmatrix}$

or $\dfrac{A^n}{n} = \begin{bmatrix} \dfrac{\cos n\theta}{n} & \dfrac{\sin n\theta}{n} \\ \dfrac{-\sin n\theta}{n} & \dfrac{\cos n\theta}{n} \end{bmatrix}$

or $\lim\limits_{n\to\infty} \dfrac{A^n}{n} = \begin{bmatrix} \lim\limits_{n\to\infty} \dfrac{\cos n\theta}{n} & \lim\limits_{n\to\infty} \dfrac{\sin n\theta}{n} \\ -\lim\limits_{n\to\infty} \dfrac{\sin n\theta}{n} & \lim\limits_{n\to\infty} \dfrac{\cos n\theta}{n} \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ = Zero matrix

10. A is symmetric

$\Rightarrow \quad A^T = A$

$\Rightarrow \quad \begin{bmatrix} 3 & 2 & b \\ a & 5 & 8 \\ -1 & c & 2 \end{bmatrix} = \begin{bmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{bmatrix}$

$\Rightarrow \quad a = 2, b = -1, c = 8$

B is skew-symmetric

$\Rightarrow \quad B^T = -B$

$\Rightarrow \quad \begin{bmatrix} d & b-a & -2 \\ 3 & e & 6 \\ a & -2b-c & -f \end{bmatrix} = \begin{bmatrix} -d & -3 & -a \\ a-b & -e & 2b+c \\ 2 & -6 & f \end{bmatrix}$

$\Rightarrow \quad d = -d, f = -f$ and $e = -e$

$\Rightarrow \quad d = f = 0$

So $A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$\Rightarrow \quad AB = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$= \begin{bmatrix} -4 & 3 & -6 \\ -31 & 54 & -26 \\ -28 & 9 & -50 \end{bmatrix}$

Exercise 13.4

1. Let $P = (AB^T - BA^T)$

$\therefore \quad P^T = (AB^T - BA^T)^T = (AB^T)^T - (BA^T)^T$

$= (B^T)^T (A)^T - (A^T)^T B^T = BA^T - AB^T$

$= -(AB^T - BA^T) = -P$

Hence, $(AB^T - BA^T)$ is a skew-symmetric matrix.

2. $(A - B)(A^2 + AB + B^2)$

$= A^3 + A^2B + AB^2 - BA^2 - BAB - B^3$

Now $A^2B = A(AB) = A(BA) = (AB)A = BAA = BA^2$

And $AB^2 = (AB)B = (BA)B$

$\therefore \quad (A - B)(A^2 + AB + B^2) = A^3 - B^3$

3. We have, $A^2 = I$

Now, A and I are commutative.

$(A - I)^3 + (A + I)^3 - 7A$

$= A^3 - 3A^2I + 3AI - I^3 + A^3 + 3A^2I + 3AI + I^3 - 7A$

$= 2A^3 + 6A - 7A$

$= 2A^2 \times A - A = 2I \times A - A = 2A - A = A$

4. $A^{p+1} = (B + C)^{p+1}$

Since $BC = CB$, we have

$(B + C)^{p+1}$

$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 + \ldots + {}^{p+1}C_{p+1} C^{p+1}$

$= {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + O + O + \ldots + O \quad (\because C^2 = O \Rightarrow C^3 = C^4 = \ldots = O)$

$= B^{p+1} + (p+1)B^p C$

$= B^p[B + (p+1)C]$

5. Let $A = \begin{bmatrix} a & p \\ b & q \\ c & r \end{bmatrix}$

$\therefore \quad A^T = \begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix}$

$\therefore \quad AA^T = \begin{bmatrix} a^2 + p^2 & ab + pq & ac + pr \\ ab + pq & b^2 + q^2 & bc + qr \\ ac + pr & bc + qr & c^2 + r^2 \end{bmatrix}$

$$\det(AA^T) = \begin{vmatrix} a & p & 0 \\ b & q & 0 \\ c & r & 0 \end{vmatrix} \begin{vmatrix} a & b & c \\ p & q & r \\ 0 & 0 & 0 \end{vmatrix} = 0$$

6. $\det.(A^2 - I) = \det.(AA - A^TA)$

$= \det.[(A - A^T)A]$

$= \det.(A - A^T)\ \det.\ A$

Now, $A - A^T$ is skew-symmetric matrix.

$\therefore \quad \det.(A - A^T) = 0$

$\Rightarrow \quad \det.(A^2 - I) = 0$

7. $(A^2 + B^2)(A - B) = A^3 - A^2B + B^2A - B^3 = O$

$\therefore \quad \det.[(A^2 + B^2)(A - B)] = 0$

$\Rightarrow \quad \det.(A^2 + B^2) \times \det.(A - B) = 0$

$\Rightarrow \quad \det.(A^2 + B^2) = 0 \qquad (\text{as } \det.(A - B) \neq 0)$

8. Let

$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

$\Rightarrow \quad f(D) = a_0I + a_1D + a_2D^2 + \ldots + a_nD^n$

$= a_0 \times \text{diag.}(1, 1, \ldots, 1) + a_1 \times \text{diag.}(d_1, d_2, \ldots, d_n)$

$+ a_2 \times \text{diag.}(d_1^2, d_2^2, \ldots, d_n^2)$

$\ldots$

$\ldots$

$\ldots$

$+ a_n \times \text{diag.}(d_1^n, d_2^n, \ldots, d_n^n)$

$= \text{diag.}(a_0 + a_1d_1 + a_2d_1^2 + \ldots + a_nd_1^n,$

$a_0 + a_1d_2 + a_2d_2^2 + \ldots + a_nd_2^n,$

$a_0 + a_1d_3 + a_2d_3^2 + \ldots + a_nd_3^n,$

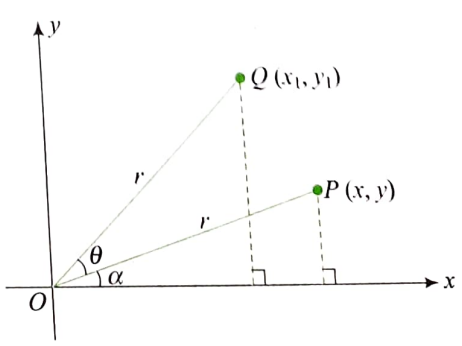
$\ldots$

$\ldots$

$a_0 + a_1d_n + a_2d_n^2 + \ldots + a_nd_n^n)$

$= \text{diag.}(f(d_1), f(d_2), \ldots, f(d_n))$

9.



From the figure, $x_1 = r\cos(\theta+\alpha)$

$= (r\cos\alpha)\cos\theta - (r\sin\alpha)\sin\theta$

$\therefore \quad x_1 = x\cos\theta - y\sin\theta$

and $y_1 = r\sin(\theta+\alpha) = (r\sin\alpha)\cos\theta + (r\cos\alpha)\sin\theta$

$\therefore \quad y_1 = x\sin\theta + y\cos\theta$

So, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore \quad A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

10. Matrix A is diagonal matrix.

$\therefore \quad A = \text{dia.}\ (a_1, a_2, \ldots, a_n)$

$\Rightarrow \quad A^2 = \text{dia.}\ ((a_1)^2, (a_2)^2, \ldots, (a_n)^2)$

Since A is idempotent, we have

$\therefore \quad A^2 = A$

$\Rightarrow \quad (a_1)^2 = a_1, (a_2)^2 = a_2, \ldots, (a_n)^2 = a_n.$

$\Rightarrow \quad a_1, a_2, \ldots, a_n = 0, 1.$

Thus, number of required matrix A is $2\times2\times2\times\ldots$ n times $= 2^n$.

11. Matrix A is diagonal matrix.

$\therefore \quad A = \text{dia.}\ (a_1, a_2, \ldots, a_n)$

$\Rightarrow \quad A^2 = \text{dia.}\ ((a_1)^2, (a_2)^2, \ldots, (a_n)^2)$

Since A involuntary, we have

$\therefore \quad A^2 = I$

$\Rightarrow \quad (a_1)^2, (a_2)^2, \ldots, (a_n)^2 = 1$

$\Rightarrow \quad a_1, a_2, \ldots, a_n = 1$

Thus, number of required matrix A is $2\times2\times2\times\ldots$ n times $= 2^n$.

12. A and B are two orthogonal matrices. Hence,

$A^TA = AA^T = I$ and $BB^T = B^TB = I$

AB will be orthogonal if

$$(AB)^T(AB) = (B^TA^T)(AB)$$
$$= B^T(A^TA)B$$
$$= B^TIB$$
$$= B^TB = I$$

Similarly, we can show that $(BA)^T(BA) = I$. Therefore, BA is also orthogonal.

Exercise 13.5

1. We have

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$\Rightarrow \quad AX = B \qquad (1)$

Clearly $|A| = -4 \neq 0$. Therefore,

$$\therefore \quad \text{adj}\ A = \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & 1 & 3 \end{bmatrix}^T = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj.}\ A}{|A|} = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\text{Now,}\ A^{-1}B = \frac{-1}{4}\begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= \frac{-1}{4}\begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From Eq. (1), we get

$X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

$\Rightarrow \quad x_1 = 1, x_2 = 3, x_3 = 5$

or $\quad y_1 = -1, y_2 = 2, y_3 = 1$

2. We have

$$AB = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad x = 1/5$

3. We have,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Cofactors are

$A_{11} = -1, A_{12} = 1, A_{13} = 1,$

$A_{21} = 1, A_{22} = -1, A_{23} = 1,$

$A_{31} = 1, A_{31} = 1, A_{32} = 1\ A_{33} = -1$

$$\therefore \quad \text{adj}\ A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$|A| = 0 - 1(-1) + 1\cdot1 = 2$

$$\therefore \quad A^{-1} = \frac{\text{adj}\ A}{|A|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{Now}\ A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\therefore \quad \frac{A^2 - 3I}{2} = \frac{1}{2}\left\{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = A^{-1}$$

Hence proved.

4. We have

$$A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9+7 & 3+5 \\ 21+35 & 7+25 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Now, $A^2 + xI = yA$

$$\Rightarrow \quad \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 16+x & 8+0 \\ 56+0 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix}$$

$16 + x = 3y, y = 8, 7y = 56, 5y = 32 + x$

Putting $y = 8$ in $16 + x = 3y$, we get $x = 24 - 16 = 8$. Clearly, $x = 8$ and $y = 8$ also satisfy $7y = 56$ and $5y = 32 + x$. Hence, $x = 8$ and $y = 8$. We have

$$|A| = \begin{vmatrix} 3 & 1 \\ 7 & 5 \end{vmatrix} = 8 \neq 0$$

So, A is invertible.

Putting $x = 8$, $y = 8$ in $A^2 + xI = yA$, we get

$$A^2 + 8I = 8A$$

$$\Rightarrow \quad A^{-1}(A^2 + 8I) = 8A^{-1}A$$

$$A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$$

$$A + 8A^{-1} = 8I$$

$[\because A^{-1}A^2 = (A^{-1}A)A = IA = A,\ A^{-1}I = A^{-1} \text{ and } A^{-1}A = I]$

$$8A^{-1} = 8I - A$$

or $$A^{-1} = \frac{1}{8}(8I - A) = \frac{1}{8}\left\{\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\right\}$$

$$= \frac{1}{8}\begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} = \begin{bmatrix} 5/8 & -1/8 \\ -7/8 & 3/8 \end{bmatrix}$$

5. Given $A^3 = O$

Now, $(I - A)(I + A + A^2)$

$$= I^2 + IA + IA^2 - AI - A^2 - A^3$$

$$= I + A + A^2 - A - A^2 - O$$

$$= I$$

$$\therefore \quad (I - A)^{-1} = I + A + A^2$$

6. $BAB = A^{-1}$

$$\Rightarrow \quad ABAB = I$$

$$\Rightarrow \quad (AB)^2 = I$$

Now,

$$AB = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha + 2\beta) & \sin(\alpha + 2\beta) \\ \sin(\alpha + 2\beta) & -\cos(\alpha + 2\beta) \end{bmatrix}$$

and $$(AB)^2 = \begin{bmatrix} \cos(\alpha + 2\beta) & \sin(\alpha + 2\beta) \\ \sin(\alpha + 2\beta) & -\cos(\alpha + 2\beta) \end{bmatrix} \times \begin{bmatrix} \cos(\alpha + 2\beta) & \sin(\alpha + 2\beta) \\ \sin(\alpha + 2\beta) & -\cos(\alpha + 2\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2(\alpha + 2\beta) + \sin^2(\alpha + 2\beta) & 0 \\ 0 & \cos^2(\alpha + 2\beta) + \sin^2(\alpha + 2\beta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

$$BA^4B = A^{-1}$$

or $$A^4B = B^{-1}A^{-1} = (AB)^{-1} = AB$$

or $$A^4 = A \qquad (1)$$

Now, $$A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

and $$A^4 = \begin{bmatrix} \cos 4\alpha & -\sin 4\alpha \\ \sin 4\alpha & \cos 4\alpha \end{bmatrix}$$

Hence, from Eq. (1),

$$\begin{bmatrix} \cos 4\alpha & -\sin 4\alpha \\ \sin 4\alpha & \cos 4\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

or $$4\alpha = 2\pi + \alpha$$

or $$\alpha = \frac{2\pi}{3}$$

7. Given that $CB = D$. Now,

$$AX = B$$

or $$CAX = CB = D$$

or $$X = (CA)^{-1}D$$

Now, $$CA = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 10 \\ 7 & 7 & 13 \\ 4 & 3 & 8 \end{bmatrix}$$

and $$(CA)^{-1} = \begin{bmatrix} -\frac{17}{5} & 2 & 1 \\ \frac{4}{5} & 0 & -1 \\ \frac{7}{5} & -1 & 0 \end{bmatrix}$$

$$\Rightarrow \quad (CA)^{-1}D = \begin{bmatrix} -\frac{17}{5} & 2 & 1 \\ \frac{4}{5} & 0 & -1 \\ \frac{7}{5} & -1 & 0 \end{bmatrix}\begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

8. We have, $A^2 - 4A + 3I = O$

$$\Rightarrow \quad A(A + 3I) - 7A + 3I = O$$

$$\Rightarrow \quad A(A + 3I) - 7(A + 3I) = -24I$$

$$\Rightarrow \quad (A + 3I)(A - 7I) = -24I$$

$$\Rightarrow \quad (A + 3I)\left(\frac{7}{24}I - \frac{A}{24}\right) = I$$

$$\Rightarrow \quad (A + 3I)^{-1} = \frac{7}{24}I - \frac{A}{24}$$

9. $$\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1}\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$$

$$= \left(\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 & 0 \\ 0 & z_2\bar{z}_2 + z_1\bar{z}_1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} |z_1|^2 + |z_2|^2 & 0 \\ 0 & |z_1|^2 + |z_2|^2 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

10. A is skew-symmetric, then $A^T = -A$.

$$\therefore \quad (A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -A^{-1}$$

Thus, A^{-1} is skew-symmetric.

11. We have $AA^T = I$

Now, $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^TA)^{-1} = I^{-1} = I$

Thus, A^{-1} is orthogonal.

12. We have $A = (I + S)(I - S)^{-1}$

$\therefore \quad A^T = [(I - S)^{-1}]^T (I + S)^T = [(I - S)^T]^{-1} (I + S)^T$

But $(I - S)^T = I^T - S^T = I + S \qquad (\text{as } S^T = -S)$

and $(I + S)^T = I^T + S^T = I - S$

$\therefore \quad A^T = (I + S)^{-1}(I - S)$

$\therefore \quad A^TA = (I + S)^{-1}(I - S)(I + S)(I - S)^{-1}$

$= (I + S)^{-1}(I + S)(I - S)(I - S)^{-1}$

$= I$

Thus, A is orthogonal.

13. $(\text{adj. } A)^{-1} = \dfrac{\text{adj.}(\text{adj. } A)}{|\text{adj. } A|} = \dfrac{|A|^{n-2} A}{|A|^{n-1}} = \dfrac{A}{|A|} = (\text{adj. } A^{-1})$

14. $A = \begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix}$

We write,

$$\begin{bmatrix} a & b \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ c & \left(\dfrac{1+bc}{a}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ 0 & 1 \end{bmatrix} A \qquad \left(R_1 \to \frac{R_1}{a}\right)$$

$$\text{or} \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & \dfrac{1}{a} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ \dfrac{-c}{a} & 1 \end{bmatrix} A \qquad (R_2 \to R_2 - cR_1)$$

$$\text{or} \quad \begin{bmatrix} 1 & \dfrac{b}{a} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a} & 0 \\ -c & a \end{bmatrix} A \qquad (R_2 \to aR_2)$$

$$\text{or} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix} A \qquad \left(R_1 \to R_1 - \frac{b}{a}R_2\right)$$

$$\Rightarrow \quad A^{-1} = \begin{bmatrix} \dfrac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

15. Let

$P^{-1}AP = B$

$\therefore \quad B - \lambda I = P^{-1}AP - \lambda I$

$= P^{-1}AP - P^{-1}\lambda IP$

$= P^{-1}(A - \lambda I)P$

$\Rightarrow \quad |B - \lambda I| = |P^{-1}||A - \lambda I||P|$

$= |A - \lambda I||P^{-1}||P|$

$= |A - \lambda I||P^{-1}P|$

$= |A - \lambda I||I| = |A - \lambda I|$

Thus, the two matrices A and B have the same characteristic determinants and hence the same characteristic equations and the same characteristic roots. The same thing may also be seen in another way. Now,

$AX = \lambda X$

$\Rightarrow \quad P^{-1}AX = \lambda P^{-1}X$

$\Rightarrow \quad (P^{-1}AP)(P^{-1}X) = \lambda(P^{-1}X)$

16. Since A is an idempotent matrix, we have

$A^2 = A$

Let X be a latent vector of the matrix A corresponding to the latent root λ so that

$AX = \lambda X \qquad (1)$

or $(A - \lambda I)X = O$

such that

$X \neq O$

On pre-multiplying Eq. (1) by A, we get

$A(AX) = A(\lambda X) = \lambda(AX)$

i.e., $(AA)X = \lambda(AX)$

or $AX = \lambda^2 X$

or $\lambda X = \lambda^2 X \qquad (\because A^2 = A)$

or $(\lambda^2 - \lambda)X = O \qquad (\because AX = \lambda X)$

or $\lambda^2 - \lambda = 0$

or $\lambda(\lambda - 1) = 0 \qquad (\because X \neq 0)$

$\Rightarrow \quad \lambda = 0, \lambda = 1$

17. Since α is a characteristic root of a nonsingular matrix, therefore $\alpha \neq 0$. Also α is a characteristic root of A implies that there exists a nonzero vector X such that

$AX = \alpha X$

or $(\text{adj } A)(AX) = (\text{adj } A)(\alpha X)$

or $[(\text{adj } A)A]X = \alpha(\text{adj } A)X$

or $|A|IX = \alpha(\text{adj } A)X \qquad [\because (\text{adj } A)A = |A|I]$

or $|A|X = \alpha(\text{adj } A)X$

or $\dfrac{|A|}{\alpha}X = (\text{adj } A)X$

or $(\text{adj } A)X = \dfrac{|A|}{\alpha}X$

Since X is a nonzero vector, $|A/\alpha|$ is a characteristic root of the matrix adj A.

Exercises

Single Correct Answer Type

1. (2) Let $A = [a_{ij}]$. Since A is skew-symmetric, we have

$a_{ii} = 0$ and $a_{ij} = -a_{ij}$ $(i \neq j)$

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j.

$\therefore \quad a_{ij} = 0$ for all $i \neq j$

Hence, $a_{ij} = 0$ for all i and j, i.e., A is a null matrix.

2. (4) $\text{tr}(A) = \sum_{i=j} a_{ij}$

$= (a_{11} + a_{22} + a_{33} + \cdots + a_{1010})$

$= (\omega^2 + \omega^4 + \omega^6 + \cdots + \omega^{20})$

$= \omega^2(1 + \omega^2 + \omega^4 + \cdots + \omega^{18})$

$= \omega^2[(1 + \omega + \omega^2) + \cdots + (1 + \omega + \omega^2) + 1]$

$= \omega^2 \times 1$

$= \omega^2$

3. (2) $B = A_1 + 3A_3^3 + \cdots + (2n-1)(A_{2n-1})^{2n-1}$

$B^T = -[A_1 + 3A_3^3 + \cdots + (2n-1)(A_{2r-1})^{2r-1}]$

$= -B$

Hence, B is skew-symmetric.

4. (3) $[1\ x\ y]\begin{bmatrix}1 & 3 & 1\\ 0 & 2 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix}=10$

or $[1\ \ 3+2x\ \ 1-x+y]\begin{bmatrix}1\\ x\\ y\end{bmatrix}=[0]$

or $[1+3x+2x^2+y-xy+y^2]=[0]$

or $2x^2+y^2+y+3x-xy+1=0$

If $y=0$, $2x^2+3x+1=0$

or $(2x+1)(x+1)=0$

or $x=-1/2, -1$ (rational roots)

If $y=-1$, $2x^2+4x+1=0$

or $x=\dfrac{-4\pm\sqrt{12}}{4}=\dfrac{-2\pm\sqrt{3}}{2}$ (irrational roots)

5. (4) (i) is false.

If $A=\begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, then $AB=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=O$

(ii) is true as the product AB is an identity matrix, if and only if B is inverse of the matrix A.

(iii) is false since matrix multiplication in not commutative.

6. (4) $A=\text{diag}\,(d_1, d_2, \ldots, d_n)$

Given, $A^3=A$

$\Rightarrow$ $\text{diag}\,(d_1^3, d_2^3, \ldots, d_n^3)=\text{diag}\,(d_1, d_2, \ldots, d_n)$

$\Rightarrow$ $d_1^3=d_1, d_2^3=d_2, \ldots, d_n^3=d_n$

Hence, all $d_1, d_2, d_3, \ldots, d_n$ have three possible values ±1, 0. Each diagonal element can be selected in three ways. Hence, the number of different matrices is 3^n.

7. (4) $A\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$ (1)

$A^2\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}$ (2)

Let A be given by $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. The first equation gives

$a-b=-1$ (3)

$c-d=2$ (4)

For the second equation,

$$A^2\begin{bmatrix}1\\ -1\end{bmatrix}=A\left(A\begin{bmatrix}1\\ -1\end{bmatrix}\right)=A\left(\begin{bmatrix}-1\\ 2\end{bmatrix}\right)=\begin{bmatrix}1\\ 0\end{bmatrix}$$

This gives

$-a+2b=1$ (5)

$-c+2d=0$ (6)

Eqs. (3) + (5) gives $b=0$ and $a=-1$

Eqs. (4) + (6) gives $d=2$ and $c=4$

So the sum $a+b+c+d=5$.

8. (3) $AB=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}\begin{bmatrix}\cos^2\phi & \cos\phi\sin\phi\\ \cos\phi\sin\phi & \sin^2\phi\end{bmatrix}$

$=\begin{bmatrix}\cos^2\theta\cos^2\phi+\cos\theta\cos\phi\sin\theta\sin\phi & \cos^2\theta\cos\phi\sin\phi+\cos\theta\sin\theta\sin^2\phi\\ \cos\theta\sin\theta\cos^2\phi+\sin^2\theta\cos\phi\sin\phi & \cos\theta\cos\phi\sin\theta\sin\phi+\sin^2\theta\sin^2\phi\end{bmatrix}$

$=\begin{bmatrix}\cos\theta\cos\phi(\cos(\theta-\phi)) & \cos\theta\sin\phi(\cos(\theta-\phi))\\ \sin\theta\cos\phi(\cos(\theta-\phi)) & \sin\theta\sin\phi(\cos(\theta-\phi))\end{bmatrix}$

$=(\cos(\theta-\phi))\begin{bmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi\\ \sin\theta\cos\phi & \sin\theta\sin\phi\end{bmatrix}$

Now, $AB=O\Rightarrow\cos(\theta-\phi)=0$

$\Rightarrow$ $\theta-\phi=(2n+1)\pi/2, n\in Z$.

9. (4) If A is nth root of I_2, then $A^n=I_2$. Now,

$$A^2=\begin{bmatrix}a & b\\ 0 & a\end{bmatrix}\begin{bmatrix}a & b\\ 0 & a\end{bmatrix}=\begin{bmatrix}a^2 & 2ab\\ 0 & a^2\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}a^2 & 2ab\\ 0 & a^2\end{bmatrix}\begin{bmatrix}a & b\\ 0 & a\end{bmatrix}=\begin{bmatrix}a^3 & 3a^2b\\ 0 & a^3\end{bmatrix}$$

Thus, $A^n=\begin{bmatrix}a^n & na^{n-1}b\\ 0 & a^n\end{bmatrix}$

Now,

$$A^n=I\Rightarrow\begin{bmatrix}a^n & na^{n-1}b\\ 0 & a^n\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$\Rightarrow$ $a^n=1, b=0$

10. (2) We have

$$\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

or $\begin{bmatrix}\alpha^2+\beta\gamma & 0\\ 0 & \alpha^2+\beta\gamma\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

or $\alpha^2+\beta\gamma-1=0$

11. (2) We have,

$A=iB$

or $A^2=(iB)^2=i^2B^2=-B^2=-\begin{bmatrix}2 & -2\\ -2 & 2\end{bmatrix}=-2B$

or $A^4=(-2B)^2=4B^2=4(2B)=8B$

or $(A^4)^2=(8B)^2$

or $A^8=64B^2=128B$

12. (2) Since the product matrix is 3×3 matrix and the pre-multiplier of A is a 3×2 matrix, therefore A is 2×3 matrix. Let

$A=\begin{bmatrix}l & m & n\\ x & y & z\end{bmatrix}$. Then the given equation becomes

$$\begin{bmatrix}2 & -1\\ 1 & 0\\ -3 & 4\end{bmatrix}\begin{bmatrix}l & m & n\\ x & y & z\end{bmatrix}=\begin{bmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{bmatrix}$$

or $\begin{bmatrix}2l-x & 2m-y & 2n-z\\ l & m & n\\ -3l+4x & -3m+4y & -3n+4z\end{bmatrix}=\begin{bmatrix}-1 & -8 & -10\\ 1 & -2 & -5\\ 9 & 22 & 15\end{bmatrix}$

or $2l-x=-1, 2m-y=-8, 2n-z=-10,$

$l=1, m=-2, n=-5$

$\Rightarrow$ $x=3, y=4, z=0, l=1, m=-2, n=-5$

$\Rightarrow$ $A=\begin{bmatrix}l & m & n\\ x & y & z\end{bmatrix}=\begin{bmatrix}1 & -2 & -5\\ 3 & 4 & 0\end{bmatrix}$

13. (1) $A(x)A(y) = (1-x)^{-1}(1-y)^{-1}\begin{bmatrix}1 & -x\\ -x & 1\end{bmatrix}\begin{bmatrix}1 & -y\\ -y & 1\end{bmatrix}$

$= (1+xy-(x+y))^{-1}\begin{bmatrix}1+xy & -(x+y)\\ -(x+y) & 1+xy\end{bmatrix}$

$= \left(1-\frac{x+y}{1+xy}\right)^{-1}\begin{bmatrix}1 & -\frac{x+y}{1+xy}\\ -\frac{x+y}{1+xy} & 1\end{bmatrix} = A(z)$

14. (2) Since $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and given $A=\begin{bmatrix}0 & \tan\alpha/2\\ -\tan\alpha/2 & 0\end{bmatrix}$

$\therefore \quad I-A=\begin{bmatrix}1 & -\tan\alpha/2\\ \tan\alpha/2 & 1\end{bmatrix}$ (1)

Now, $(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}1 & \tan\alpha/2\\ -\tan\alpha/2 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}1 & \tan\alpha/2\\ -\tan\alpha/2 & 1\end{bmatrix}\begin{bmatrix}\frac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\frac{2\tan\alpha/2}{1+\tan^2\alpha/2}\\ \frac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \frac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$

$=\begin{bmatrix}\frac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}+\frac{2\tan^2\alpha/2}{1+\tan^2\alpha/2} & -\frac{2\tan\alpha/2}{1+\tan^2\alpha/2}+\frac{\tan\alpha/2(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}\\ \frac{-\tan\alpha/2(1-\tan^2\alpha/2)}{1+\tan^2\alpha/2}+\frac{2\tan\alpha/2}{1+\tan^2\alpha/2} & \frac{2\tan^2\alpha/2}{1+\tan^2\alpha/2}+\frac{1-\tan^2\alpha/2}{1+\tan^2\alpha/2}\end{bmatrix}$

$=\begin{bmatrix}\frac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} & -\frac{\tan\alpha/2(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\\ \frac{\tan\alpha/2(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)} & \frac{(1+\tan^2\alpha/2)}{(1+\tan^2\alpha/2)}\end{bmatrix}$

$=\begin{bmatrix}1 & -\tan\alpha/2\\ \tan\alpha/2 & 1\end{bmatrix}$

$= I-A$ [Using (1)]

15. (1) Let $X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

$\Rightarrow \quad X^2=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}$

$\Rightarrow \quad a^2+bc=1$ and $ab+bd=1 \Rightarrow b(a+d)=1$

$ac+cd=2 \Rightarrow c(a+d)=2 \Rightarrow 2b=c$

Also,

$bc+d^2=3 \Rightarrow d^2-a^2=2$

$\Rightarrow \quad (d-a)(a+d)=2 \quad \Rightarrow \quad d-a=2b$

$a+d=1/b$

$\Rightarrow \quad 2d=2b+1/b, \quad 2a=1/b-2b$

$d=b+1/2b, \quad a=1/(2b)-b$

$c=2b$

$\Rightarrow \quad \left(b^2+\frac{1}{4b^2}+1\right)+2b^2=3$

$\Rightarrow \quad 3b^2+\frac{1}{4b^2}=2$

$\Rightarrow \quad 12b^2-8b^2+1=0$

$\Rightarrow \quad (6b^2-1)(2b^2-1)=0$

or $\quad b=\pm\frac{1}{\sqrt{6}}$ or $b=\pm\frac{1}{\sqrt{2}}$

Therefore, matrices are

$$\begin{pmatrix}0 & 1/\sqrt{2}\\ \sqrt{2} & \sqrt{2}\end{pmatrix},\begin{pmatrix}0 & -1/\sqrt{2}\\ -\sqrt{2} & -\sqrt{2}\end{pmatrix},\begin{pmatrix}2/\sqrt{6} & -1/\sqrt{6}\\ 2/\sqrt{6} & 4\sqrt{6}\end{pmatrix}$$

16. (3) We have,

$$A^2=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a^2+bc & ab+db\\ ac+cd & bc+d^2\end{bmatrix}$$

As A satisfies $x^2+k=0$, we have

$A^2+kI=O$

or $\quad \begin{bmatrix}a^2+bc+k & (a+d)b\\ (a+d)c & bc+d^2+k\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

or $\quad a^2+bc+k=0, bc+d^2+k=0$

and $\quad (a+d)b=(a+d)c=0$

As $\quad bc\neq 0, b\neq 0, c\neq 0$, so

$a+d=0$

or $\quad a=-d$

Also,

$k=-(a^2+bc)$

$=-(d^2+bc)$

$=-((-ad)+bc)$

$=|A|$

17. (1) $BC=\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix}\Rightarrow B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$\Rightarrow \quad \text{tr}(A)+\text{tr}\left(\frac{ABC}{2}\right)+\text{tr}\left(\frac{A(BC)^2}{4}\right)+\text{tr}\left(\frac{A(BC)^3}{8}\right)+\cdots+\infty$

$=\text{tr}(A)+\text{tr}\left(\frac{A}{2}\right)+\text{tr}\left(\frac{A}{2^2}\right)+\cdots$

$=\text{tr}(A)+\frac{1}{2}\text{tr}(A)+\frac{1}{2^2}\text{tr}(A)\cdots$

$=\frac{\text{tr}(A)}{1-(1/2)}$

$=2\text{tr}(A)=2(2+1)=6$

18. (3) We know that $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}^n=\begin{bmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{bmatrix}$

$\therefore \quad \begin{bmatrix}\cos\frac{2\pi}{7} & -\sin\frac{2\pi}{7}\\ \sin\frac{2\pi}{7} & \cos\frac{2\pi}{7}\end{bmatrix}^k=\begin{bmatrix}\cos\frac{2k\pi}{7} & -\sin\frac{2k\pi}{7}\\ \sin\frac{2k\pi}{7} & \cos\frac{2k\pi}{7}\end{bmatrix}$

$=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, if $k=7m$, where $m\in N$

Hence, the least value of k is 7.

19. (1) $(A-\lambda I)(B-\lambda I)=(B-\lambda I)(A-\lambda I)$

or $AB-\lambda(A+B)I+\lambda^2I^2=BA-\lambda(B+A)I+\lambda^2I^2$

or $AB=BA$

20. (3) Given, $A^2=2A-I$

Now, $A^3=A(A^2)$

$=A(2A-I)$

$=2A^2-A$

$=2(2A-I)-A$

$=3A-2I$

$A^4=A(A^3)$

$=A(3A-2I)$

$=3A^2-2A$

$=3(2A-I)-2A$

$=4A-3I$

Following this, we can say $A^n=nA-(n-1)I$.

21. (1) We have, $A^2=O$, $A^k=O$, $\forall\ k\geq 2$

Thus,

$(A+I)^{50}=I+50A$

$\Rightarrow$ $a=1, b=0, c=0, d=1$

22. (2) Z is idempotent, then

$Z^2=Z\Rightarrow Z^3, Z^4, \ldots, Z^n=Z$

$\therefore$ $(I+Z)^n={}^nC_0I^n+{}^nC_1I^{n-1}Z+{}^nC_2I^{n-2}Z^2+\ +{}^nC_nZ^n$

$={}^nC_0I+{}^nC_1Z+{}^nC_2Z+{}^nC_3Z+\ +{}^nC_nZ$

$=I+({}^nC_1+{}^nC_2+{}^nC_3+\ +{}^nC_n)Z$

$=I+(2^n-1)Z$

23 (1) $AB=A+B$

or $B=AB-A=A(B-I)$

or $\det(B)=\det(A)\det(B-I)=0$

or $\det(B)=0$

24. (2) $|A^{2005}-6A^{2004}|=|A|^{2004}|A-6I|$

$=2^{2004}\begin{vmatrix}0 & 11\\ 2 & -2\end{vmatrix}=(-22)\,2^{2004}=(-11)\,(2)^{2005}$

25. (3) $A^2=I$

or $A^2-I=O$

or $(A+I)(A-I)=0$

Therefore, either $|A+I|=0$ or $|A-I|=0$. If $|A-I|\neq 0$, then $(A+I)(A-I)=O\Rightarrow A+I=O$ which is not so.

$\therefore$ $|A-I|=0$ and $A-I\neq O$

26. (1) If possible assume that A is nonsingular, then A^{-1} exists. Thus,

$AB=O\Rightarrow A^{-1}(AB)=(A^{-1}A)B=O$

$\Rightarrow$ $IB=O$ or $B=O$ (contradiction)

Hence, both A and B must be singular.

27. (3) Given that

$X=AB+BA\Rightarrow X=X^T$ ($\because$ $A=A^T$ and $B=B^T$)

and $Y=AB-BA$

$\Rightarrow$ $Y=-Y^T$

Now, $(XY)^T=Y^TX^T=-YX$.

28. (2) Given $A, B, A+I, A+B$ are idempotent. Hence,

$A^2=A, B^2=B, (A+I)^2=A+I$ and $(A+B)^2=A+B$

$\Rightarrow$ $A^2+B^2+AB+BA=A+B$

$\Rightarrow$ $A+B+AB+BA=A+B$

$\Rightarrow$ $AB+BA=O$

29. (2) $A^3+A=O$

$\Rightarrow$ $A(A^2+I)=O$

$\Rightarrow$ $|A(A^2+I)|=0$

$\Rightarrow$ $|A|=0$ or $|A^2+I|=0$

If $|A|=0$, then $xy=0$

$A^2+I=\begin{bmatrix}0 & x\\ y & 0\end{bmatrix}\begin{bmatrix}0 & x\\ y & 0\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$=\begin{bmatrix}xy & 0\\ 0 & xy\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

$=\begin{bmatrix}xy+1 & 0\\ 0 & xy+1\end{bmatrix}$

$|A^2+I|=0$

$\Rightarrow$ $(xy+1)^2=0$

$\Rightarrow$ $xy=-1$

So, sum of possible values of xy is -1.

30. (1) Matrix $\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$ is orthogonal if

$$\sum a_i^2=\sum b_i^2=\sum c_i^2=1;\ \sum a_ib_i=\sum b_ic_i=\sum c_ia_i=0$$

31. (2) Let $C=BA$, then

$C^T=(BA)^T=A^TB^T$

$=-AB^T$ (as A is skew-symmetric)

$=BA^T$ (as $AB^T+BA^T=O$)

$=B(-A)$

$=-BA=-C$

Therefore, C is skew-symmetric.

32. (1) For involuntary matrix,

$A^2=I$

$\Rightarrow$ $|A^2|=|I|\Rightarrow|A|^2=1\Rightarrow|A|=\pm1$

For idempotent matrix,

$A^2=A$

$\Rightarrow$ $|A^2|=|A|\Rightarrow|A|^2=|A|\Rightarrow|A|=0$ or 1

For orthogonal matrix,

$AA^T=I$

$\Rightarrow$ $|AA^T|=|I|\Rightarrow|A|\,|A^T|=1\Rightarrow|A|^2=1\Rightarrow|A|=\pm1$

Thus, if matrix A is idempotent it may not be invertible.

33. (2) We have, $AB=A(\text{adj }A)=|A|\,I_n$

$\therefore$ $AB+KI_n=|A|I_n+KI_n$

$=(|A|+K)I_n$

$\Rightarrow$ $|AB+KI_n|=|(|A|+K)I_n|$ ($\because$ $|\alpha I_n|=\alpha^n$)

$=(|A|+K)^n$

34. (1) $|B|=\begin{vmatrix}q & -b & y\\ -p & a & -x\\ r & -c & z\end{vmatrix}$ (Multiplying R_2 by -1)

$=-\begin{vmatrix}q & -b & y\\ p & -a & x\\ r & -c & z\end{vmatrix}$ (Multiplying C_2 by -1)

$=\begin{vmatrix}q & b & y\\ p & a & x\\ r & c & z\end{vmatrix}$ (Changing R_1 with R_2)

$$= -\begin{vmatrix} p & a & x \\ q & b & y \\ r & c & z \end{vmatrix}$$

$$= -\begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix}$$

Hence, $|A| = -|B|$, obviously when $|A| \neq 0$, $|B| \neq 0$.
Also, $|\text{adj } B| = |B|^2 = (-|A|)^2 = |A|^2$.

35. (1) We have,

$$A(\alpha, \beta)^{-1} = \frac{1}{e^{\beta}} \begin{bmatrix} e^{\beta}\cos\alpha & -e^{\beta}\sin\alpha & 0 \\ e^{\beta}\sin\ \alpha & e^{\beta}\cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= A(-\alpha, -\beta)$$

36. (1) We have,

$$|A| = (a + ib)(a - ib) - (-c + id)(c + id)$$
$$= a^2 + b^2 + c^2 + d^2 = 1$$

and $\text{adj}(A) = \begin{bmatrix} a - ib & -c - id \\ c - id & a + ib \end{bmatrix}$

Then $A^{-1} = \begin{bmatrix} a - ib & -c - id \\ c - id & a + ib \end{bmatrix}$

37. (2) Let $A = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}$

$$\Rightarrow \quad \text{adj}(A) = \begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$$

$$\Rightarrow \quad A^{-2} = (A^{-1})^2 = \frac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}\frac{1}{25}\begin{bmatrix} 5 & 0 \\ a & 5 \end{bmatrix}$$

$$= \frac{1}{625}\begin{bmatrix} 25 & 0 \\ 10a & 25 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{25} & 0 \\ \frac{2a}{125} & \frac{1}{25} \end{bmatrix}$$

Now,

$$\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} \frac{1}{25} & 0 \\ \frac{2a}{125} & \frac{1}{25} \end{bmatrix}$$

$$\Rightarrow \quad x = 2a/125$$

38. (3) $f(x) = \dfrac{1+x}{1-x}$

or $\quad (1 - x) f(x) = 1 + x$

$$\Rightarrow \quad (I - A) f(A) = (I + A)$$

$$\Rightarrow \quad f(A) = (I - A)^{-1}(I + A)$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\right)$$

$$= \frac{\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{-4}$$

$$= \frac{\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}}{-4}$$

$$= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

39. (2) $A^{-1} = \dfrac{\text{adj. } A}{\det. A}$

$$\Rightarrow \quad \begin{bmatrix} 2A+1 & -5 \\ -4 & A \end{bmatrix}^{-1} = \frac{\begin{bmatrix} A & 5 \\ 4 & 2A+1 \end{bmatrix}}{2A^2 + A - 20}$$

$$\Rightarrow \quad \frac{1}{2A^2 + A - 20}\begin{bmatrix} A & 5 \\ 4 & 2A+1 \end{bmatrix}\begin{bmatrix} A-5 & B \\ 2A-2 & C \end{bmatrix}$$

$$= \begin{bmatrix} 14 & D \\ E & F \end{bmatrix} \text{ (from given equation)}$$

$$\Rightarrow \quad \frac{A^2 + 5A - 10}{2A^2 + A - 20} = 14$$

$$\Rightarrow \quad 27A^2 + 9A - 270 = 0$$

$$\Rightarrow \quad 3A^2 + A - 30 = 0$$

$$\Rightarrow \quad 3A^2 - 9A + 10A - 30 = 0$$

$$\Rightarrow \quad (3A + 10)(A - 3) = 0$$

$$\Rightarrow \quad A = 3 \quad \text{or} \quad -\frac{10}{3}$$

Therefore, sum of possible values of A is $3 + \dfrac{10}{3} = \dfrac{19}{3}$.

40. (1) As $B = -A^{-1}BA$, we get

$$AB = -BA \text{ or } AB + BA = O$$

Now,

$$(A + B)^2 = (A + B)(A + B)$$
$$= A^2 + BA + BA + B^2$$
$$= A^2 + O + B^2$$
$$= A^2 + B^2$$

41. (2) $|A| = \begin{vmatrix} 1 & \tan x \\ -\tan x & 1 \end{vmatrix} = 1 + \tan^2 x \neq 0$

So A is invertible. Also,

$$\text{adj } A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{1}{|A|}\text{adj } A$

$$= \frac{1}{(1 + \tan^2 x)}\begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$$

$$\therefore \quad A^T A^{-1} = \begin{bmatrix} 1 & -\tan x \\ \tan x & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{1+\tan^2 x} & \frac{-\tan x}{1+\tan^2 x} \\ \frac{\tan x}{1+\tan^2 x} & \frac{1}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\tan^2 x}{1+\tan^2 x} & \frac{-2\tan x}{1+\tan^2 x} \\ \frac{2\tan x}{1+\tan^2 x} & \frac{1-\tan^2 x}{1+\tan^2 x} \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2x & -\sin 2x \\ \sin 2x & \cos 2x \end{bmatrix}$$

42. (2) We know that $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$

$\Rightarrow \quad |\text{adj}(\text{adj}(\text{adj } A))| = |\text{adj } A|^{(n-1)^2}$

$= |A|^{(n-1)^3}$

$= 2^8 = 256$

43. (2) We have,

$$I = AA^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & c+1 \\ 0 & 1 & 2\ (c+1) \\ 4(1-a) & 3\ (a-1) & 2+ac \end{bmatrix}$$

Comparing the elements of AA^{-1} with those of I, we have

$c + 1 = 0$ or $c = -1$

$a - 1 = 0$ or $a = 1$

44. (2) Since A is orthogonal, we have

$AA^T = I$

or $|AA^T| = 1$

or $|A^2| = 1$

or $|A| = \pm 1$

Now, $|\text{adj. (adj. } A)| = |A|^{(n-1)^2}$

45. (2) $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

$$\Rightarrow \quad A^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$$

$$\Rightarrow \quad A^3 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^3 & 0 \\ 0 & b^3 \end{pmatrix}$$

$$\Rightarrow \quad A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

$$\Rightarrow \quad (A^n)^{-1} = \frac{1}{a^n b^n}\begin{pmatrix} b^n & 0 \\ 0 & a^n \end{pmatrix} = \begin{pmatrix} a^{-n} & 0 \\ 0 & b^{-n} \end{pmatrix}$$

$$\Rightarrow \quad \lim_{n\to\infty} (A^n)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ as } a > 1 \text{ and } b > 1$$

46. (1) From given data, $|A| = 2^4$

$\Rightarrow \quad |\text{adj. (adj. } A)| = (2^4)^9 = 2^{36}$

$$\Rightarrow \quad \left\{\frac{\det.(\text{adj.}(\text{adj. } A))}{7}\right\} = \left\{\frac{2^{36}}{7}\right\} = \left\{\frac{(7+1)^{12}}{7}\right\} = \frac{1}{7}$$

47. (1) A is involuntary. Hence,

$A^2 = I \Rightarrow A = A^{-1}$

Also, $(kA)^{-1} = \frac{1}{k}(A)^{-1}$

or $\left(\frac{1}{2}A\right)^{-1} = 2(A)^{-1}$ or $2A$

48. (2) Given,

$B = A^{-1} A^T$

$\Rightarrow \quad B^T = (A^{-1} A^T)^T = A \times (A^{-1})^T$

$\Rightarrow \quad B \times B^T = A^{-1}A^T \times A \times (A^{-1})^T$

$= A^{-1} \times (A^T \times A)(A^{-1})^T$

$= A^{-1}(A \times A^T)(A^{-1})^T$

$= (A^{-1}A) \times (A^{-1}A)^T = I \times I^T = I$

49. (2) $P^T P = I$

$Q = PAP^T$

$\therefore \quad x = P^T Q^{1000} P = P^T (PAP^T)^{1000} P$

$= P^T PAP^T (PAP^T)^{999} P$

$= IAP^T \cdot PAP^T (PAP^T)^{998} P$

$= AIAP^T (PAP^T)^{998} P$

$= A^2 P^T PAP^T (PAP^T)^{997} P$

$= A^3 P^T ((PAP^T)^{997} P$

$\vdots$

$= A^{1000} = I$ ($\because A$ is involuntary)

Hence, $x^{-1} = I$.

50. (3) Given $B^r = I$ or $B^r B^{-1} = IB^{-1}$ or $B^{r-1} = B^{-1}$

$\Rightarrow \quad A^{-1}B^{r-1}A - A^{-1}B^{-1}A = A^{-1}B^{-1}A - A^{-1}B^{-1}A = O$

51. (3) $\text{adj.}(Q^{-1}BP^{-1}) = \text{adj.}(P^{-1})\text{adj.}(B)\text{adj.}(Q^{-1})$

$$= \frac{P}{|P|} A \frac{Q}{|Q|} = PAQ$$

52. (2) We have,

$(A - 2I)(A - 4I) = O$

or $A^2 - 2A - 4A + 8I = O$

or $A^2 - 6A + 8I = O$

or $A^{-1}(A^2 - 6A + 8I) = A^{-1}O$

or $A - 6I + 8A^{-1} = O$

or $A + 8A^{-1} = 6I$

or $\frac{1}{6}A + \frac{4}{3}A^{-1} = 1$

53. (2) $(I - A)f(A) = I + A$

or $f(A) = (I + A)(I - A)^{-1}$

$= (I + A)(I + A + A^2)$

$= I + A + A^2 + A + A^2 + A^3$

$= I + 2A + 2A^2$

54. (1) $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

55. (3) $A^2 - A + I = 0$

or $I = A - A^2$

$IA^{-1} = AA^{-1} - A^2A^{-1}$

$\Rightarrow \quad A^{-1} = I - A$

56. (2) We have,

$[F(x)G(y)]^{-1} = [G(y)]^{-1}[F(x)]^{-1}$

$= G(-y)F(-x)$

57. (3) $(A^{-1}BA)^2 = (A^{-1}BA)\,(A^{-1}BA)$

$= A^{-1}B(AA^{-1})BA$

$= A^{-1}BIBA = A^{-1}B^2A$

$(A^{-1}BA)^3 = (A^{-1}B^2A)\,(A^{-1}BA)$

$= A^{-1}B^2(AA^{-1})\,BA$

$= A^{-1}B^3A$ and so on

$\therefore \quad (A^{-1}BA)^n = A^{-1}B^nA$

58. (2) (kI_n) adj. $(kI_n) = |kI_n|I_n$ [using $A(\text{adj. } A) = |A|I$]

adj. $(kI_n) = k^{n-1}I_n$

$|\text{adj. } (kI_n)| = k^{n(n-1)}$

59. (3) A adj. $A = |A|\,I$

$|A| = xyz - 8x - 3(z - 8) + 2(2 - 2y)$

$|A| = xyz - (8x + 3z + 4y) + 28$

$= 60 - 20 + 28 = 68$

$\Rightarrow \quad A\,(\text{adj. } A) = 68\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}68 & 0 & 0\\ 0 & 68 & 0\\ 0 & 0 & 68\end{bmatrix}$

60. (1) $|A| = 1(0 - 10) - 2(2 - 0) + 3(4 - 0)$

$= -10 - 4 + 12 = -2$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow$ Unique solution

61. (1) We know that $A - A^T$ is skew-symmetric.

Now, it is given that $|A - A^T| \neq 0$.

So, order of A cannot be odd.

Thus, order of A is 2.

Now, $B = \text{adj. } A$

$\therefore \quad AB = BA = |A|I$

$\Rightarrow \quad (AB)^{-1} = (BA)^{-1}$

$\Rightarrow \quad B^{-1}A^{-1} = A^{-1}B^{-1}$

$\therefore \quad \text{adj. } (B^2A^{-1}B^{-1}A) = \text{adj. } (B^2B^{-1}A^{-1}A)$

$= \text{adj. } B = \text{adj. } (\text{adj. } A) = |A|^{2-2}A = A$

62. (4) We know that det.(adj. A) = (det. A)$^{n-1}$, where n is order of matrix A.

For $n = 3$, $\det.(\text{adj. } A^{-1}) = (\det. A^{-1})^2 = \dfrac{1}{(\det. A)^2} = 9$

63. (4) $AA^T = 2I$ ...(1)

$\Rightarrow \quad |A|^2 = 8$

Now, $A^{-1} = A^T - A\ \text{adj.}(2B^{-1})$ ...(2)

Multiplying with A, we get

$I = AA^T - A^2\ \text{adj.}(2B^{-1})$

$\therefore \quad I = 2I - A^2\ \text{adj.}(2B^{-1})$

$\Rightarrow \quad A^2\ \text{adj.}(2B^{-1}) = I$

$\Rightarrow \quad |A^2|\,|\text{adj.}(2B^{-1})| = 1$

$\Rightarrow \quad 8|2B^{-1}|^2 = 1$

$\Rightarrow \quad 8 \cdot \dfrac{64}{|B|^2} = 1$

$\Rightarrow \quad |B|^2 = 64 \times 8$

$\therefore \quad |B| = \pm(8 \times 2\sqrt{2}) = \pm 16\sqrt{2}$

64. (1) We have,

$2A^{-1} = A^T$

$\Rightarrow \quad |2A^{-1}| = |A^T|$

$\Rightarrow \quad 2^5\dfrac{1}{|A|} = |A|$

$\Rightarrow \quad |A|^2 = 2^5$

$\therefore \quad |\text{adj. } (\text{adj. } (\text{adj. } A))| = |A|^{4^3} = (|A|^2)^{32} = 2^{160}$

$\dfrac{2^{160}}{7} = \dfrac{2(2^3)^{53}}{7} = \dfrac{2(7+1)^{53}}{7} = m + \dfrac{2}{7}$, where m is integer

Thus, required remainder is 2.

65. (3) $(PQ)^{-1} = R$

$\Rightarrow \quad Q^{-1} \times P^{-1} = R$

$\Rightarrow \quad Q^{-1} = RP = \begin{bmatrix}-1 & 0 & 1\\ 1 & 1 & 3\\ 2 & 0 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ 0 & 1 & -1\\ 3 & 1 & 1\end{bmatrix}$

$= \begin{bmatrix}2 & -1 & 0\\ 10 & 6 & 3\\ 8 & 6 & 4\end{bmatrix}$

Multiple Correct Answers Type

1. (2), (3)

$\det(-A) = (-1)^n \det(A)$

$\det(A^{-1}) = \dfrac{1}{\det(A)} = 1$

$\det(\text{adj } A) = |A|^{n-1} = 1$

$|\omega A| = \omega^n|A| = 1$ only when $n = 3k$, $k \in Z$.

2. (2), (4)

$a_{ij} = \begin{cases} x; & \text{if } i = j,\ x \in R\\ 1; & \text{if } |i - j| = 1\\ 0; & \text{otherwise}\end{cases}$

$\Rightarrow \quad A = \begin{bmatrix}x & 1 & 0\\ 1 & x & 1\\ 0 & 1 & x\end{bmatrix}$

$\Rightarrow \quad |A| = x^3 - 2x$

If $f(x) = x^3 - 2x$

$\Rightarrow \quad f'(x) = 3x^2 - 2 = (\sqrt{3}x - \sqrt{2})\,(\sqrt{3}x + \sqrt{2})$

So $x = \dfrac{\sqrt{2}}{\sqrt{3}}$ point of minima and $x = \dfrac{-\sqrt{2}}{\sqrt{3}}$ is maxima.

3. (1), (4)

Given, $(A + B)^2 = A^2 + B^2 + 2AB$

or $(A + B)\,(A + B) = A^2 + B^2 + 2AB$

or $A^2 + AB + BA + B^2 = A^2 + B^2 + 2AB$ or $BA = AB$

or $\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}\begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\ 2 & 1\end{bmatrix}\begin{bmatrix}a & 1\\ b & -1\end{bmatrix}$

or $\begin{bmatrix}a+2 & -a+1\\ b-2 & -b-1\end{bmatrix} = \begin{bmatrix}a-b & 1+1\\ 2a+b & 2-1\end{bmatrix}$

The corresponding elements of equal matrices are equal.

$a+2=a-b, -a+1=2 \Rightarrow a=-1$

$b-2=2a+b, -b-1=1 \Rightarrow b=-2$

$\Rightarrow \quad a=-1, b=-2$

4. (1), (2), (3)

Given, $AB=A$, $BA=B$

$\Rightarrow \quad B\times AB=B\times A$

or $\quad (BA)B=B$

or $\quad B^2=B$

Also,

$A\times BA=AB$

$\Rightarrow \quad (AB)A=A$

$\Rightarrow \quad A^2=A$

Now $(A^T)^2=(A^T\times A^T)=(A\times A)^T=(A^2)^T=A^T$

Similarly, $(B^T)^2=B^T$

Hence, A^T and B^T are idempotent.

5. (1), (2), (3)

We have, $|A(\theta)|=1$

Hence, A is invertible.

$$A(\pi+\theta)=A(\pi+\theta)=\begin{bmatrix}\sin(\pi+\theta) & i\cos(\pi+\theta)\\ i\cos(\pi+\theta) & \sin(\pi+\theta)\end{bmatrix}$$

$$=\begin{bmatrix}-\sin\theta & -i\cos\theta\\ -i\cos\theta & -\sin\theta\end{bmatrix}=-A(\theta)$$

$$\text{adj}(A(\theta))=\begin{bmatrix}\sin\theta & -i\cos\theta\\ -i\cos\theta & \sin\theta\end{bmatrix}$$

$$\Rightarrow \quad A(\theta)^{-1}=\begin{bmatrix}\sin\theta & -i\cos\theta\\ -i\cos\theta & \sin\theta\end{bmatrix}=A(\pi-\theta)$$

6. (1), (4)

$(B^TAB)^T=B^TA^T(B^T)^T=B^TA^TB=B^TAB$ if A is symmetric.

Therefore, B^TAB is symmetric if A is symmetric.

Also, $(B^TAB)^T=B^TA^TB=B^T(-A)B=-(B^TA^TB)$

Therefore, B^TAB if A is skew-symmetric if A is skew-symmetric.

7. (1), (3), (4)

B is an idempotent matrix

$\therefore \quad B^2=B$

Now, $A^2=(I-B)^2$

$=(I-B)(I-B)$

$=I-IB-IB+B^2$

$=I-B-B+B^2$

$=I-2B+B^2$

$=I-2B+B$

$=I-B$

$=A$

Therefore, A is idempotent. Again,

$AB=(I-B)B=IB-B^2=B-B^2=B^2-B^2=O$

Similarly, $BA=B(I-B)=BI-B^2=B-B=O$.

8. (1), (2)

Let $\quad i=k=1$ (say). Then,

$A_iA_k=A_kA_i=A_1A_1$

$$A_iA_k=A_1A_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$$

$$A_2A_2=\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$$

$\therefore \quad A_iA_k+A_kA_i=I+I=2I$

If $i\neq k$ let $i=1$ and $k=2$, then

$$A_iA_k=A_1A_2=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{bmatrix}$$

Also, $$A_2A_1=\begin{bmatrix}0&0&0&i\\0&0&-i&0\\0&i&0&0\\-i&0&0&0\end{bmatrix}\times\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-i\end{bmatrix}$$

$\Rightarrow \quad A_1A_2+A_2A_1=O$

9. (1), (2), (3)

Applying $R_3\to R_3-R_2$, $R_2\to R_2-R_1$, we get

$$|A|=3\begin{vmatrix}a_1&a_2&a_3\\d&d&d\\d&d&d\end{vmatrix}=0$$

where d is the common difference of the A.P.

Therefore, the given system of equations has infinite number of solutions. Also,

$|B|=a_1^2+a_2^2\neq 0$

10. (1), (2), (4)

$$A=\begin{bmatrix}\cos\alpha&\sin\alpha&0\\\cos\beta&\sin\beta&0\\\cos\gamma&\sin\gamma&0\end{bmatrix}\begin{bmatrix}\cos\alpha&\cos\beta&\cos\gamma\\\sin\alpha&\sin\beta&\sin\gamma\\0&0&0\end{bmatrix}$$

Clearly, A is symmetric and $|A|=0$; hence, singular and not invertible. Also,

$AA^T\neq I$

11. (1), (2), (3)

All are properties of diagonal matrix.

12. (1), (2), (3), (4)

(1) Possible matrices are

$$\begin{bmatrix}1&0\\0&3\end{bmatrix},\begin{bmatrix}3&0\\0&1\end{bmatrix},\begin{bmatrix}2&0\\0&2\end{bmatrix},\begin{bmatrix}4&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&4\end{bmatrix}$$

(2) Using 0, 0, 2, 2, there are two matrices which are invertible which are $\begin{bmatrix}2&0\\0&2\end{bmatrix}$ and $\begin{bmatrix}0&2\\2&0\end{bmatrix}$.

Using 0, 0, 1, 3, there are four matrices which are invertible.

Using 0, 1, 1, 2, there are twelve matrices which are invertible.

Using 0, 0, 0, 4 and using 1, 1, 1, 1 no matrix is formed, which is invertible.

$\therefore$ Total number of matrices = 18

(3) $|4-(-4)|=8$

(4) There are five matrices, which are either symmetric or skew symmetric and whose determinant is divisible by 2.

These are $\begin{bmatrix}2&0\\0&2\end{bmatrix},\begin{bmatrix}0&2\\2&0\end{bmatrix},\begin{bmatrix}0&0\\0&4\end{bmatrix},\begin{bmatrix}4&0\\0&0\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix}$

13. (1), (2), (3)

$$S=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} \text{ or } S^{-1}=\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

We have,

$$SA=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}\begin{bmatrix}b+c&c-a&b-a\\c-b&c+a&a-b\\b-c&a-c&a+b\end{bmatrix}$$

$$=\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}$$

$$\therefore\quad SAS^{-1}=\begin{bmatrix}0&2a&2a\\2b&0&2b\\2c&2c&0\end{bmatrix}\frac{1}{2}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}0&a&a\\b&0&b\\c&c&0\end{bmatrix}\begin{bmatrix}-1&1&1\\1&-1&1\\1&1&-1\end{bmatrix}$$

$$=\begin{bmatrix}2a&0&0\\0&2b&0\\0&0&2c\end{bmatrix}$$

$$= \text{diag}\,(2a, 2b, 2c)$$

14. (1), (2), (3)

$B=P^{-1}AP$

$\Rightarrow\quad B^2=(P^{-1}AP)(P^{-1}AP)=P^{-1}A^2P$

$\Rightarrow\quad B^n=P^{-1}A^nP\ \forall n\in N$

$|B|=|P^{-1}AP|=|P^{-1}||A||P|=|A|$

$\therefore$ If $|A|\neq 0$ then $|B|\neq 0$

$P^{-1}(A-\lambda I)P=P^{-1}AP-P^{-1}\lambda P=B-\lambda I$

15. (1), (2), (4)

$$A^2=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}$$

We have,

$$A^2-4A-5I_3$$

$$=\begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}-4\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}-5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=O$$

or $\quad 5I_3=A^2-4A=A(A-4I_3)$

or $\quad I_3=A\cdot\frac{1}{5}(A-4I_3)$ or $A^{-1}=\frac{1}{5}(A-4I_3)$

Note that $|A|=5$. Since $|A^3|=|A|^3=5^3\neq 0$, A^3 is invertible. Similarly, A^2 is invertible.

16. (1), (2), (3)

$$A^2=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}$$

$$A^3=\begin{bmatrix}1&0&0\\1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\2&0&1\\1&1&0\end{bmatrix}$$

$$A^3-A^2=\begin{bmatrix}0&0&0\\1&-1&1\\0&1&-1\end{bmatrix}$$

and $\quad A-I=\begin{bmatrix}0&0&0\\1&-1&1\\0&1&-1\end{bmatrix}$

$\Rightarrow\quad A^3-A^2=A-I$ (1)

Now, $\det.(A^n-I)=\det.((A-I)(I+A+A^2+\ldots+A^{n-1}))$

$=\det.(A-I)\times\det.(I+A+A^2+\ldots+A^{n-1})$

$=0$

From (1), $A^4-A^3=A^2-A$ (2)

Again from (1), $A^5-A^4=A^3-A^2=A-I$

Thus, if n is even, $A^n-A^{n-1}=A^2-A$ (3)

If n is odd, $A^n-A^{n-1}=A-I$ (4)

Consider that n is even.

$\therefore\quad A^n-A^{n-1}=A^2-A$ [from (3)]

and $\quad A^{n-1}-A^{n-2}=A-I$ [from (4)]

Adding these, we get

$A^n-A^{n-2}=A^2-I$

$\Rightarrow\quad A^n=A^{n-2}+A^2-I$

$=(A^{n-4}+A^2-I)+A^2-I$

$=(A^{n-6}+A^2-I)+2(A^2-I)$

$\ldots\quad\ldots\quad\ldots$

$\ldots\quad\ldots\quad\ldots$

$$=(A^2)+\frac{n-2}{2}(A^2-I)$$

$$\therefore\quad A^n=\left(\frac{n}{2}\right)A^2-\left(\frac{n-2}{2}\right)I$$

$\therefore\quad A^{200}=100A^2-99I$

$$= 100\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix} - 99\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 100 & 1 & 0\\ 100 & 0 & 1\end{bmatrix}$$

17. (1), (2)

$A^T = A$ and $B^T = -B$

(1) $(ABA^T)^T = AB^TA^T = -(ABA^T)$

(2) $AB^T + BA^T = -AB + BA$

Now, $(BA - AB)^T = (BA)^T - (AB)^T$

$= A^TB^T - B^TA^T = -AB + BA = AB^T + BA^T$

(3) $((A + B)(A - B))^T = (A - B)^T(A + B)^T$

$= (A^T - B^T)(A^T + B^T) = (A + B)(A - B)$

(4) $((A + I)(B - I))^T = (B^T - I)(A^T + I)$

$= -(-B + I)(A + I) = (B - I)(A + I)$

18. (1), (3)

$$\sin A = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix} \text{ and } \cos A = \begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$\therefore \quad |\sin A| = \cos^2\theta + \sin^2\theta = 1.$

Hence, $\sin A$ is invertible.

$$\text{Also, } (\sin A) \times (\sin A)^T = \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos^2\theta + \sin^2\theta & 0\\ 0 & \cos^2\theta + \sin^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= I$$

Hence, $\sin A$ is orthogonal. Also,

$$2\sin A\cos A = 2\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\sin\theta & \cos\theta\\ \cos\theta & \sin\theta\end{bmatrix}$$

$$= 2\begin{bmatrix}2\sin\theta\cos\theta & \cos^2\theta + \sin^2\theta\\ \cos^2\theta - \sin^2\theta & 0\end{bmatrix}$$

$$= 2\begin{bmatrix}\sin 2\theta & 1\\ \cos 2\theta & 0\end{bmatrix} \neq \sin 2A$$

19. (1), (3), (4)

Given, $A^2 + A + 2I = O$

$\Rightarrow \quad A^2 + A = -2I$

$\Rightarrow \quad |A^2 + A| = |-2I|$

$\Rightarrow \quad |A||A + I| = (-2)^n$

$\Rightarrow \quad |A| \neq 0$

Therefore, A is nonsingular; hence, its inverse exists. Also, multiplying the given equation both sides with A^{-1}, we get

$$A^{-1} = -\frac{1}{2}(A + I)$$

20. (1), (2), (3)

$$\because \quad |A| = \begin{vmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{vmatrix} = 3(-3 + 4) + 3(2 - 0) + 4(-2 - 0) = 1$$

$\therefore \quad \text{adj}(\text{adj } A) = |A|^{3-2}A = A$ and $|\text{adj}(\text{adj } A)| = |A| = 1$

Also,

$|\text{adj } A| = |A|^{3-1} = |A|^2 = 1^2 = 1$

21. (1), (3)

We have,

$$\begin{bmatrix}1 & \tan\theta\\ -\tan\theta & 1\end{bmatrix}^{-1} = \frac{1}{1 + \tan^2\theta}\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}a & -b\\ b & a\end{bmatrix} = \cos^2\theta\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}\begin{bmatrix}1 & -\tan\theta\\ \tan\theta & 1\end{bmatrix}$$

$$= \cos^2\theta\begin{bmatrix}1 - \tan^2\theta & -2\tan\theta\\ 2\tan\theta & 1 - \tan^2\theta\end{bmatrix}$$

$$= \begin{bmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{bmatrix}$$

$\therefore \quad a = \cos 2\theta, b = \sin 2\theta$

22. (1), (2), (3)

$|A^{-1}| = -1 \Rightarrow |A| = -1$

Now, use $\text{adj } A = |A|A^{-1}$ and $A = (A^{-1})^{-1}$

23. (1), (2)

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

$$\therefore \quad (\text{adj } A)^{-1} = \frac{\text{adj}(\text{adj } A)}{|\text{adj } A|}$$

$$= \frac{|A|^{n-2}A}{|A|^{n-1}}$$

$$= \frac{A}{|A|}$$

Also $A(\text{adj } A) = |A|I$

or $\quad A^{-1}(\text{adj } A^{-1}) = |A^{-1}|I$

or $\quad A^{-1}(\text{adj } A^{-1}) = \frac{I}{|A|}$

or $\quad AA^{-1}(\text{adj } A^{-1}) = \frac{A \cdot I}{|A|}$

or $\quad I(\text{adj } A^{-1}) = \frac{A}{|A|}$

or $\quad (\text{adj } A^{-1}) = \frac{A}{|A|} = (\text{adj } A)^{-1}$

24. (1), (2), (4)

See theory.

25. (1), (2), (3)

If $|A| \neq 0$, then

$AB = AC$

or $\quad A^{-1}AB = A^{-1}AC$

or $\quad B = C$

Also, if A is orthogonal matrix, then, $AA^T = I$

$\Rightarrow \quad |AA^T| = 1 \Rightarrow |A|^2 = 1 \Rightarrow A$ is invertible

26. (1), (2), (3)

Given that A and B commute, we have

$AB = BA \qquad (\because A \text{ and } B \text{ are symmetric}) \quad (1)$

Also,

$A^T = A, B^T = B$

$(A^{-1}B)^T = B^T(A^{-1})^T = BA^{-1} \quad (2)$

$(\because \text{if } A \text{ is symmetric, } A^{-1} \text{ is also symmetric})$

Also from Eq. (1),

$$ABA^{-1} = B \quad (3)$$

or $A^{-1}ABA^{-1} = A^{-1}B$

or $IBA^{-1} = A^{-1}B$

or $BA^{-1} = A^{-1}B$

Hence, from Eq. (2),

$$(A^{-1}B)^T = A^{-1}B$$

Thus, $A^{-1}B$ is symmetric. Similarly, AB^{-1} is also symmetric.

Also,

$$BA = AB$$

or $(BA)^{-1} = (AB)^{-1}$

or $A^{-1}B^{-1} = B^{-1}A^{-1}$

or $(A^{-1}B^{-1})^T = (B^{-1}A^{-1})^T$

$= (A^{-1})^T(B^{-1})^T$

$= A^{-1}B^{-1}$

Hence, $A^{-1}B^{-1}$ is symmetric.

27. (1), (3)

$A^3 = 8I$

$\Rightarrow (A - 2I)(A^2 + 2A + 4I) = O$

Also, $B^3 = I$

$\Rightarrow (B - I)(B^2 + B + I) = 0$

Now, $\det.(AB - A - 2B + 2I) \neq 0$

$\Rightarrow \det.((A - 2I)(B - I)) \neq 0$

So, $B - I$ and $A - 2I$ are non-singular matrices.

$\Rightarrow A^2 + 2A + 4I = O$ and $B^2 + B + I = O$

28. (3), (4)

$A^{n+2} - B^{n+2} = (A + B)(A^{n+1} - B^{n+1}) - AB(A^n - B^n)$

$\Rightarrow A^n - B^n = (A + B)(A^n - B^n) - AB(A^n - B^n)$

$\Rightarrow I = A + B - AB \quad [\because A^n - B^n \text{ is invertible}]$

$\Rightarrow (I - A)(I - B) = O$

Now, $A, B \neq I$.

$\therefore I - A \neq O$ and $I - B \neq O$.

If $(I - A)$ is non-singular, then $(I - A)^{-1}$ exists.

$\therefore (I - A)^{-1}(I - A)(I - B) = O$

$\Rightarrow I - B = O$; this is contradiction to the given condition.

Thus, $(I - A)$ is singular.

With similar reasons, $(I - B)$ is also singular.

29. (1), (2), (3), (4)

We have $A^2 = I$ and $B^2 = I$

$\therefore A = A^{-1}$ and $B = B^{-1}$

If A and B are inverse to each other, then $A = B^{-1} = B$.

$$C = \frac{AB + BA}{2}$$

$$\Rightarrow C^2 = \frac{(AB)^2 + (BA)^2 + AB \cdot BA + BA \cdot AB}{4}$$

Now, $ABBA = AIA = AA = I$

Similarly, $BAAB = I$

Also $(AB)^2 = ABAB$

$= AABB \quad$ (if $AB = BA$)

$= A^2B^2$

$= I$

Similarly $(BA)^2 = I$

$\therefore C^2 = I$

Thus, if $AB = BA$, then $C^2 = I$

$D = AB - BA$

$\Rightarrow D^2 = (AB)^2 + (BA)^2 - AB \times BA - BA \times AB$

$= I + I - I - I \quad$ (If $AB = BA$)

$= O$

Thus, if $AB = BA$, then $D^n = O$.

$(A + B)^5 = A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5$

$= A + 5B + 10A + 10B + 5A + B$

$= 16(A + B)$

30. (1), (2)

Given that $AB = B^T$ and $B = \text{adj. } A$

$\Rightarrow A = B^TB^{-1}$

$\Rightarrow |A| = |B^T| \times |B^{-1}| = 1 \quad \ldots(i)$

Now, adj. $A = \text{adj}(B^TB^{-1})$

$= \text{adj}(B^{-1}) \times \text{adj}(B^T)$

$= (\text{adj } B)^{-1} \times \text{adj}(B^T)$

$\Rightarrow (\text{adj. } B)^T = (\text{adj. } B) \times (\text{adj. } A) = (\text{adj. } B)\, B = |B|\, I$

$\Rightarrow (\text{adj. } B)^T = |B| \cdot I$

$\Rightarrow (\text{adj.}(\text{adj. } A))^T = |\text{adj. } A|\, I$

$\therefore |A|^{n-2} A^T = |A|^{n-1} I$

$\Rightarrow A^T = |A| I$

$\Rightarrow A^T = I$

$\Rightarrow A = I$

$\therefore B = \text{adj. } A = \text{adj. } I = I$

31. (3), (4)

We know that

$\det.(A) = a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$

$= (1)(-2) + (3)(-1) + (2)(3\alpha)$

$= 6\alpha - 5$

Also, $(\det.(A))^2 = \det.(\text{adj. } A) = -10 + 17\alpha - 6\alpha^2$

$\therefore (6\alpha - 5)^2 = -10 + 17\alpha - 6\alpha^2$

$\Rightarrow 42\alpha^2 - 77\alpha + 35 = 0$

$\Rightarrow 6\alpha^2 - 11\alpha + 5 = 0$

$\Rightarrow (\alpha - 1)(6\alpha - 5) = 0$

$\Rightarrow \alpha = 1, 5/6$

For $\alpha = 1$, $\det.(A) = 6 - 5 = 1$

For $\alpha = 5/6$, $\det.(A) = 0$

32. (1), (4)

Given that $B = A - 2I \quad \ldots(i)$

And $A^3 - 6A^2 + 7A - 8I = O$

$\therefore (A - 2I)^3 = 5A \quad \ldots(ii)$

$\therefore B^3 = 5A$

$\Rightarrow |B^3| = |5A| = 5^3|A| = 5^3 \times 8$

$\Rightarrow |B| = 10$

Now, from (i), we get

$\therefore A^{-1}B = I - 2A^{-1}$

$\therefore |\text{adj. } (I - 2A^{-1})| = |I - 2A^{-1}|^2$

$= |A^{-1}B|^2 = \left(\frac{|B|}{|A|}\right)^2$

$= \left(\frac{5}{4}\right)^2$

$= \frac{25}{16}$

$$\text{adj.}\left(\left(\frac{B}{2}\right)^{-1}\right) = \left(\text{adj.}\left(\frac{B}{2}\right)\right)^{-1} = \frac{\text{adj.}\left(\text{adj.}\dfrac{B}{2}\right)}{\left|\text{adj.}\dfrac{B}{2}\right|}$$

$$= \frac{\left|\frac{B}{2}\right|\frac{B}{2}}{\left|\frac{B}{2}\right|^2} = \frac{\frac{B}{2}}{\left|\frac{B}{2}\right|} = \frac{4B}{|B|} = \frac{2B}{5}$$

33. (1), (2), (3)

(1) Characteristic determinant $= \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$

Thus, characteristic equation is

$\lambda^2 - 1 = 0$

$\therefore \quad \lambda = 1, -1$

(2) Characteristic determinant $= \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix}$

Thus, characteristic equation is

$\lambda^2 - (-i^2) = 0$ or $\lambda^2 - 1 = 0$

$\therefore \quad \lambda = 1, -1$

(3) Characteristic determinant $= \begin{vmatrix} 1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix}$

Thus, characteristic equation is

$\lambda^2 - 1 = 0$

$\therefore \quad \lambda = 1, -1$

(4) Characteristic determinant $= \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix}$

Thus, characteristic equation is

$(\lambda - 1)^2 = 0$

$\therefore \quad \lambda = 1$

Linked Comprehension Type

1. (2) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \quad A^2 - (a+d)A + (ad-bc)I$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} - \begin{bmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{bmatrix} + \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= O$$

2. (2) If $A = O$, $\text{tr}(A) = 0$. Suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$|A| = 0$ and $A^2 - (a+d)A + (ad-bc)I = O$

$\Rightarrow \quad a + d = 0 = \text{tr}(A)$

3. (4) $(I+A)^{100} = {}^{100}C_0 I^{100} + {}^{100}C_1 I^{99} A + {}^{100}C_2 I^{98} A^2 + \quad + {}^{100}C_{100} A^{100}$

$= I + 100A + O + O + \quad + O$

$= I + 100A$

4. (3) $AB = A \Rightarrow |AB| = |A|$ (1)

$\Rightarrow \quad |A| = 0$ or $|B| = 1$

$BA = B \Rightarrow |BA| = |B|$ (2)

$\Rightarrow \quad |A| = 1$ or $|B| = 0$

If $|A| = 0$, then from Eq. (2), $|B| = 0$

If $|B| = 0$, then from Eq. (1), $|A| = 0$

5. (3) $AB = A$, $BA = B$

$ABA = A^2$ or $A(BA) = A^2$ or $AB = A^2$ or $A = A^2$

Similarly, $B^2 = B$

$(A+B)^2 = A^2 + B^2 + AB + BA$

$= A + B + A + B = 2(A+B)$

$(A+B)^3 = (A+B)^2(A+B) = 2(A+B)^2 = 2^2(A+B)$

$\Rightarrow \quad (A+B)^7 = 2^6(A+B) = 64\,(A+B)$

6. (3) $(A+I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$= I + 5A + 10A + 10A + 5A + A$

$(\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \quad = A)$

$= I + 31A$

7. (4) 8. (3)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Rightarrow \quad B = \begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$= \frac{1}{|A|}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} a_{12}+a_{13} & a_{11}+a_{13} & a_{11}+a_{12} \\ a_{22}+a_{23} & a_{21}+a_{23} & a_{21}+a_{22} \\ a_{32}+a_{33} & a_{31}+a_{33} & a_{31}+a_{32} \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} 0 & |A| & |A| \\ |A| & 0 & |A| \\ |A| & |A| & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\Rightarrow \quad |A^{-1}B| = 2$

or $\quad |A^{-1}||B| = 2$

or $\quad |B| = 2|A|$

9. (2) 10. (4), 11. (2)

$A^n - A^{n-2} = A^2 - I \Rightarrow A^{50} = A^{48} + A^2 - I$

Further,

$A^{48} = A^{46} + A^2 - I$

$A^{46} = A^{44} + A^2 - I$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$A^4 = A^2 + A^2 - I$

$A^{50} = 25A^2 - 24I$

Here,

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A^{50} = \begin{bmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{bmatrix} - 24 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

$\therefore \quad |A^{50}| = 1$

Also, $\text{tr}(A^{50}) = 1 + 1 + 1 = 3$. Further,

$$\begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 25 \\ 25 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \cup_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Similarly,

$$\cup_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \cup_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \cup = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ i.e., } |\cup| = 1$$

12. (4), 13. (1)

$$\begin{bmatrix} - & R_1 & - \\ - & R_2 & - \\ - & R_3 & - \end{bmatrix}_B \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}_A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix}_C \qquad (1)$$

$\therefore \quad (\text{det. } B)\,(\text{det. } A) = 3$

$\therefore \quad (\text{det. } B) = 3 \qquad [\text{as det. A} = 1]$

$\text{det.}(2A^{100}B^3 - A^{99}B^4)$

$= \text{det.}(A^{99}(2A - B)B^3)$

$= (\text{det. } A)^{99} \times \text{det. } (2A - \text{B}) \times (\text{det } B)^3$

Now from (1), we get

$$B = A^{-1}C = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ -1 & -3 & 1 \end{bmatrix}$$

$$\therefore \quad 2A - B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 7 & 7 & 1 \end{bmatrix}$$

$\therefore \quad \text{det.}(2A - B) = -1$

$\therefore \quad \text{det.}(2A^{100}B^3 - A^{99}B^4) = (1)^{99}(-1)(3)^3 = -27$

14. (4) $BB^T = I$

$\therefore \quad |BB^T| = 1$

$\Rightarrow \quad |B||B^T| = 1$

$\Rightarrow \quad |B| = 1$

$A(\text{adj. } A + \text{adj. } B) = B$

$\Rightarrow \quad |A|I + A \times \text{adj } B = B$

$\Rightarrow \quad I + A \times \text{adj } B = B$

$\Rightarrow \quad I \times B + A \times (\text{adj } B) \times B = B^2$

$\Rightarrow \quad B + A|B|I = B^2$

$\Rightarrow \quad B + A = B^2 \qquad (1)$

$\Rightarrow \quad |A + B| = |B|^2 = 1$

15. (1) $B + A = B^2$

$\Rightarrow \quad B^TB + B^TA = B^TB^2$

$\Rightarrow \quad I + B^TA = B$

$\Rightarrow \quad B^TA = B - I \qquad (2)$

Now from (1), we get

$B^2 - B = A$

$\Rightarrow \quad B(B - I) = A$

Multiplying both sides by B^{-1}, we get

$\Rightarrow \quad B - I = AB^{-1}$

So, from (1) and (2), we get

$AB^{-1} = B^TA$

$\Rightarrow \quad AB^{-1} = B^{-1}A \qquad (\because BB^T = I \text{ then } B^T = B^{-1})$

16. (3) As second row of all the options is same, we have to look at the elements of the first row. Let the left inverse be $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$. Then

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad a + b + 2c = 1$

$-a + b + 3c = 1$, i.e., $b = \dfrac{1 - 5c}{2}, a = \dfrac{1 + c}{2}$

Thus, matrices in the options (1), (2), and (4) are the inverses and matrix in option (3) is not the left inverse.

17. (4) Let right inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $a - c + 2e = 1$

$b - d + 2f = 0$

$2a - c + e = 0$

$2b - d + f = 1$

This system of equations has infinite solutions.

18. (3) By observation there cannot be any left inverse for options (2) and (4). So we will check for options (1) and (3) only.

For option (1), let the left inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $a - 3b = 1$, $2a + 2b = 0$ and $4a + b = 0$ which is not possible. For option (3),

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a + 2b + 5c = 1, 4a - 3b + 4c = 0, d + 2e + 5f = 0,$

$4d - 3e + 4f = 1$

Therefore, there are infinite number of left inverses.

$$\begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad a + 4d = 1, 2a - 3d = 0$ and $5a + 4d = 0$

which is not possible. Therefore, there is no right inverse.

Matrix Match Type

1. **a → s; b → p; c → q; d → r.**

a. Since A is idempotent, we have

$A^2 = A$

$\Rightarrow A^3 = AA^2 = AA = A^2 = A,\ A^4 = AA^3 = AA = A^2 = A$

$\Rightarrow A^n = A$

$\Rightarrow (I-A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 +$

$= I + (-{}^nC_1 + {}^nC_2 - {}^nC_3 + \ )A$

$= I + [({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \ ) - {}^nC_0]A$

$= I - A$

b. A is involuntary. Hence,

$A^2 = I$

$\Rightarrow A^3 = A^5 = \ = A$ and $A^2 = A^4 = A^6 = \ = I$

$\Rightarrow (I-A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 +$

$= {}^nC_0 I - {}^nC_1 A + {}^nC_2 I - {}^nC_3 A + {}^nC_4 I -$

$= ({}^nC_0 + {}^nC_2 + {}^nC_4 + \ )I - ({}^nC_1 A + {}^nC_3 + {}^nC_5 + \ )A$

$= 2^{n-1}(I-A)$

$\Rightarrow [(I-A)^n]A^{-1} = 2^{n-1}(I-a)A^{-1} = 2^{n-1}(A^{-1} - I)$

c. If A is nilpotent of index 2, then

$A^2 = A^3 = A^4 \ = A^n = O$

$\Rightarrow (I-A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 +$

$= I - nA + O + O +$

$= I - nA$

d. A is orthogonal. Hence,

$AA^T = I$

$\Rightarrow (A^T)^{-1} = A$

2. **a → r; b → s; c → p, r; d → p, q, r, s.**

a. Since A is idempotent, $A^2 = A^3 = A^4 = \ = A$. Now,

$(A+I)^n = I + {}^nC_1 A + {}^nC_2 A^2 + \ + {}^nC_n A^n$

$= I + {}^nC_1 A + {}^nC_2 A + \ + {}^nC_n A$

$= I + ({}^nC_1 + {}^nC_2 + \ + {}^nC_n)A$

$= I + (2^n - 1)A$

$\Rightarrow 2^n - 1 = 127$

$\Rightarrow n = 7$

b. We have,

$(I-A)(I + A + A^2 + \ + A^7)$

$= I + A + A^2 + \ + A^7 + (-A - A^2 - A^3 - A^4 \ - A^8)$

$= I - A^8$

$= I$ (if $A^8 = O$)

c. Here matrix A is skew-symmetric and since

$|A| = |-A^T| = (-1)^n |A|$, so $|A|(1 - (-1)^n) = 0$.

As n is odd, hence $|A| = 0$. Hence, A is singular.

d. If A is symmetric, A^{-1} is also symmetric for matrix of any order.

3. **a → q; b → p; c → s; d → r.**

a. $|A| = 2 \Rightarrow |2A^{-1}| = 2^3/|A| = 4$

b. $|\text{adj}(\text{adj}(2A))| = |2A|^4 = 2^{12}|A|^4 = 2^{12}/2^{12} = 1$

c. $(A+B)^2 = A^2 + B^2$

$\Rightarrow AB + BA = O$

$\Rightarrow |AB| = |-BA| = -|BA| = -|AB|$

$\Rightarrow |AB| = 0$

$\Rightarrow |B| = 0$

d. Product ABC is not defined.

4. **(3)** Let $A = [A_{ij}]_{3\times 3}$, where $a_{ij} = (k)^{i+j}$

So, $A = \begin{bmatrix} k^2 & k^3 & k^4 \\ k^3 & k^4 & k^5 \\ k^4 & k^5 & k^6 \end{bmatrix}$

a. If A is singular, then $|A| = 0$

$\Rightarrow k^2 \cdot k^3 \cdot k^4 \begin{vmatrix} 1 & 1 & 1 \\ k & k & k \\ k^2 & k^2 & k^2 \end{vmatrix} = 0,$

$\Rightarrow k \in I$

b. If A is null matrix, then $k \in \{0\}$

c. There is no value of k for A to be skew-symmetric matrix which is not null-matrix.

$\therefore \ k \in \phi$

d. If $A^2 = 3A$, then

$$\begin{bmatrix} k^2 & k^3 & k^4 \\ k^3 & k^4 & k^5 \\ k^4 & k^5 & k^6 \end{bmatrix}\begin{bmatrix} k^2 & k^3 & k^4 \\ k^3 & k^4 & k^5 \\ k^4 & k^5 & k^6 \end{bmatrix} = \begin{bmatrix} 3k^2 & 3k^3 & 3k^4 \\ 3k^3 & 3k^4 & 3k^5 \\ 3k^4 & 3k^5 & 3k^6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} k^4+k^6+k^8 & k^5+k^7+k^9 & k^6+k^8+k^{10} \\ k^5+k^7+k^9 & k^6+k^8+k^{10} & k^7+k^9+k^{11} \\ k^6+k^8+k^{10} & k^7+k^9+k^{11} & k^8+k^{10}+k^{12} \end{bmatrix}$$

$$= \begin{bmatrix} 3k^2 & 3k^3 & 3k^4 \\ 3k^3 & 3k^4 & 3k^5 \\ 3k^4 & 3k^5 & 3k^6 \end{bmatrix}$$

$\Rightarrow k \in \{-1, 0, 1\}$

5. **(3)** **a.** $|M_r| = \dfrac{1}{r-1} - \dfrac{1}{r}$

$$\lim_{n\to\infty}\sum_{r=2}^{n}\left(\frac{1}{r-1} - \frac{1}{r}\right)$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right]$$

$= 1 - 0 = 1$

b. $(A+B)^2 = A^2 + B^2$

$\Rightarrow AB + BA = O$

$\Rightarrow AB = -BA$

$\Rightarrow |AB| = |-BA|$

$\Rightarrow |A||B| = -|B||A|$ [A and B are odd ordered matrices]

$\Rightarrow |B| = -|B|$ ($|A| = 2$)

$\Rightarrow |B| = 0$

c. $k^2 = |C| = (\det.A)^2 = 4^2 \Rightarrow k = 4$

d. $A^4 = -4I \Rightarrow \lambda = 4$

Numerical Value Type

1. **(0)** $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow A^4 = A^2 \cdot A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$\Rightarrow \quad A^8 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix}$$

and $A^6 = A^4 \cdot A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix}$

Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$

$$A^8 + A^6 + A^4 + A^2 + I$$

$$\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}$$

$$(A^8 + A^6 + A^4 + A^2 + I)\, V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

or $\begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

or $\begin{bmatrix} 121x \\ 121y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$

$\Rightarrow \quad x = 0$ and $y = 1/11$

$\Rightarrow \quad V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1/11 \end{bmatrix}$

2. (0.25) $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix.

$$\Rightarrow \quad \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

or $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

or $\begin{bmatrix} a^2+bc & ab+b-ab \\ ac+c-ac & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

or $\begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$

or $a^2 + bc = a$

$a - a^2 = bc = 1/4$ (given)

$f(a) = 1/4$

3. (2) $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

$$\Rightarrow \quad A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad A^2 = I \Rightarrow A^4 = A^6 = A^8 = \ = I$

Now $A^x = I$

$\Rightarrow \quad x = 2, 4, 6, 8, \ldots$

$\therefore \quad \Sigma(\cos^x\theta + \sin^x\theta)$

$= (\cos^2\theta + \sin^2\theta) + (\cos^4\theta + \sin^4\theta) + (\cos^6\theta + \sin^6\theta) +$

$= (\cos^2\theta + \cos^4\theta + \cos^6\theta + \ ) + (\sin^4\theta + \sin^4\theta + \sin^6\theta + \ )$

$$= \frac{\cos^2\theta}{1-\cos^2\theta} + \frac{\sin^2\theta}{1-\sin^2\theta}$$

$= \cot^2\theta + \tan^2\theta$

which has minimum value 2

4. (1) $A = \begin{bmatrix} 1 & \tan x \\ -\tan x & 1 \end{bmatrix}$

Hence, det. $A = \sec^2 x$

$\therefore \quad$ det $A^T = \sec^2 x$

Now $f(x) = \text{det.}\,(A^T A^{-1})$

$= (\text{det.}\,A^T)\,(\text{det.}\,A^{-1})$

$= (\text{det.}\,A^T)\,(\text{det.}\,A)^{-1}$

$$= \frac{\text{det.}(A^T)}{\text{det.}(A)} = 1$$

Hence, $f(x) = 1$.

5. (2) $\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & k \end{vmatrix} = 0$

or $1(3k - 16) - 2(k - 12) + 2(4 - 9) = 0$

or $3k - 16 - 2k + 24 - 10 = 0$

or $k = 2$

6. (6) Given $A^2 = A$

Now $I = (I - 0.4A)(I - \alpha A)$

$= I - I\alpha A - 0.4AI + 0.4\alpha A^2$

$= I - A\alpha - 0.4A + 0.4\alpha A$

$= I - A(0.4 + \alpha) + 0.4\alpha A$

$\Rightarrow \quad 0.4\alpha = 0.4 + \alpha$

$\Rightarrow \quad \alpha = -2/3$

$\Rightarrow \quad |9\alpha| = 6$

7. (7) We have $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now $\text{tr}\cdot(AB) = \text{tr}\cdot(C)$

$\Rightarrow \quad 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2 \quad \forall x \in R$ (Identity)

$\Rightarrow \quad 3ax^2 + 6cx + b = 6x^2 + 6x + 4$

$\Rightarrow \quad a = 2, c = 1, b = 4$

8. (8) In a skew symmetric matrix, diagonal elements are zero.

Also $a_{ij} + a_{ji} = 0$

Hence, number of matrices $= 2 \times 2 \times 2 = 8$

9. (0.4) Given that $AA^T = 4I$

$\Rightarrow \quad |A|^2 = 4$

or $|A| = \pm 2$

So $A^T = 4A^{-1} = 4\,\dfrac{adj\,A}{|A|}$

$$\Rightarrow \quad \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \frac{4}{|A|}\begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

Now $a_{ij} = \frac{4}{|A|} c_{ij}$

$\Rightarrow \quad -2c_{ij} = \frac{4}{|A|} c_{ij} (\text{as } a_{ij} + 2c_{ij} = 0)$

$\Rightarrow \quad |A| = -2$

Now $|A + 4I| = |A + AA^T|$

$= |A|\,|I + A^T|$

$= -2\,|(I + A)^T|$

$= -2\,|I + A|$

$\Rightarrow \quad |A + 4I| + 2|A + I| = 0,$

so on comparing, we get $5\lambda = 2 \Rightarrow \lambda = \frac{2}{5}$

10. (0) For idempotent matrix, $A^2 = A$

$\Rightarrow \quad A^{-1}A^2 = A^{-1}A \qquad (\because \ A \text{ is nonsingular})$

or $\quad A = I$

Thus, nonsingular idempotent matrix is always a unit matrix.

$\therefore \quad l^2 - 3 = 1 \quad \Rightarrow \quad l = \pm 2$

$m^2 - 8 = 1 \quad \Rightarrow \quad m = \pm 3$

$n^2 - 15 = 1 \quad \Rightarrow \quad n = \pm 4$

and $p = q = r = 0$

Hence, the required sum is 0.

11. (64) A diagonal matrix is commutative with every square matrix if it is scalar matrix; so every diagonal element is 4. Therefore,

$|A| = 64$

12. (4) $|\text{adj } A^{-1}| = |A^{-1}|^2 = \frac{1}{|A|^2}$

$\Rightarrow \quad |(\text{adj } A^{-1})^{-1}| = \frac{1}{|adj\ A^{-1}|}$

$= |A|^2 = 2^2 = 4$

13. (3) $A^2 + B = I \qquad (1)$

$\therefore \quad A^2B + B^2 = B$

$\Rightarrow \quad B^2 = B$

So, from (1), we get

$A^2 + B^2 = I$

$\Rightarrow \quad \text{tr. } (A^2 + B^2) = 3$

14. (6) $AA^T = I$

$\therefore \quad a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$

This is possible when

(i) $a = \pm 1, b = c = 0$

(ii) $a = 0, b = \pm 1, c = 0$

(iii) $a = 0, b = 0, c = \pm 1$

So, there are six such matrices A.

15. (38) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Given that $|A| = 19$.

$$B = \begin{bmatrix} a_{12} + a_{13} & a_{11} + a_{13} & a_{11} + a_{12} \\ a_{22} + a_{23} & a_{21} + a_{23} & a_{21} + a_{22} \\ a_{32} + a_{33} & a_{31} + a_{33} & a_{31} + a_{32} \end{bmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 19 \times 2 = 38$$

16. (6) $M^3 = M^2M = (I - M)\,M = M - M^2 = 2M - I$

Similarly,

$M^4 = 2I - 3M$

$M^5 = 5M - 3I$

$M^6 = 5I - 8M$

$\therefore \quad n = 6$

17. (4) $$\det.B = \begin{vmatrix} a_{11} & \frac{a_{12}}{3} & \frac{a_{13}}{3^2} \\ 3a_{21} & a_{22} & \frac{a_{23}}{3} \\ 9a_{32} & 3a_{32} & a_{33} \end{vmatrix} = 3 \times 9 \begin{vmatrix} a_{11} & \frac{a_{12}}{3} & \frac{a_{13}}{3^2} \\ a_{21} & \frac{a_{22}}{3} & \frac{a_{23}}{9} \\ a_{32} & \frac{a_{32}}{3} & \frac{a_{33}}{9} \end{vmatrix}$$

(Dividing C_2 by 3 and C_3 by 9)

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{32} & a_{32} & a_{33} \end{vmatrix}$$

$= \text{det. } A = 2$

Similarly, det. C = det. $A = 2$

$\therefore \quad \det.(BC) = (\text{det. } B)(\text{det. } C) = 2 \times 2 = 4$

18. (36) $|A| + \left|\frac{1}{2}A\right| + \left|\frac{1}{4}A\right| + \left|\frac{1}{8}A\right| + \ldots$

$\Rightarrow \quad |A| + \frac{1}{4}|A| + \frac{1}{16}|A| + \ldots$

$\Rightarrow \quad \frac{|A|}{1 - \frac{1}{4}} = \frac{4}{3}|A| = 36$

19. (20) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore \quad A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

$\therefore \quad (\text{tr. } A)^2 - (\text{tr. } A^2) = (a + d)^2 - (a^2 + bc + bc + d^2)$

$= 2(ad - bc) = 2\text{det. } A = 20$

20. (36) $(\text{adj.}(B^{-1}A^{-1}))^{-1} = (\text{adj.}(AB)^{-1})^{-1}$

$= ((\text{adj.}(AB))^{-1})^{-1} \qquad [\text{as adj. } A^{-1} = (\text{adj. } A)^{-1}]$

$= \text{adj.}(AB)$

$\therefore \quad |(\text{adj.}(AB))| = |AB|^2 = |A|^2|B|^2 = 9 \times 4 = 36$

21. (27) We have $A = PQ^{-1}$, $B = QR^{-1}$, $C = RP^{-1}$

$\therefore \quad ABC = I, BCA = I, CAB = I$

$\therefore \quad \det.(ABC + BCA + CAB) = \det.(3I) = 3^3 \times \det.(I) = 27$

22. (11) $A = \text{adj. } B$

$\Rightarrow \quad |A| = |\text{adj. } B| = |B|^2$

$\therefore \quad |A| = (4)^2 \Rightarrow |A| = 16$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{vmatrix} = 16$$

$\Rightarrow \quad 1(0) - x(-2) + 3(-2) = 16$

$\Rightarrow \quad 2x = 22$

$\Rightarrow \quad x = 11$

23. (9) $\det.(\text{adj.}(\text{adj.}(ABC))) = 2^{16} \times 3^8 \times 7^4$

$|A|^4 |B|^4 |C|^4 = 2^{16} \times 3^8 \times 7^4$

$\Rightarrow \quad (xyz)^4 \times 2^8 \times 3^8 \times 2^4 = 2^{16} \times 3^8 \times 7^4$

$\Rightarrow \quad (xyz)^4 = 2^4 \times 7^4$

$\Rightarrow \quad xyz = 2^1 \times 7^1$

Number of possible solutions $= {}^3C_2 \times {}^3C_2 = 9$

24. (12) adj. $A = -A$

$\Rightarrow \quad A \cdot \text{adj. } A = -A^2 = |A| I = -I$

$\Rightarrow \quad A^2 = I$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\Rightarrow \quad A^2 = \begin{bmatrix} a^2+bc & (a+d)b \\ (a+d)c & d^2+bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

On comparing both sides, we get

$a^2 + bc = 1, \quad (a+d)b = 0, \quad (a+d)c = 0, \quad d^2 + bc = 1$

Case I: When $(a + d) \neq 0$

$\Rightarrow \quad b = 0 = c$ and $a = 1, d = 1$

or $\quad a = -1, d = -1$

$\therefore \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

But both are rejected as det. $A = -1$ (given.)

Case II: When $(a + d) = 0 \Rightarrow d = -a$

(i) If $a = 1, d = -1 \Rightarrow bc = 0$

For $b = 0$, c can be $-1, 0, 1$.

For $b = 1$, c can be 0 only.

For $b = -1$, c can be 0 only.

So, 5 matrices are possible.

(ii) If $a = -1, d = 1$

$\Rightarrow \quad bc = 0$

For $b = 0, c = -1, 0, 1$.

For $b = 1, c = 0$ only.

For $b = -1, c = 0$ only.

So, 5 matrices are possible.

(iii) If $a = 0, \ d = 0$

$\Rightarrow \quad bc = 1$

$\therefore \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

So, 2 matrices are possible.

Therefore, total number of matrices is 12.

Archives

JEE Main

Single Correct Answer Type

1. (2) $|\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$

$\text{adj } (\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$

Thus both statements are true, but statement 2 is not correct explanation of statement 1.

2. (1) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$\therefore \quad \det.(A) = a_1b_2c_3 + a_2b_3c_1 + b_1c_2a_3 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2$

If any of the terms is non-zero, then $\det.(A)$ will be non-zero and all the elements of that term will be unity.

Now there are 6 elements remaining out of which any one can be unity.

Hence number of non-singular matrices

= (no. of ways of choosing any one triplet) × (no. of ways of choosing any one element)

$= {}^6C_1 \times {}^6C_1$

3. (4) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a, b, c, d \neq 0$

$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

Given $A^2 = I$

$\Rightarrow \quad a^2 + bc = 1, \ bc + d^2 = 1$

$ab + bd = ac + cd = 0$

$c \neq 0$ and $b \neq 0 \Rightarrow a + d = 0$

Trace $A = a + d = 0$

$\therefore \quad |A| = ad - bc = -a^2 - bc = -1$

4. (3) $(A(BA))^T = (BA)^T A^T$

$= (A^T B^T) \cdot A^T$

$= (AB) A$

$= A(BA)$

Similarly $((AB)A)^T = (AB)A$

Clearly both statements are true but statement 2 is not a correct explanation of statement 1.

5. (4) $A(u_1 + u_2) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Now $|A| = 1$

$A^{-1} = \dfrac{1}{|A|} \text{ adj } A$

$u_1 + u_2 = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

6. (3) We have $P^3 = Q^3$ and $P^2Q = PQ^2$

Subtracting, we get

$P^3 - P^2Q = Q^3 - Q^2P$

$P^2(P - Q) + Q^2(P - Q) = O$

$(P^2 + Q^2)(P - Q) = O$

If $|P^2 + Q^2| \neq O$ then $P^2 + Q^2$ is invertible

$\Rightarrow \quad P - Q = O$ (contradiction)

Hence $|P^2 + Q^2| = 0$

7. (2) Given P is adjoint of A.

$\therefore \quad |P| = |A|^2$

$\therefore \quad \begin{vmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{vmatrix} = |A|^2 = 4^2$ (Given)

$\therefore \quad 2\alpha - 6 = 16$

$\therefore \quad \alpha = 11.$

8. (2) $B = A^{-1} A'$

$\Rightarrow \quad AB = A'$

$ABB' = A'B' = (BA)'$

$= (A^{-1} A'A)' = (A^{-1} AA')' = A$

$\Rightarrow \quad BB' = I$

9. (4) $AA^T = 9I$

$$\Rightarrow \begin{bmatrix}1 & 2 & 2\\ 2 & 1 & -2\\ a & 2 & b\end{bmatrix}\begin{bmatrix}1 & 2 & a\\ 2 & 1 & 2\\ 2 & -2 & b\end{bmatrix} = 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}9 & 0 & a+4+2b\\ 0 & 9 & 2a+2-2b\\ a+4+2b & 2a+2-2b & a^2+4+b^2\end{bmatrix} = \begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$\Rightarrow a+4+2b=0,\ 2a+2-2b=0,\ a^2+4+b^2=9$

$\Rightarrow a=-2,\ b=-1$

10. (1) $A \text{ adj } A = AA^T$

$\Rightarrow |A|I = AA^T$

$$\Rightarrow (10a+3b)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}5a & -b\\ 3 & 2\end{bmatrix}\begin{bmatrix}5a & 3\\ -b & 2\end{bmatrix}$$

$\Rightarrow 25a^2+b^2 = 10a+3b,\ 15a-2b=0,\ 10a+3b=13$

$$\Rightarrow 10a+\frac{3.15a}{2}=13$$

$\Rightarrow 65a = 2\times 13$

$\Rightarrow a = \frac{2}{5}$

$\Rightarrow 5a = 2$

$\Rightarrow 2b = 6$

$\Rightarrow b = 3$

$\therefore 5a+b=5$

11. (3) Given $A = \begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix} = \begin{bmatrix}16 & -9\\ -12 & 13\end{bmatrix}$$

$$\Rightarrow 3A^2+12A = \begin{bmatrix}48 & -27\\ -36 & 39\end{bmatrix}+\begin{bmatrix}24 & -36\\ -48 & 12\end{bmatrix}$$

$$= \begin{bmatrix}72 & -63\\ -84 & 51\end{bmatrix}$$

$$\text{adj}\,(3A^2+12A) = \begin{bmatrix}51 & 63\\ 84 & 72\end{bmatrix}$$

JEE ADVANCED

Single Correct Answer Type

1. (1) Three planes cannot intersect at two distinct points.

2. (1) For being nonsingular,

$$\begin{vmatrix}1 & a & b\\ \omega & 1 & c\\ \omega^2 & \omega & 1\end{vmatrix} \neq 0$$

or $ac\omega^2 - (a+c)\,\omega + 1 \neq 0$

Hence, number of possible triplets of (a, b, c) is 2.

i.e., $(\omega, \omega^2, \omega)$ and (ω, ω, ω).

3. (4) $|Q| = \begin{vmatrix}2^2a_{11} & 2^3a_{12} & 2^4a_{13}\\ 2^3a_{21} & 2^4a_{22} & 2^5a_{23}\\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33}\end{vmatrix}$

$$= 2^2\cdot 2^3\cdot 2^4\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 2a_{21} & 2a_{22} & 2a_{23}\\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33}\end{vmatrix}$$

$$= 2^9\cdot 2\cdot 2^2\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

$= 2^{12}|P|$

$= 2^{13}$

4. (2) $P^2 = \begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 8 & 1 & 0\\ 48 & 8 & 1\end{bmatrix}$

$$P^3 = \begin{bmatrix}1 & 0 & 0\\ 8 & 1 & 0\\ 48 & 8 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 16 & 4 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 12 & 1 & 0\\ 96 & 12 & 1\end{bmatrix}$$

$$\therefore P^n = \begin{bmatrix}1 & 0 & 0\\ 4n & 1 & 0\\ 8(n^2+n) & 4n & 1\end{bmatrix}$$

$$\therefore P^{50} = \begin{bmatrix}1 & 0 & 0\\ 200 & 1 & 0\\ 8\times 50(51) & 200 & 1\end{bmatrix}$$

$P^{50} - Q = I$

$\therefore$ Equating we get

$200 - q_{21} = 0 \Rightarrow q_{21} = 200$

$400\times 51 - q_{31} = 0$

$\Rightarrow q_{31} = 400\times 51$

$200 - q_{32} = 0 \Rightarrow q_{32} = 200$

$$\Rightarrow \frac{q_{31}+q_{32}}{q_{21}} = \frac{400\times 51+200}{200} = 2(51)+1 = 103$$

5. (1) Let $M = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$

$\therefore \text{tr}(M^TM) = a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2+i^2$

$= 5$, where entries are $\{0, 1, 2\}$

Only two cases are possible

Case I: five entries are 1 and other four are 0.

$\therefore$ Number of matrices $= {}^9C_5\times 1 = 126$

Case II: One entry is 2, one entry is 1 and others are 0.

$\therefore$ Number of matrices $= {}^9C_2\times 2! = 72$

So, total number of matrices $= 126+72 = 198$

Multiple Correct Answers Type

1. (3)

Given $MN = NM$

$M^2N^2\,(M^TN)^{-1}\,(MN^{-1})^T$

$= M^2N^2\,N^{-1}\,(M^T)^{-1}\,(N^{-1})^T\,.\,M^T$

$= M^2N.\,(M^T)^{-1}(N^{-1})^T\,M^T$

$= -M^2N(M)^{-1}\,(N^T)^{-1}\,M^T$

$= +M^2NM^{-1}N^{-1}\,M^T$

$= -MNMM^{-1}\,N^{-1}\,M$ $\quad (\because MN = NM)$

$= -MNN^{-1}M = -M^2$

2. (2), (3), (4)

$$P=\begin{bmatrix}\omega^2 & \omega^3 & \omega^4 & \cdots & \omega^{n+1}\\ \omega^3 & \omega^4 & \omega^5 & \cdots & \omega^{n+2}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \omega^{n+1} & \omega^{n+2} & \cdots & \cdots & \omega^{2n}\end{bmatrix}$$

or

$$P^2=\begin{bmatrix}\omega^4+\omega^6\ldots & \omega^5+\omega^7+\ldots & \cdots & \cdots\\ \omega^5+\omega^7+\ldots & \cdots & \cdots & \cdots\\ \vdots & \vdots & \vdots & \cdots\\ \omega^{n+2}+\omega^{n+4}\ldots & \cdots & \cdots & \omega^{2n+2}+\omega^{2n+4}\ldots\end{bmatrix}$$

= Null matrix if n is a multiple of 3.

3. (3), (4)

(1) $(N^TMN)^T = N^TM^TN = N^TMN$ if M is symmetric and is $-N^TMN$ if M is skew symmetric

(2) $(MN-NM)^T = N^TM^T - M^TN^T = NM - MN = -(MN-NM)$.

So, $(MN-NM)$ is skew symmetric

(3) $(MN)^T = N^TM^T = NM \neq MN$ if M and N are symmetric.

So, MN is not symmetric

(4) (adj. M) (adj. N) = adj(NM) ≠ adj (MN).

4. (3), (4)

Let $M=\begin{bmatrix}a & c\\ c & b\end{bmatrix}$ (where $a, b, c \in I$)

(1) If the first column of M is the transpose of the second row of M, then

$[a \quad c] = [c \quad b]$

$\therefore \quad a = b = c$

Thus, det.$(M) = ab - c^2 = 0$

Hence, M is not invertible.

(2) If the second row of M is the transpose of the first column of M, then

$[c \quad b] = [a \quad c]$

$\therefore \quad a = b = c$

Thus, det.$(M) = ab - c^2 = 0$

Hence, M is not invertible.

(3) If $M=\begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$, with $a, b \neq 0$, then

det.$(M) = ab \neq 0$

Hence, M is invertible.

(4) If product of elements in main diagonal which (ab) is not perfect square, then

det.$(M) = ab - c^2 \neq 0$

Hence, M is invertible.

5. (1), (2)

$M^2 = N^4$

$\Rightarrow \quad M^2 - N^4 = O$

$\Rightarrow \quad (M-N^2)(M+N^2) = O$ (as M, N commute)

Also, $M \neq N^2$.

det.$((M-N^2)(M+N^2)) = 0$

as $M \neq N^2 \Rightarrow$ det. $(M+N^2) = 0$

Also, det. $(M^2 + MN^2) =$ (det. M)(det. $(M+N^2)) = 0$

Hence, there exist non-null U such that $(M^2 + MN^2)\,U = O$.

6. (3), (4)

(1) $(Y^3Z^4 - Z^4Y^3)^T = (Y^3Z^4)^T - (Z^4Y^3)^T$

$= (Z^T)^4(Y^T)^3 - (Y^T)^3(Z^T)^4$

$= -Z^4Y^3 + Y^3Z^4$

$\Rightarrow$ $Y^3Z^4 - Z^4Y^3$ is symmetric but not skew-symmetric

(2) $(X^{44} + Y^{44})^T = (X^T)^{44} + (Y^T)^{44}$

$= (-X)^{44} + (-Y)^{44}$

$= X^{44} + Y^{44}$

$\Rightarrow$ $X^{44} + Y^{44}$ is symmetric but not skew-symmetric

(3) $(X^4Z^3 - Z^3X^4)^T = (X^4Z^3)^T - (Z^3X^4)^T$

$= (Z^T)^3(X^T)^4 - (X^T)^4(Z^T)^3$

$= Z^3X^4 - X^4Z^3$

$= -(X^4Z^3 - Z^3X^4)$

$\Rightarrow$ $X^4Z^3 - Z^3X^4$ is skew-symmetric

(4) $(X^{23} + Y^{23})^T = (X^T)^{23} + (Y^T)^{23}$

$= (-X)^{23} + (-Y)^{23}$

$= -X^{23} - Y^{23}$

$\Rightarrow$ $X^{23} + Y^{23}$ is skew-symmetric

7. (2), (3)

$PQ = kI$

$\therefore \quad Q = kP^{-1}I = kP^{-1}$

$$\therefore \quad Q = k\frac{\text{adj}(P)}{|P|} = \frac{k}{(12\alpha+20)}\begin{bmatrix}- & - & -\\ - & - & (-3\alpha-4)\\ - & - & -\end{bmatrix}$$

Comparing q_{23}, we get

$$\frac{-k}{8} = \frac{-k(3\alpha+4)}{(12\alpha+20)}$$

$\Rightarrow \quad \alpha = -1$

Also $PQ = kI$

$\Rightarrow$ det(P) . det$(Q) = k^3$

$\therefore \quad (12\alpha+20)\,\frac{k^2}{2} = k^3$

$\therefore \quad k = 6\alpha + 10 = 4$

$\therefore \quad 4\alpha - k + 8 = 0$

det $(P$ adj $(Q))$ = det (P) det(adj $(Q))$

$= 8 \times (\text{det}(Q))^2$

$= 8 \times 8^2$

$= 2^9$

det $(Q$ adj $(P))$ = det (Q) det(adj $(P))$

$= 2^9$

8. (1), (3)

$A = B^2 \Rightarrow |A| = |B|^2 =$ +ve

(1) $\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{vmatrix} = 1(-1) < 0$

Matrix B can not be possible.

(2) $\begin{vmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{vmatrix} = 1\,(1-0) > 0$

Matrix B can be possible.

(3) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1 < 0$

Matrix B can not be possible.

(4) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0$

Matrix B can be I (identity matrix).

9. (1), (4)

Let the given equations represent planes P_1, P_2 and P_3.

$$\Delta = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 0$$

Since no pair of planes is parallel, there are infinite number of solutions.

Let $\alpha P_1 + \beta P_2 = P_3$

$\therefore \quad P_1 + 7P_2 = 13P_3$

$\therefore \quad b_1 + 7b_2 = 13b_3$

(1) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$.
Since $\Delta \neq 0$, system has at least one solution for any set of values of b_1, b_2 and b_3.

(2) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
Since $\Delta = 0$ and $b_1 + 7b_2 \neq 13b_3$, system of equations has no solution.

(3) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$.
Since planes are parallel, there is no solution for any set of values of b_1, b_2 and b_3.

(4) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$
Since $\Delta \neq 0$, system has at least one solution for any set of values of b_1, b_2 and b_3.

Linked Comprehension Type

1. (1) Let the matrix be

$$\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$$

We have five entries as 1 and remaining four entries as 0.

Since matrix is symmetric, we must have even number of zeros for $i \neq j$. We have two cases.

(i) Two entries in diagonal are zero. We can select two places from three (in diagonal) in ${}^3C_2$ ways. Now we have to select elements for upper triangle. For upper triangle, we have three places of which one entry is '0' and two are '1'. One place from three can be selected in ${}^3C_1$ ways. Hence, the number of matrices is ${}^3C_2 \times {}^3C_1 = 9$.

(ii) If all the entries in the principal diagonal are 1, we have two '0' and one '1' in upper triangle. Hence, the number of matrices is 3. Therefore, total number of matrices is 12.

2. (2) $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{bmatrix}$

Either $b = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix}$$

Either $a = 0$ or $c = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$

Either $a = 0$ or $b = 0 \Rightarrow |A| \neq 0$

$\Rightarrow$ two matrices

$$A = \begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix}$$

$a = b = 0 \Rightarrow |A| = 0$

$a = c = 0 \Rightarrow |A| = 0$

$b = c = 0 \Rightarrow |A| = 0$

Therefore, there will be only six matrices.

3. (2) The six matrices A for which $|A| = 0$ are:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$ (inconsistent)

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ (inconsistent)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ (infinite solutions)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (inconsistent)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (inconsistent)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$ (inconsistent)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$ (inconsistent)

4. (4) We must have $a^2 - b^2 = kp$

$\Rightarrow \quad (a + b)(a - b) = kp$

$\Rightarrow$ either $a - b = 0$ or $a + b$ is multiple of p

when $a = b$; number of matrices is p

and when $a + b =$ multiple of $p \Rightarrow a, b$ has $p - 1$

$\therefore$ Total number of matrices $= p + p - 1 = 2p - 1$.

5. (3)

6. (4)

7. (4) $a + 8b + 7c = 0$

$9a + 2b + 3c = 0$

$a + b + c = 0$

Solving these, we get

$b = 6a \Rightarrow c = -7a$

now $2x + y + z = 0$

$\Rightarrow \quad 2a + 6a + (-7a) = 1$

$\Rightarrow \quad a = 1, b = 6, c = -7.$

8. (1) $a = 2$, b and c satisfies (E)

$b = 12, c = -14$

$$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$$

$$= \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}}$$

$$= 3\omega + 1 + 3\omega^2$$

$$= -2.$$

9. (2) $ax^2 + bx + c = 0$

$\Rightarrow \quad x^2 + 6x - 7 = 0$

$\Rightarrow \quad \alpha = 1, \beta = -7$

$$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{1} - \frac{1}{7}\right)^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n$$

$$= \frac{1}{1 - \frac{6}{7}} = 7$$

Numerical Value Type

1. (4) $|A| = (2k+1)^3$, $|B| = 0$ (since B is a skew-symmetric matrix of order 3)

$\Rightarrow$ $\det(\text{adj } A) = |A|^{n-1} = ((2k+1)^3)^2 = 10^6$

$\Rightarrow \quad 2k + 1 = 10 \quad \text{or} \quad 2k = 9$

$\Rightarrow \quad [K] = 4.$

2. (9) Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$M\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \quad b = -1, e = 2, h = 3$$

$$M\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \quad a = 0, d = 3, g = 2$$

$$M\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix} = g + h + i = 12 \Rightarrow \quad i = 7$$

Therefore, sum of diagonal elements = 9.

3. (1) $Z = \dfrac{-1 + i\sqrt{3}}{2} = \omega$

$$P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}\begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$$

$$= \begin{bmatrix} (-\omega)^{2r} + (\omega^{2s})^2 & \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} \\ \omega^{2s}(-\omega)^r + \omega^r\omega^{2s} & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$$= \begin{bmatrix} \omega^{4s} + \omega^{2r} & \omega^{2s}(\omega^r + (-\omega)^r) \\ \omega^{2s}(\omega^r + (-\omega)^r) & \omega^{4s} + \omega^{2r} \end{bmatrix}$$

$= -I$ (Given)

$\therefore \quad \omega^{4s} + \omega^{2r} = -1$ and

$\omega^{2s}(\omega^r + (-\omega)^r) = 0$

$\therefore \quad \omega^r + (-\omega)^r = 0$

$\therefore \quad r = 1$ and $s = 1$

So only one pair

Chapter 14

Concept Application Exercises

Exercise 14.1

1. Let E be the event in which all three coins show tail and F be the event in which a coin shows tail. Therefore, $F = \{HHT, HTH, THH, HTT, THT, TTH, TTT\}$ and $E = \{TTT\}$. Hence, the required probability is

$$\therefore \quad P(E/F) = \frac{P(E \cap F)}{P(E)} = \frac{1}{7}$$

2. $$P[B/(A \cup B')] = \frac{P(B \cap (A \cup B'))}{P(A \cup B')}$$

$$= \frac{P(A \cap B)}{P(A) + P(B') - P(A \cap B')}$$

$$= \frac{P(A) - P(A \cap B')}{P(A) + P(B') - P(A \cap B')}$$

$$= \frac{0.7 - 0.5}{0.8} = \frac{1}{4}$$

3. Since 4 has appeared on the first, so we are required 4 or 5 or 6 on second dice; hence, required probability is $3/6 = 1/2$.

4. $P(E) = \frac{4}{8} = \frac{1}{2}$ [$\because$ favorable cases are THH, HTH, HHT, HHH]

$P(F) = \frac{4}{8} = \frac{1}{2}$ [$\because$ favorable cases are HTH, HHT, HTT, HHH]

$$P(E \cap F) = \frac{3}{8}$$

$$P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{3/8}{1/2} = \frac{3}{4}$$

5. Let M be the event that man is chosen and E be the event that chosen one is employed. From the concept of reduced sample space, we immediately get

$$P(M/E) = \frac{460}{600} = \frac{23}{30}$$

Also, $$P(E) = \frac{600}{900} = \frac{2}{3}$$

$$P(E \cap M) = \frac{460}{900} = \frac{23}{45}$$

$$\Rightarrow \quad P(E/M) = \frac{23/45}{2/3} = \frac{23}{30}$$

Exercise 14.2

1. Required probability

$$= P(WBWB) + P(BWBW)$$

$$= P(W)\,P(B/W)P(W/WB)\,P(B/WBW) + P(B)P(W/B)\,P(B/BW)\,P(W/BWB)$$

$$= \frac{5}{8}\times\frac{3}{7}\times\frac{4}{6}\times\frac{2}{5} + \frac{3}{8}\times\frac{5}{7}\times\frac{2}{6}\times\frac{4}{5}$$

$$= \frac{1}{14} + \frac{1}{14} = \frac{1}{7}$$

2. We must have one ace in $(n-1)$ attempts and one ace in the nth attempt.

Let event A be one ace card and $(n-2)$ other than ace cards are drawn in first $(n-1)$ draws.

$$\therefore \quad P(A) = \frac{{}^4C_1 \times {}^{48}C_{n-2}}{{}^{52}C_{n-1}}$$

Let event B be nth card is ace.

Therefore, required probability is

$$P(A \cap B) = P(A)\,P(B/A)$$

Where, $$P(B/A) = \frac{{}^3C_1}{{}^{[52-(n-1)]}C_1} = \frac{3}{53-n}$$

$$\therefore \quad P(A \cap B) = \frac{4 \times 48!}{(n-2)!(50-n)!} \times \frac{(n-1)!(53-n)!}{52!} \times \frac{3}{53-n}$$

$$= \frac{(n-1)(52-n)(51-n)}{50 \times 49 \times 17 \times 13}$$

3. Since the question is multiple choice in which one or more than one options are correct, correct options may be one, two, three or four.

$\therefore$ Total number of ways of ticking one or more alternatives out of four $= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

Out of these 15 combinations, only one combination is correct.

$\therefore$ Probability of ticking the alternative correctly at the first trial,

$$P(T_1) = \frac{1}{15}$$

Probability of ticking the answer correctly in second trial,

$$P(T_1' \cap T_2) = P(T_1')\,P(T_2/T_1')$$

$$= \frac{14}{15} \times \frac{1}{14} = \frac{1}{15}$$

Probability of ticking the answer correctly in third trial,

$$P(T_1' \cap T_2' \cap T_3) = P(T_1')\,P(T_2'/T_1')\,P(T_3/T_1'T_2')$$

$$= \frac{14}{15} \times \frac{13}{14} \times \frac{1}{13} = \frac{1}{15}$$

Thus, the probability that the candidate will get marks on the question if he is allowed up to three trials

$$= \frac{1}{15} + \frac{1}{15} + \frac{1}{15} = \frac{1}{5}$$

Exercise 14.3

1. $P(A) = \frac{3}{8}$ and $P(B) = \frac{1}{2}$

$$P(A)\,P(B) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16}$$

$$P(A \cap B) = \frac{2}{8} = \frac{1}{4} \neq P(A)\,P(B)$$

Hence, A and B are not independent.

2. It is given that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$,

and $P(A \cap B) = \frac{1}{4}$

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$\Rightarrow \quad P(A' \cup B') = \frac{1}{4}$

$\Rightarrow \quad P((A \cap B)') = \frac{1}{4} \qquad [\because A' \cup B' = (A \cap B)']$

$\Rightarrow \quad 1 - P(A \cap B) = \frac{1}{4}$

or $\quad P(A \cap B) = \frac{3}{4} \qquad (1)$

However,

$$P(A) \cdot P(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24} \qquad (2)$$

Here, $\frac{3}{4} \neq \frac{7}{24}$

$\therefore \quad P(A \cap B) \neq P(A) \cdot P(B)$

Therefore, A and B are not independent events.

3. Probability of first card to be a king, $P(A) = 4/52$ and probability of second to be a king, $P(B) = 3/51$. Hence, required probability is

$$P(A \cap B) = \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

4. Here $P(A) = 0.4$ and $P(\bar{A}) = 0.6$. Probability that A does not happen at all is $(0.6)^3$. Thus, required probability is $1 - (0.6)^3 = 0.784$.

5. P (six balls drawn follow the pattern $WBWBWB$)

$$= \left(\frac{3}{6}\right)^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$\therefore$ P(six balls drawn follow the pattern $BWBWBW$)

$$= \left(\frac{3}{6}\right)^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$$

$\therefore$ Required probability $= \frac{1}{64} + \frac{1}{64} = \frac{1}{32}$

6. Let probability of success in a trial is 'p'

Then $P(X = 4) = {}^{10}C_4 p^4 (1-p)^6$

$P(X) = {}^{10}C_4 [4p^3(1-p)^6 - 6(1-p)^5 p^4]$

$= {}^{10}C_4 p^3 (1-p)^5 (4 - 10p)$

For maximum $P'(X) = 0$

$\Rightarrow \quad P(X) = \frac{2}{5}$

$\Rightarrow \quad P'(X) = \frac{3}{5}$

7. Let E_1 be the events that man will be selected and E_2 the events that woman will be selected. Then,

$$P(E_1) = \frac{1}{4}, \text{ so } P(\bar{E}_1) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(E_2) = \frac{1}{3}, \text{ so } P(\bar{E}_2) = 1 - \frac{1}{3} = \frac{2}{3}$$

Clearly E_1 and E_2 are independent events. So,

$$P(\bar{E}_1 \cap \bar{E}_2) = P(\bar{E}_1) \times P(\bar{E}_2) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$

8. $P(K) = \frac{7}{15} \therefore P(\bar{K}) = 1 - \frac{7}{15} = \frac{8}{15}$

$P(H) = \frac{7}{10} \therefore P(\bar{H}) = 1 - \frac{7}{10} = \frac{3}{10}$

Hence, required probability is

$$P(\bar{K} \cap \bar{H}) = P(\bar{K})\, P(\bar{H}) = \frac{8}{15} \times \frac{3}{10} = \frac{24}{150} = \frac{4}{25}$$

9. Probability for an incorrect digit is p. Hence, probability for a correct digit is $1 - p$. Hence, probability for 8 correct digit is $(1-p)^8$. Hence, required probability is $1 - (1-p)^8$.

10. Required probability is

$$P(\bar{A}_1 \cap A_2 \cap A_3) + P(A_1 \cap \bar{A}_2 \cap A_3)$$

$$= P(\bar{A}_1)P(A_2)P(A_3) + P(A_1)P(\bar{A}_2)P(A_3)$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} + \frac{1}{8} = \frac{2}{8} = \frac{1}{4}$$

11. Let p be the chance of cutting a spade and q be the chance of not cutting a spade from the pack of 52 cards.

Then $p = 13/52 = 1/4$ and $q = 1 - 1/4 = 3/4$.

Now, A will win a prize if the cuts spade at 1st, 4th, 7th, 10th turns etc. Note that A will get a second chance if A, B and C all fail to cut spade once and then A cuts a spade at the 4th turn.

Similarly, he will cut a spade at the 7th turn when each of A, B, C fail to cut spade twice etc. Hence, A's chance of winning the prize

$$= p + q^3 p + q^6 p + q^9 p + \ldots$$

$$= \frac{p}{1-q^3} = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^3} = \frac{16}{37}$$

Similarly, B's chance

$$= (qp + q^4 p + q^7 p + \ldots)$$

$$= q(p + q^3 p + q^6 p) = 12/37$$

and C's chance $= 1 - (16/37) - (12/37) = 9/37$

12. Let

W: event of drawing a white ball at any draw; and

B: event of drawing a black ball at any draw

Then

$$P(W) = \frac{a}{a+b} \text{ and } P(B) = \frac{b}{a+b}$$

$P(A$ wins the game$) = P(W$ or BBW or $BBBBW$ or ...)

$= P(W) + P(BBW) + P(BBBBW) + \ldots$

$= P(W) + P(B)\,P(B)\,P(W) + P(B)\,P(B)\,P(B)\,P(B)\,P(W) + \ldots$

$= P(W) + P(W)\,(P(B))^2 + P(W)\,(P(B))^4 + \ldots$

$$= \frac{P(W)}{1 - (P(B))^2} = \frac{a(a+b)}{a^2 + 2ab} = \frac{a+b}{a+2b}$$

Also, $P(B$ wins the game$) = 1 - \frac{a+b}{a+2b} = \frac{a}{a+2b}$

According to the given condition,

$\Rightarrow \quad \frac{a+b}{a+2b} = 3 \times \frac{b}{a+2b}$

$\Rightarrow \quad a = 2b$

$\Rightarrow \quad a : b = 2 : 1$

Exercise 14.4

1. The urn contains 5 red and 5 black balls.

Let a red ball be drawn in the first attempt. Therefore,

$$P(\text{drawing a red ball}) = \frac{5}{10} = \frac{1}{2}$$

If two red balls are added to the urn, then the urn contains 7 red and 5 black balls. Thus,

$$P(\text{drawing a red ball}) = \frac{7}{12}$$

Let a black ball be drawn in the first attempt. Then

$$P(\text{drawing a black ball in the first attempt}) = \frac{5}{10} = \frac{1}{2}$$

If two black balls are added to the urn, then the urn contains 5 red and 7 black balls.

$$P(\text{drawing a red ball}) = \frac{5}{12}$$

Therefore, probability of drawing the second ball as red is

$$\frac{1}{2} \times \frac{7}{12} + \frac{1}{2} \times \frac{5}{12} = \frac{1}{2}\left(\frac{7}{12} + \frac{5}{12}\right) = \frac{1}{2} \times 1 = \frac{1}{2}$$

2. If third ball is red, then in first two draws, there will be either no red ball or one red ball.

Let events

R_0 = in first two draws, no red ball is drawn

R_1 = in first two draws, one red ball is drawn.

R = in third draw, red ball is drawn.

So, from total probability theorem,

$$P(R) = P(R_0)\,P(R/R_0) + P(R_1)\,P(R/R_1)$$

$$= \frac{{}^6C_2}{{}^8C_2} \cdot \frac{{}^2C_1}{{}^6C_1} + \frac{{}^6C_1{}^2C_1}{{}^8C_2} \cdot \frac{{}^1C_1}{{}^6C_1}$$

$$= \frac{15}{28} \cdot \frac{2}{6} + \frac{6 \times 2}{28} \cdot \frac{1}{6}$$

$$= \frac{1}{4}$$

3. We have $P(G) = 1/3$ and $P(B) = 2/3$

$P(A/G) = 0.25$ and $P(A/B) = 0.28$

Here, A is the event that student chosen will get a first class.

So, using total probability theorem,

$$P(A) = P(B)\,P(A/B) + P(G)\,P(A/G)$$

$$= \frac{2}{3} \times 0.28 + \frac{1}{3} \times 0.25$$

$$= 0.27$$

4. Let C be the event of the selected number being composite and E be the event of there being no remainder.

$$P(C) = \frac{n(C)}{n(S)} = \frac{15}{25} = \frac{3}{5},\ P(\bar{C}) = \frac{2}{5}$$

$$P(E/C) = \frac{4}{15},\ P(E/\bar{C}) = \frac{1}{10}$$

$$P(E) = P(C)\,P(E/C) + P(\bar{C})\,P(E/\bar{C}).$$

$$= \frac{3}{5} \times \frac{4}{15} + \frac{2}{5} \times \frac{1}{10} = \frac{4}{25} + \frac{1}{25} = \frac{1}{5} = 0.2$$

5. Let A_1 and A_2 be the events that the specific home is left unlocked, and is left locked, respectively. Then $P(A_1) = 0.4$, $P(A_2) = 0.6$.

Let A be the event that the real estate man get into the specific home. Then

$$P(A/A_1) = 1,\ P(A/A_2) = \frac{{}^7C_2}{{}^8C_3} = \frac{3}{8}$$

$$\Rightarrow \quad P(A) = P(A_1)\,P(A/A_2) + P(A_2)\,P(A/A_2)$$

$$= (0.4)\,(1) + (0.6)\,(3/8) = \frac{4}{10} + \frac{18}{80} = \frac{5}{8}$$

6. Let W_1 (B_1) be the event that a white (a black) ball is drawn in the first draw and let W be the event that white ball is drawn in the second draw. Then

$$P(W) = P(B_1)\,P(W/B_1) + P(W_1)\,P(W/W_1)$$

$$= \frac{n}{m+n}\,\frac{m}{m+n+k} + \frac{m}{m+n}\,\frac{m+k}{m+n+k}$$

$$= \frac{m(n+m+k)}{(m+n)(m+n+k)}$$

$$= \frac{m}{m+n}$$

7. Let us define the following events

A: 4 white balls are drawn in first six draws

B: 5 white balls are drawn in first six draws

C: 6 white balls are drawn in first six draws

E: exactly one white ball is drawn in next two draws (i.e., one white and one red)

Then

$$P(E) = P(E/A)\,P(A) + P(E/B)\,P(B) + P(E/C)\,P(C)$$

But

$P(E/C) = 0$ [as there are only 6 white balls in the bag]

$\therefore$ $P(E) = P(E/A)\,P(A) + P(E/B)\,P(B)$

$$= \frac{{}^{10}C_1 \times {}^2C_1}{{}^{12}C_2}\,\frac{{}^{12}C_2 \times {}^6C_4}{{}^{18}C_6} + \frac{{}^{11}C_1 \times {}^1C_1}{{}^{12}C_1}\,\frac{{}^{12}C_1 \times {}^6C_5}{{}^{18}C_6}$$

Exercise 14.5

1. Let A_1 be the event that missing card is spade and A_2 be event that missing card is non-spade. Then,

$$P(A_1) = \frac{1}{4},\ P(A_2) = \frac{3}{4}$$

Let A be the event that 2 spade cards are drawn from the remaining cards. Then,

$$P\left(\frac{A}{A_2}\right) = \frac{{}^{12}C_2}{{}^{51}C_2} \text{ and } P\left(\frac{A}{A_2}\right) = \frac{{}^{13}C_2}{{}^{51}C_2}$$

$$\Rightarrow \quad P(A) = P(A_1)\,P\left(\frac{A}{A_1}\right) + P(A_2)\,P\left(\frac{A}{A_2}\right)$$

$$= \frac{1}{4}\frac{{}^{12}C_2}{{}^{51}C_2} + \frac{3}{4}\frac{{}^{13}C_2}{{}^{51}C_2}$$

$$= \frac{1}{4 \times {}^{51}C_2}\left[{}^{12}C_2 + 3 \times {}^{13}C_2\right]$$

Now,

$$P\left(\frac{A_1}{A}\right) = \frac{P(A_1)P\left(\frac{A}{A_1}\right)}{P(A)}$$

$$= \frac{\frac{1}{4}\frac{{}^{12}C_2}{{}^{51}C_2}}{\frac{1}{4 \times {}^{51}C_2}\left[{}^{12}C_2 + 3 \times {}^{13}C_2\right]}$$

$$= \frac{11}{50}$$

2. Let E_1, E_2, and E_3 be the respective events of choosing a two headed coin, a biased coin, and an unbiased coin. Then

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let A be the event that the coin shows heads.

A two-headed coin will always show heads. Therefore,

$P(A/E_1) = P$ (coin showing heads, given that it is a two-headed coin) = 1

Probability of heads coming up, given that it is a biased coin = 75%

$\therefore$ $P(A/E_1)$

= P(coin showing heads, given that it is a biased coin)

$= \frac{75}{100} = \frac{3}{4}$

Since the third coin is unbiased, the probability that it shows heads is always 1/2.

$\therefore$ $P(A/E_3) = P$ (coin showing heads, given that it is an unbiased coin)

$= \frac{1}{2}$

The probability that the coin is two-headed, given that it shows heads, is given by

$P(E_1/A)$

By using Bayes' theorem, we obtain

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{2}} = \frac{4}{9}$$

3. Let E_1 and E_2 be the events such that

E_1 : A speaks truth

E_2: A speaks false

Let X be the event that a head appears.

$$P(E_1) = \frac{4}{5}$$

$$\therefore \quad P(E_2) = 1 - P(E_1) = 1 - \frac{4}{5} = \frac{1}{5}$$

If a coin is tossed, then it may result in either head (H) or tail (T).

The probability of getting a head is $\frac{1}{2}$ whether A speaks truth or not. Therefore,

$$P(X/E_1) = P(X/E_2) = \frac{1}{2}$$

The probability that there is actually a head is given by $P(E_1/X)$.

$$P(E_1/X) = \frac{P(E_1) \cdot P(X|E_1)}{P(E_1) \cdot P(X|E_1) + P(E_2) \cdot P(X|E_2)}$$

$$= \frac{\frac{4}{5} \times \frac{1}{2}}{\frac{4}{5} \times \frac{1}{2} + \frac{1}{5} \times \frac{1}{2}} = \frac{4}{5}$$

4. Let $A_i (i = 1, 2, 3, 4)$ be the event that the urn contains 2, 3, 4, or 5 white balls and E the event that two white balls are drawn. Since the four events A_1, A_2, A_3, A_4 are equally likely, therefore $P(A_i) = 1/4$, $i = 1, 2, 3, 4$. Now, the probability that the urn contains 2 white balls and both have been drawn is

$$P(E/A_1) = \frac{{}^2C_2}{{}^5C_2} = \frac{1}{10}$$

Similarly,

$$P(E/A_2) = \frac{{}^3C_2}{{}^5C_2} = \frac{3}{10},$$

$$P(E/A_3) = \frac{{}^4C_2}{{}^5C_2} = \frac{3}{5}$$

$$P(E/A_4) = \frac{{}^5C_2}{{}^5C_2} = 1$$

Hence, the required probability is

$$P(A_4/E) = \frac{P(A_4)P(E/A_4)}{\sum_{i=1}^{4} P(A_i)P(E/A_i)}$$

$$= \frac{\frac{1}{4} \times 1}{\frac{1}{4}\left(\frac{1}{10} + \frac{3}{10} + \frac{3}{5} + 1\right)} = \frac{1}{2}$$

5. Let E_1, E_2, and E_3 denote the events of selecting boxes A, B, C, respectively, and A be the event that a screw selected at random is defective. Then,

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

$$\therefore \quad P(A/E_1) = \frac{1}{5}, \; P(A/E_2) = \frac{1}{6}, \; P(A/E_3) = \frac{1}{7}$$

By Bayes's rule, the required probability is

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{7}} = \frac{42}{107}$$

6. Let A, E_1, and E_2, respectively, denote the events that a person has a heart attack, the selected person followed the course of yoga and meditation, and the person adopted the drug prescription. Therefore,

$$P(A) = 0.40$$

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(A|E_1) = 0.40 \times 0.70 = 0.28$$

$$P(A|E_2) = 0.40 \times 0.75 = 0.30.$$

Probability that the patient suffering a heart attack followed a course of mediation and yoga is given by $P(E_1|A)$.

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A|E_1)}{P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2)}$$

$$= \frac{\frac{1}{2} \times 0.28}{\frac{1}{2} \times 0.28 + \frac{1}{2} \times 0.30} = \frac{14}{29}$$

7. Let events

A : Event that first man speaks truth

B : Event that second man speaks truth

R : Day is rainy

$$\therefore \quad P(A) = \frac{4}{5}, \; P(B) = \frac{2}{3}, \; P(R) = \frac{3}{4}$$

$\therefore$ Required probability

$$= \frac{P(A \cap B) \cdot P(R)}{P(A \cap B) \cdot P(R) + P(A' \cap B') \cdot P(R')}$$

$$= \frac{\frac{4}{5} \times \frac{2}{3} \times \frac{3}{4}}{\frac{4}{5} \times \frac{2}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{1}{3} \times \frac{1}{4}} = \frac{24}{25}$$

Exercise 14.6

1. According to question,

$$^nC_6\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^{n-6} = {}^nC_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{n-8}$$

or $$^nC_6\left(\frac{1}{2}\right)^n = {}^nC_8\left(\frac{1}{2}\right)^n$$

or $$^nC_6 = {}^nC_8 = {}^nC_{n-8}$$

$\Rightarrow$ $6 = n - 8$ or $n = 14$

2. Given that $p = 1/6$, $q = 5/6$. Therefore, required probability is

$$P(0) + P(1) + P(2) = {}^4C_0\left(\frac{5}{6}\right)^4 + {}^4C_1\left(\frac{5}{6}\right)^3\frac{1}{6} + {}^4C_2\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)^2$$

$$= \frac{425}{432}$$

$$= 0.984$$

3. Let p be the probability that a student selected at random is a swimmer, which is given by $1 - 1/5 = 4/5$. The number of students selected is 5. Probability that exactly 4 students are swimmers is

$$^5C_4 p^4 q = 5\left(\frac{4}{5}\right)^4\left(\frac{1}{5}\right) = \left(\frac{4}{5}\right)^4$$

4. The probability p_1 of winning the best of three games is equal to the sum of the probability of winning two games and the probability of wining three games, which is given by

$p_1 = {}^3C_2\,(0.6)\,(0.4)^2 + {}^3C_3\,(0.4)^3$

$= 0.288 + 0.064 = 0.352$ [Using binomial distribution]

Similarly, the probability of winning the best five games is

p_2 = Probability of winning three games
+ Probability of winning four games
+ Probability of winning 5 games

$= {}^5C_3\,(0.6)^2\,(0.4)^3 + {}^5C_4\,(0.6)\,(0.4)^4 + {}^5C_5\,(0.4)^5$

$= 0.2304 + 0.0768 + 0.01024$

$= 0.31744$

As $p_1 > p_2$, A must choose the first offer, i.e., best of three games.

5. The given numbers are 00, 01, 02, ..., 99. There are total 100 numbers, out of which the numbers, the product of whose digits is 18, are 29, 36, 63, and 92.

$\therefore$ $p = P(E) = \frac{4}{100} = \frac{1}{25}$

$\Rightarrow$ $q = 1 - p = \frac{24}{25}$

From binomial distribution,

$P(E$ occurring at least 3 times)

$= P(E$ occurring 3 times) + $P(E$ occurring 4 times)

$= {}^4C_3\,p^3q + {}^4C_4\,p^4$

$$= 4 \times \left(\frac{1}{25}\right)^3\left(\frac{24}{25}\right) + \left(\frac{1}{25}\right)^4$$

$$= \frac{97}{(25)^4}$$

Exercises

Single Correct Answer Type

1. (1) $P(B_1) = \frac{{}^6C_1}{{}^{10}C_1} = \frac{6}{10} = \frac{3}{5}$

$P(B_2/B_1) = \frac{5}{9}$ (B_2 = black)

$\therefore$ $P(B_1 \cap B_2) = P(B_1)\,P(B_2/B_1) = \frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$

2. (2) Consider the following events:

A: getting a card with mark I in first draw

B: getting a card with mark I in second draw

C: getting a card with mark T in this draw

Then, the required probability is

$P(A \cap B \cap C) = P(A)\,P(B/A)\,P(C/A \cap B)$

$$= \frac{10}{20} \times \frac{9}{19} \times \frac{10}{18} = \frac{5}{38}$$

3. (1) Let the number selected be xy. Then

$x + y = 9$, $0 \le x, y \le 9$

and $xy = 0 \Rightarrow x = 0, y = 9$

or $y = 0, x = 9$

$$P(x_1 = 9/x_2 = 0) = \frac{P(x_1 = 9 \cap x_2 = 0)}{P(x_2 = 0)}$$

Now, $P(x_2 = 0) = \frac{19}{100}$

and $P(x_1 = 9 \cap x_2 = 0) = \frac{2}{100}$

$\Rightarrow$ $P(x_1 = 9/x_2 = 0) = \frac{2/100}{19/100} = \frac{2}{19}$

4. (1) $P(A \cap B') = P(A) - P(A \cap B) = 0.20$

Also,

$P(A' \cap B) = P(B) - P(A \cap B) = 0.15$

$\Rightarrow$ $P(A) + P(B) - 2P(A - B) = 0.35$

Now,

$P(A' \cap B') = 1 - P(A \cup B)$

$\Rightarrow$ $0.1 = 1 - P(A) - P(B) + P(A \cap B)$

$\Rightarrow$ $P(A) + P(B) - P(A \cap B) = 0.9$

$\Rightarrow$ $P(A \cap B) = 0.9 - 0.35 = 0.55$

and $P(A) = 0.75$, $P(B) = 0.70$

Now,

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.55}{0.70}$$

5. (2) Consider the following events:

A: Father has at least one boy

B: Father has 2 boys and one girl

Then,

A = one boy and 2 girls, 2 boys and one girl, 3 boys and no girl

$A \cap B$ = 2 boys and one girl

Now, the required probability is

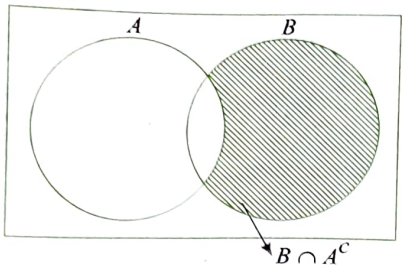
$$= P(A/B) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$$

6. (2) We have,

$$P(A)=\frac{40}{100},\ P(B)=\frac{25}{100} \text{ and } P(A\cap B)=\frac{15}{100}$$

So, $P\left(\frac{B}{A}\right)=\frac{P(A\cap B)}{P(A)}=\frac{15/100}{40/100}=\frac{3}{8}$

7. (2)



$$P(A)=\frac{1}{4},\ P(A\cup B)=\frac{1}{2}$$

$$P\left(\frac{B}{A^C}\right)=\frac{P(B\cap A^C)}{P(A^C)}$$

$$=\frac{P(A\cup B)-P(A)}{1-P(A)}$$

$$[\because P(A\cup B)+P(B)-P(A\cap B)]$$

$$=\frac{\frac{1}{2}-\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{4}\times\frac{4}{3}=\frac{1}{3}$$

8. (2) $P(S\cap F)=0.0006$, where S is the event that the motor cycle is stolen and F is the event that the moter cycle is found. Therefore,

$$P(S)=0.0015$$

$$P(F/S)=\frac{P(F\cap S)}{P(S)}=\frac{6\times10^{-4}}{15\times10^{-4}}=\frac{2}{5}$$

9. (3) Let A be the event that 11 is picked and B be the event that sum is even. The number of ways of selecting 11 along with one more odd number is $n(A\cap B)={}^7C_1$.

The number of ways of selecting either two even numbers or selecting two odd numbers is $n(B)=1+{}^8C_2$.

$$\therefore\quad P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{7}{29}=0.24$$

10. (4) A: exactly one ace

B: both aces

$E: A\cup B$

$$P(B/A\cup B)=\frac{{}^4C_2}{{}^4C_1\ {}^{12}C_1+{}^4C_2}=\frac{6}{54}=\frac{1}{9}$$

11. (2) The sum of the digits can be 7 in the following ways: 07, 16, 25, 34, 43, 52, 61, 70.

$\therefore\quad (A=7)=\{07, 16, 25, 34, 43, 52, 61, 70\}$

Similarly,

$$(B=0)=\{00, 01, 02, \ldots, 10, 20, 30, \ldots, 90\}$$

Thus,

$$(A=7)\cap(B=0)=\{09, 70\}$$

$$\therefore\quad P((A=7)\cap(B=0))=\frac{2}{100},\ P((B=0))=\frac{19}{100}$$

Hence,

$$P(A=7\mid B=0)=\frac{P((A=7)\cap(B=0))}{P(B=0)}$$

$$=\frac{\frac{2}{100}}{\frac{19}{100}}=\frac{2}{19}$$

12. (4) Consider two events as follows:

A_i: getting number i on first die

B_i: getting a number more than i on second die

The required probability is

$$P(A_1\cap B_1)+P(A_2\cap B_2)+P(A_3\cap B_3)+P(A_4\cap B_4)+P(A_5\cap B_5)=\sum_{i=1}^{5}P(A_i\cap B_i)=\sum_{i=1}^{5}P(A_i)\,P(B_i)$$

$$[\because A_i, B_i \text{ are independent}]$$

$$=\frac{1}{6}[P(B_1)+P(B_2)+\cdots+P(B_5)]$$

$$=\frac{1}{6}\left(\frac{5}{6}+\frac{4}{6}+\frac{3}{6}+\frac{2}{6}+\frac{1}{6}\right)=\frac{5}{12}$$

13. (3) 18 draws are required for 2 aces. This means in the first 17 draws, there is one ace and 16 other cards, and 18th draw produces an ace. So, the required probability is

$$\frac{{}^{48}C_{16}\times{}^4C_1}{{}^{52}C_{17}}\times\frac{3}{35}=\frac{561}{15925}$$

14. (2) The required probability is

$$\frac{n^2}{{}^{2n}C_2}\,\frac{(n-1)^2}{{}^{2n-2}C_2}\,\frac{(n-2)^2}{{}^{2n-4}C_2}\cdots\frac{2^2}{{}^4C_2}\,\frac{1^2}{{}^2C_2}$$

$$=\frac{(1\times2\times3\times4\times\cdots\times(n-1)n)^2}{\frac{(2n)!}{2^n}}=\frac{2^n(n!)^2}{(2n)!}=\frac{2^n}{{}^{2n}C_n}$$

15. (4) Let the probability for getting an odd number be p. Therefore, the probability for getting an even number is $2p$.

$$\therefore\quad p+2p=1 \text{ or } 3p=1 \text{ or } p=\frac{1}{3}$$

Sum of two numbers is even means either both are odd or both are even. Therefore, the required probability is

$$\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{2}{3}=\frac{1}{9}+\frac{4}{9}=\frac{5}{9}$$

16. (2) If A, B, C represent events that the student is successful in tests I, II, III, respectively, then the probability that the student is successful is

$$P[(A\cap B\cap C')\cup(A\cap B'\cap C)\cup(A\cap B\cap C)]$$

$$=P(A\cap B\cap C')+P(A\cap B'\cap C)+P(A\cap B\cap C)$$

$$=P(A)\,P(B)\,P(C')+P(A)\,P(B')\,P(C)+P(A)\,P(B)\,P(C)$$

$$[\therefore A, B, C \text{ are independent events}]$$

$$=pq\left(1-\frac{1}{2}\right)+p(1-q)\frac{1}{2}+pq\frac{1}{2}$$

$$=pq+\frac{1}{2}p-\frac{1}{2}pq$$

$$=\frac{1}{2}(pq+p)$$

$$\therefore\quad \frac{1}{2}p(1+q)=\frac{1}{2}$$

or $\quad p(1+q)=1$

17. (1) $P(A)=\frac{1}{2}, P(B)=\frac{1}{3}$, and $P(C)=\frac{1}{4}$

$\therefore \quad P(\bar{A})=1-\frac{1}{2}=\frac{1}{2}, P(\bar{B})=1-\frac{1}{3}=\frac{2}{3}$

$P(\bar{C})=1-\frac{1}{4}=\frac{3}{4}$

Therefore, the required probability is

$$1-P(\bar{A})\,P(\bar{B})\,P(\bar{C}) = 1-\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}$$

$$=1-\frac{1}{4}=\frac{3}{4}$$

18. (1) We are given that

$P(A\cap B)=P(A)\,P(B)$

$P(B\cap C)=P(B)P(C)$

$P(C\cap A)=P(C)P(A)$

$P(A\cap B\cap C)=P(A)(B)P(C)$

We have,

$$P(A\cap(B\cap C))=P(A\cap B\cap C)=P(A)P(B)P(C)$$
$$=P(A)P(B\cap C)$$

$\Rightarrow$ A and $B\cap C$ are indepedent

Therefore, S_2 is true. Also,

$$P[(A\cap(B\cup C)]=P[(A\cap B)\cup(A\cap C)]$$
$$=P(A\cap B)+P(A\cap C)-P[(A\cap B)\cap(A\cap C)]$$
$$=P(A\cap B)+P(A\cap C)-P(A\cap B\cap C)$$
$$=P(A)P(B)+P(A)\,P(C)-P(A)\,P(B)\,P(C)$$
$$=P(A)[P(B)+P(C)-P(B)P(C)]$$
$$=P(A)[P(B)+P(C)-P(B\cap C)]$$
$$=P(A)P(B\cup C)$$

Therefore, A and $B\cup C$ are independent.

19. (1) The ratios of the ships A, B, and C for arriving safely are 2:5, 3:7, and 6:11, respectively. Therefore, the probability of ship A for arriving safely is $2/(2+5)=2/7$.

Similarly, for B the probability is $3/(3+7)=3/10$ and for C the probability is $C=6/(6+11)=6/17$.

Therefore, the probability of all the ships for arriving safely is $(2/7)\times(3/10)\times(6/17)=18/595$.

20. (2) Let event A be drawing 9 cards which are not ace and B be drawing an ace card. Therefore, the required probability is $P(A\cap B)=P(A)\times P(B)$

Now, there are four aces and 48 other cards. Hence,

$$P(A)=\frac{{}^{48}C_9}{{}^{52}C_9}$$

After having drawn 9 non-ace cards, the 10^{th} card must be ace. Hence,

$$P(B)=\frac{{}^{4}C_1}{{}^{42}C_1}=\frac{4}{42}$$

Hence,

$$P(A\cap B)=\frac{{}^{48}C_9}{{}^{52}C_9}\,\frac{4}{42}$$

21. (3) Out of 5 horses, only one is the wining horse. The probability that Mr. A selected that losing horse is $4/5\times3/4$. Therefore, the required probability is

$$1-\frac{4}{5}\times\frac{3}{4}=1-\frac{3}{5}=\frac{2}{5}$$

22. (4) We have,

$$P(\overline{A\cup B})=\frac{1}{4}, P(A\cap B)=\frac{1}{4}$$

$$P(\bar{A})=\frac{1}{4}\Rightarrow P(A)=1-\frac{1}{4}=\frac{3}{4}$$

$$\therefore \quad P(\overline{A\cup B}) = 1-P(A\cup B)$$
$$=1-[P(A)+P(B)-P(A\cap B)]$$

$$\Rightarrow \quad \Rightarrow\frac{1}{6}=1-\frac{3}{4}-P(B)+\frac{1}{4}$$

$$\Rightarrow \quad P(B)=1-\frac{1}{2}-\frac{1}{6}=\frac{6-3-1}{6}=\frac{2}{6}=\frac{1}{3}$$

Since $P(A\cap B)=P(A)P(B)$ and $P(A)\neq P(B)$, therefore A and B are independent but not equally likely.

23. (3) Total number of the students is 80. Total number of girls is 25. Total number of boys is 55. There are 10 rich, 70 poor, 20 intelligent students in the class. Therefore, required probability is

$$\underset{(I)}{\frac{1}{4}}\times\underset{(R)}{\frac{1}{8}}\times\underset{(G)}{\frac{25}{80}}=\frac{5}{512}$$

24. (1) $P(A\cap C)=P(A)\,P(C)$

or $\quad \frac{1}{20}=\frac{1}{5}P(C)$

or $\quad P(C)=\frac{1}{4}$

Now,

$$P(B\cup C)=\frac{1}{6}+\frac{1}{4}-P(B\cap C)$$

$$P(B\cap C)=\frac{5}{12}-\frac{3}{8}=\frac{1}{24}\,P(B)\,P(C)$$

Therefore, B and C are independent.

25. (3) $P(A\cup B)=P(A)+P(B)-P(A)\times P(B)$

$\Rightarrow \quad 0.7=0.4+p-0.4p$

$\therefore \quad 0.6p=0.3$ or $p=\frac{1}{2}$

26. (1) The required probability is

$$=\frac{2}{9}\times\frac{1}{8}\times\frac{4}{7}\times\frac{3}{6}\times\frac{2}{5}\times\frac{1}{4}\times1\times1\times1=\frac{1}{1260}$$

27. (3) The total number of digits in any number at the unit's place is 10. Therefore,

$n(S)=10$

If the last digit in product is 1, 3, 5 or 7, then it is necessary that the last digit in each number must be 1, 3, 5 or 7. Therefore,

$n(A)=4$

$\therefore \quad P(A)=\frac{4}{10}=\frac{2}{5}$

Hence, the required probability is $(2/5)^4=16/625$.

28. (3) The probabilities of solving the question by these three students are 1/3, 2/7 and 3/8, respectively. Therefore,

$$P(A)=\frac{1}{3};\ P(B)=\frac{2}{7};\ P(C)=\frac{3}{8}$$

Then probability of question solved by only one student is

$P((A\bar{B}\bar{C}$ or $\bar{A}\,B\bar{C}$ or $\bar{A}\,\bar{B}\,C))$

$$=P(A)\,P(\bar{B})\,P(\bar{C})+P(\bar{A})\,P(B)\,P(\bar{C})+P(\bar{A})\,P(\bar{B})\,P(C)$$

$$= P(A)\,P(\bar{B})\,P(\bar{C}) + P(\bar{A})\,P(B)\,P(\bar{C}) + P(\bar{A})\,P(\bar{B})\,P(C)$$
$$= \frac{1}{3}\times\frac{5}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{2}{7}\times\frac{5}{8} + \frac{2}{3}\times\frac{5}{7}\times\frac{3}{8}$$
$$= \frac{25+20+30}{168} = \frac{25}{56}$$

29. (2) Probability of getting 2 heads in the first 5 trials is

$${}^5C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{10}{32} = \frac{5}{16}$$

Therefore, the probability that third head appears on the sixth trial is $5/16 \times 1/2 = 5/32$.

30. (2) Let H denote the head, T the tail and * any of the head or tail. Then, $P(H) = 1/2$, $P(T) = 1/2$ and $P(*) = 1$. For at least four consecutive heads, we should have any of the following patterns:

	Probability
(i) $H\,H\,H\,H * * *$	$(1/2)^4 \times 1 = 1/16$
(ii) $T\,H\,H\,H\,H * *$	$(1/2)^5 = 1/32$
(iii) $* \,T\,H\,H\,H\,H *$	$(1/2)^5 = 1/32$
(iv) $* * \,T\,H\,H\,H\,H$	$(1/2)^5 = 1/32$

Since all the above cases are mutually exclusive, the probability of getting at least four consecutive heads (on adding) is $1/16 + 3/32 = 5/32$.

31. (4) The probability that the first critic favours the book is

$$P(E_1) = \frac{5}{5+2} = \frac{5}{7}$$

The probability that the second critic favours the book is

$$P(E_2) = \frac{4}{4+3} = \frac{4}{7}$$

The probability that the third critic favours the book is

$$P(E_3) = \frac{3}{3+4} = \frac{3}{7}$$

Majority will be in favor of the book if at least two critics favor the book. Hence, the probability is

$$P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$
$$= P(E_1)P(E_2)P(\bar{E}_3) + P(E_1)P(\bar{E}_2)\,P(E_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1)P(E_2)P(E_3)$$
$$= \frac{5}{7}\times\frac{4}{7}\times\left(1-\frac{3}{7}\right) + \frac{5}{7}\times\left(1-\frac{4}{7}\right)\times\frac{3}{7} + \left(1-\frac{5}{7}\right)\times\frac{4}{7}\times\frac{3}{7} + \frac{5}{7}\times\frac{4}{7}\times\frac{3}{7}$$
$$= \frac{209}{343}$$

32. (3) Let us assume that A wins after n deuces, $n = 0, 1, 2, 3, \ldots$. The probability of a deuce is $(2/3) \times (2/3) + (1/3) \times (1/3) = (5/9)$. [$A$ wins his serve, then B wins his serve or A loses his serve.] So, the probability that A wins game after n deuces is $(5/9)^n \times (2/3) \times (1/3)$. [After nth deuce, A serves and wins, then B serves and loses.] Therefore, the required probability of 'A' winning the game is

$$\sum_{n=0}^{\infty}\left(\frac{5}{9}\right)^n \times\frac{2}{3}\times\frac{1}{3} = \frac{1}{1-\frac{5}{9}}\times\frac{2}{9} = \frac{1}{2}$$

33. (2) Die marked with 1, 2, 2, 3, 3, 3 is thrown 3 times.

$$P(1) = \frac{1}{6},\ P(2) = \frac{2}{6},\ P(3) = \frac{3}{6}$$
$$P(S) = P(4 \text{ or } 6)$$
$$= P(112 (3 \text{ cases}) \text{ or } 123 (6 \text{ cases}) \text{ or } 222)$$
$$= 3\times\frac{1}{6}\times\frac{1}{6}\times\frac{2}{6} + 6\frac{1}{6}\times\frac{2}{6}\times\frac{3}{6} + \frac{2}{6}\times\frac{2}{6}\times\frac{2}{6}$$
$$= \frac{6+36+8}{216} = \frac{50}{216} = \frac{25}{108}$$

34. (2) Let

$$P(S) = P(1 \text{ or } 2) = 1/3$$
$$P(F) = P(3 \text{ or } 4 \text{ or } 5 \text{ or } 6) = 2/3$$
$$P(A \text{ wins}) = P[(S\,S \text{ or } S\,F\,S\,S \text{ or } S\,F\,S\,F\,S\,S \text{ or } \ldots)$$
or $(F\,S\,S \text{ or } F\,S\,F\,S\,S \text{ or } \ldots)]$

$$= \frac{\frac{1}{9}}{1-\frac{2}{9}} + \frac{\frac{2}{27}}{1-\frac{2}{9}}$$
$$= \frac{1}{9}\times\frac{9}{7} + \frac{2}{27}\times\frac{9}{7}$$
$$= \frac{1}{7} + \frac{2}{21} = \frac{3+2}{21} = \frac{5}{21}$$

$$P(A \text{ winning}) = \frac{5}{21},\ P(B \text{ winning}) = \frac{16}{21}$$

35. (1) $P(a) = 0.3, P(b) = 0.5, P(c) = 0.2$. Hence, a, b, c are exhaustive.

P(same horse wins all the three races)
$= P(aaa \text{ or } bbb \text{ or } ccc)$
$= (0.3)^3 + (0.5)^3 + (0.2)^3$
$= \frac{27+125+8}{1000} = \frac{160}{1000} = \frac{4}{25}$

P(each horse wins exactly one race)
$= P(abc \text{ or } acb \text{ or } bca \text{ or } bac \text{ or } cab \text{ or } cba)$
$= 0.3 \times 0.5 \times 0.2 \times 6 = 0.18 = \frac{9}{50}$

36. (1) The probability of getting a head in a single toss of a coin is $p = 1/2$ (say). The probability of getting 5 or 6 in a single throw of a die is $q = 2/6 = 1/3$ (say). Therefore, the required probability is

$$p + (1-p)(1-q)p + (1-p)(1-q)(1-p)(1-q)p + \cdots$$
$$= p + (1-p)\,(1-q)\,p + (1-p)^2\,(1-q)^2\,p + \cdots$$
$$= \frac{p}{1-(1-p)(1-q)}$$
$$= \frac{1/2}{1-1/2\times 2/3} = \frac{3}{4}$$

37. (3) Let the probability that a man aged x dies in a year p. Thus, the probability that a man aged x does not die in a year is $1-p$. The probability that all n men aged x do not die in a year is $(1-p)^n$. Therefore, the probability that at least one man dies in a year is $1-(1-p)^n$. The probability that out of n men, A_1 dies first is $1/n$. Since this event is independent of the event that at least one man dies in a year, the probability that A_1 dies in the year and he is the first one to die is $1/n\,[1-(1-p)n]$.

38. (1) For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round. Therefore, the required probability is $30/31 \times 14/15 \times 6/7 \times 2/3 = 16/31$.

39. (1) Let A denote the event that a sum of 5 occurs, B the event that a sum of 7 occurs, and C the event that neither a sum of 5 nor a sum of 7 occurs. We have,

$$P(A) = \frac{4}{36} = \frac{1}{9}, P(B) = \frac{6}{36} = \frac{1}{6}$$

$$P(C) = \frac{26}{36} = \frac{13}{18}$$

Thus, probability that A occurs before B is

$$P[A \text{ or } (C \cap A) \text{ or } (C \cap C \cap A) \text{ or } \ldots]$$
$$= P(A) + P(C \cap A) + P(C \cap C \cap A) + \ldots$$
$$= P(A) + P(C)P(A) + P(C)^2P(A) + \ldots$$
$$= \frac{1}{9} + \left(\frac{13}{18}\right) \times \frac{1}{9} + \left(\frac{13}{18}\right)^2 \frac{1}{9} + \cdots$$
$$= \frac{1/9}{1 - 13/18} = \frac{2}{5}$$

40. (4) Let p_i denote the probability that out of 10 tosses, head occurs i times and no two heads occur consecutively. It is clear that $i > 5$.

For $i = 0$, i.e., no head, $p_0 = 1/2^{10}$.

For $i = 1$, i.e., one head, $p_1 = {}^{10}C_1 (1/2)^1 (1/2)^9 = 10/2^{10}$.

Now for $i = 2$, we have 2 heads and 8 tails. Then, we have 9 possible places for heads. For example, see the construction:

$x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x\,T\,x$

Here x represents possible places for heads. Therefore,

$$p_2 = {}^9C_2 \left(\frac{1}{2}\right)^2 (1/2)^8 = 36/2^{10}$$

Similarly,

$$p_3 = {}^8C_3/2^{10} = 56/2^{10}$$
$$p_4 = {}^7C_2/2^{10} = 35/2^{10}$$
$$p_5 = {}^6C_5/2^{10} = 6/2^{10}$$

$$\therefore \quad p = p_0 + p_1 + p_2 + p_3 + p_4 + p_5$$
$$= \frac{1 + 10 + 36 + 56 + 35 + 6}{2^{10}} = \frac{144}{2^{10}} = \frac{9}{64}$$

41. (4) According to the given condition,

$${}^nC_3 \left(\frac{1}{2}\right)^n = {}^nC_4 \left(\frac{1}{2}\right)^n,$$

where n is the number of times die is thrown.

$$\therefore \quad {}^nC_3 = {}^nC_4 \Rightarrow n = 7$$

Thus, the required probability is

$$= {}^7C_1 \left(\frac{1}{2}\right)^7 = \frac{7}{2^7} = \frac{7}{128}$$

42. (3) Possibilities of getting 9 are (5, 4), (4, 5), (6, 3), (3, 6).

$$\therefore \quad p = \frac{4}{36} = \frac{1}{9} \text{ and } q = 1 - \frac{1}{9} = \frac{8}{9}$$

Therefore, the required probability is

$${}^3C_2 q^1 p^2 = (3)\left(\frac{8}{9}\right)\left(\frac{1}{9}\right)^2 = \frac{8}{243}$$

43. (2) Here $p = 19/20$, $q = 1/20$, $n = 5$, $r = 5$. The required probability is

$$= {}^5C_5 \left(\frac{19}{20}\right)^5 \left(\frac{1}{20}\right)^0 = \left(\frac{19}{20}\right)^5$$

44. (4) Let X denote the largest number on the 3 tickets drawn.

Then, $P(X \le 7) = (7/20)^3$ and $P(X \le 6) = (6/20)^3$. Then, the required probability is

$$P(X = 7) = \left(\frac{7}{20}\right)^3 - \left(\frac{6}{20}\right)^3$$

45. (3) The required probability is

$$= \left[{}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4\right]^2 + \left[{}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3\right]^2$$
$$+ \left[{}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2\right]^2 + \left[{}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1\right]^2$$
$$+ \left[{}^4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0\right]^2 = \frac{35}{128}$$

46. (3) The required probability is

1 – Probability of getting equal number of heads and tails

$$= 1 - {}^{2n}C_n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n}$$
$$= 1 - \frac{(2n)!}{n!n!}\left(\frac{1}{4}\right)^n$$
$$= 1 - \frac{(2n)!}{(n!)^2} \times \frac{1}{4^n}$$

47. (3) In any trial P(getting white ball) $= P(W) = 1/2$

P(getting black ball) $= P(B) = 1/2$

Required event will occur if in the first six trial, 3 white balls are drawn in any of the 3 trials from six.

In remaining three trials, black must be drawn.

Three trials in which black balls is drawn can be selected in ${}^6C_3 = 20$ ways.

$\therefore$ Required probability = $P(WWWBBB$ in any order$) \times P(W)$

$$= 20 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2} = \frac{5}{32}$$

48. (4) Player should get (HT, HT, HT, ...) or (TH, TH, ...) at least $2n$ times. If the sequence starts from first place, then the probability is $1/2^{2n}$ and if starts from any other place, then the probability is $1/2^{2n+1}$. Hence, required probability is

$$2\left(\frac{1}{2^{2n}} + \frac{m}{2^{2n+1}}\right) = \frac{m+2}{2^{2n}}$$

49. (3) The probability that A gets r heads in three tosses of a coin is

$$P(X = r) = {}^3C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{3-r} = {}^3C_r \left(\frac{1}{2}\right)^3$$

The probability that A and B both get r heads in three tosses of a coin is

$${}^3C_r \left(\frac{1}{2}\right)^3 {}^3C_r \left(\frac{1}{2}\right)^3 = ({}^3C_r)^2 \left(\frac{1}{2}\right)^6$$

Hence, the required probability is

$$\sum_{r=0}^{3} ({}^3C_r)^2 \left(\frac{1}{2}\right)^6 = \left(\frac{1}{2}\right)^6 \{1 + 9 + 9 + 1\} = \frac{20}{64} = \frac{5}{16}$$

50. (2) Let the probability of getting a tail in a single trial be $p = 1/2$. The number of trials be $n = 100$ and the number of trials in 100 trials be X.

We have,

$$P(X=r) = {}^{100}C_r p^r q^{n-r}$$

$$= {}^{100}C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{100-r} = {}^{100}C_r \left(\frac{1}{2}\right)^{100}$$

Now,

$$P(X=1) + P(X=3) + \cdots + P(X=49)$$

$$= {}^{100}C_1 \left(\frac{1}{2}\right)^{100} + {}^{100}C_3 \left(\frac{1}{2}\right)^{100} + \cdots + {}^{100}C_{49} \left(\frac{1}{2}\right)^{100}$$

$$= ({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49}) \left(\frac{1}{2}\right)^{100}$$

But ${}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{99} = 2^{99}$

Also, ${}^{100}C_{99} = {}^{100}C_1$

${}^{100}C_{97} = {}^{100}C_3, \ldots {}^{100}C_{51} = {}^{100}C_{49}$

Thus, $2({}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49}) = 2^{99}$

or ${}^{100}C_1 + {}^{100}C_3 + \cdots + {}^{100}C_{49} = 2^{98}$

Therefore, probability of required event is

$$\frac{2^{98}}{2^{100}} = \frac{1}{4} 2^{98}/2^{100} = 1/4$$

51. (3) In the first 9 throws, we should have three sixes and six non-sixes; and a six in the 10^{th} throw, and thereafter it does not matter whatever face appears. So, the required probability is

$${}^9C_3 \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^6 \times \frac{1}{6} \times \underbrace{1 \times 1 \times 1 \times \cdots \times 1}_{10 \text{ times}} = \frac{84 \times 5^6}{6^{10}}$$

52. (2) Consider the following events:

A_1: A speaks truth

A_2: B speaks truth

Then, $P(A_1) = 60/100 = 3/5$, $P(A_2) = 70/100 = 7/10$.

For the required event, either both of them should speak the truth or both of them should tell a lie. Thus, the required probability is

$$= P((A_1 \cap A_2) \cup (\bar{A}_1 \cap \bar{A}_2))$$

$$= P(A_1 \cap A_2) + P(\bar{A}_1 \cap \bar{A}_2)$$

$$= P(A_1)\,P(A_2) + P(\bar{A}_1)\,(\bar{A}_2)$$

$$= \frac{3}{5} \times \frac{7}{10} + \left(1 - \frac{3}{5}\right)\left(1 - \frac{7}{10}\right) = 0.54$$

53. (4) Probability that one test is held $= 2 \times \frac{1}{5} \times \frac{4}{5} = \frac{8}{25}$

Probability that test is held on both days $= \frac{1}{5} \times \frac{1}{5} = \frac{1}{25}$

$\therefore$ Probability that the student misses at least one test

$$= \frac{8}{25} + \frac{1}{25} = \frac{9}{25}$$

54. (4) Let A and B, respectively, be the events that urn A and urn B are selected. Let R be the event that the selected ball is red. Since the urn is chosen at random, Therefore,

$$P(A) = P(B) = \frac{1}{2}$$

and $P(R) = P(A)P(R/A) + P(B)P(R/B)$

$$= \frac{1}{2} \times \frac{5}{10} + \frac{1}{2} \times \frac{4}{10} = \frac{9}{20}$$

55. (2) $P(4 \text{ biased coin}) = \frac{1}{3}$

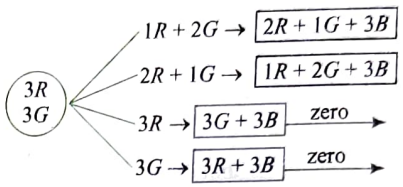
$P(5 \text{ biased coin}) = \frac{1}{4}$

Hence, the required probability is

$$= \frac{1}{3} \frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9} + \frac{2}{3} \frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9} \frac{1}{{}^{11}C_1}$$

$$= \frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

56. (3)



The required probability is

$$\frac{{}^3C_1\,{}^3C_2}{{}^6C_3} \frac{{}^2C_1\,{}^1C_1\,{}^3C_1}{{}^6C_3} + \frac{{}^3C_2\,{}^3C_1}{{}^6C_3} \frac{{}^1C_1\,{}^2C_1\,{}^3C_1}{{}^6C_3}$$

$$= 2\frac{9}{20} \times \frac{6}{20}$$

$$= \frac{27}{100}$$

57. (3) $P(4 \text{ biased coins}) = \frac{1}{3}$

$P(5 \text{ biased coins}) = \frac{1}{4}$

The required probability is

$$\frac{1}{3} \frac{{}^4C_3\,{}^{16}C_6}{{}^{20}C_9} \frac{1}{{}^{11}C_1} + \frac{2}{3} \frac{{}^5C_4\,{}^{15}C_5}{{}^{20}C_9} \frac{1}{{}^{11}C_1}$$

$$= \frac{2}{33}\left[\frac{{}^{16}C_6 + 5\,{}^{15}C_5}{{}^{20}C_9}\right]$$

58. (4) In the first case, the urn contains 3 red and n white balls. The probability that color of both the balls matches is

$$\frac{{}^3C_2 + {}^nC_2}{{}^{n+3}C_2} = \frac{1}{2}$$

or $$\frac{6 + n(n-1)}{(n+3)(n+2)} = \frac{1}{2}$$

or $2(n^2 - n + 6) = n^2 + 5n + 6$

or $n^2 - 7n + 6 = 0$

$\Rightarrow$ $n = 1$ or 6 (1)

In the second case,

$$\frac{3}{n+3} \frac{3}{n+3} + \frac{n}{n+3} \frac{n}{n+3} = \frac{5}{8}$$

Solving, we get

$$n^2 - 10n + 9 = 0$$

$\Rightarrow$ $n = 9$ or 1. (2)

From Eqs. (1) and (2), we have $n = 1$.

59. (2) Let $P(m)$, $P(p)$, $P(c)$ be the probability of selecting a book of maths, physics, and chemistry, respectively. Clearly,

$$P(m) = P(p) = P(c) = \frac{1}{3}$$

Again let $P(s_1)$ and $P(s_2)$ be the probability that he solves the first as well as second problem, respectively. Then

$$P(s_1) = P(m) \times P\left(\frac{s_1}{m}\right) + P(p) \times P\left(\frac{s_1}{p}\right) + P(c) \times P\left(\frac{s_1}{c}\right)$$

$$\Rightarrow \quad P(s_1) = \frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{4}{5} = \frac{19}{30}$$

Similarly,

$$P(S_2) = \frac{1}{3} \times \left(\frac{1}{2}\right)^2 + \frac{1}{3} \times \left(\frac{3}{5}\right)^2 + \frac{1}{3} \times \left(\frac{4}{5}\right)^2 = \frac{125}{300}$$

$$\Rightarrow \quad P\left(\frac{S_2}{S_1}\right) = \frac{\frac{125}{300}}{\frac{19}{30}} = \frac{25}{38}$$

60. (1) $P(A) = P(B) = P(C)$ and $P(A) + P(B) + P(C) = 1$

$$\therefore \quad P(A) = P(B) = P(C) = \frac{1}{3}.$$

Also,

$$P(X) = \frac{5}{12}, P(X / A) = \frac{3}{8}, P(X / B) = \frac{1}{4}$$

We have,

$$P(X) = P(A)\, P(X/A) + P(B)\, P(X/B) + P(C)\, P(X/C)$$

$$\therefore \quad \frac{5}{12} = \frac{1}{3}\left\{\frac{3}{8} + \frac{1}{4} + P(X / C)\right\}$$

$$\Rightarrow \quad P(X/C) = \frac{5}{8}$$

61. (3) Let A denote the event that target is hit when x shells are fired at point I. Let P_1 and P_2 denote the event that the target is at point I and II, respectively. We have $P(P_1) = 8/9$, $P(P_2) = 1/9$, $P(A/P_1) = 1 - (1/2)^x$, $P(A/P_2) = 1 - (1/2)^{55-x}$.

Now from total probability theorem,

$$P(A) = P(P_1)\, P(A/P) + P(P_2)\, P(A/P_2)$$

$$= \frac{1}{9}\left(8 - 8\left(\frac{1}{2}\right)^x + 1 - \left(\frac{1}{2}\right)^{55-x}\right)$$

$$= \frac{1}{9}\left(9 - 8\left(\frac{1}{2}\right)^x - \left(\frac{1}{2}\right)^{55-x}\right)$$

Now,

$$\frac{dP(A)}{dx} = \frac{1}{9}\left(-8\left(\frac{1}{2}\right)^x \ln\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^{55-x} \ln\left(\frac{1}{2}\right)\right)$$

(Note that in this step, it is being assumed that $x \in R^+$)

$$= \frac{1}{9}\ln\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{55-x}\left(1 - \left(\frac{1}{2}\right)^{2x-58}\right)$$

> 0 if $x < 29$

< 0 if $x > 29$

Therefore, $P(A)$ is maximum at $x = 29$. Thus, 29 shells must be fired at point I.

62. (1) Let E_i denote the event that the bag contains i black and $(10 - i)$ white balls $(i = 0, 1, 2, \ldots, 10)$. Let A denote the event that the three balls drawn at random from the bag are black. We have,

$$P(E_i) = \frac{1}{11} \; (i = 0, 1, 2, \ldots, 10)$$

$$P(A/E_i) = 0 \text{ for } i = 0, 1, 2 \text{ and } P(A/E_i) = \frac{{}^iC_3}{{}^{10}C_3} \text{ for } i \geq 3$$

$$\Rightarrow \quad P(A) = \frac{1}{11} \times \frac{1}{{}^{10}C_3}\left[{}^3C_3 + {}^4C_3 + \cdots + {}^{10}C_3\right]$$

But

$${}^3C_3 + {}^4C_3 + {}^5C_3 + \cdots + {}^{10}C_3 = {}^4C_4 + {}^4C_3 + {}^5C_3 + \cdots + {}^{10}C_3$$
$$= {}^5C_4 + {}^5C_3 + {}^6C_3 + \cdots + {}^{10}C_3$$
$$\vdots$$
$$= {}^{11}C_4$$

$$\Rightarrow \quad p(A) = \frac{1}{11} \times \frac{1}{{}^{10}C_3} \times {}^{11}C_4 = \frac{\frac{11 \times 10 \times 9 \times 8}{4!}}{11 \times \frac{10 \times 9 \times 8}{3!}} = \frac{1}{4}$$

$$\therefore \quad P(E_9/A) = \frac{P(E_9)\, P(A/E_9)}{P(A)} = \frac{\frac{1}{11} \times \frac{{}^9C_3}{{}^{10}C_3}}{\frac{1}{4}} = \frac{14}{55}$$

63. (2) We define the following events:

A_1: Selecting a pair of consecutive letters from the word LONDON

A_2: Selecting a pair of consecutive letters from the word CLIFTON

E: Selecting a pair of letters 'ON'

Then, $P(A_1 \cap E) = 2/5$ as there are 5 pairs of consecutive letters out of which 2 are ON and $P(A_2 \cap E) = 1/6$ as there are 6 pairs of consecutive letters of which 1 is ON. Therefore, the required probability is

$$P\left(\frac{A_1}{E}\right) = \frac{P(A_1 \cap E)}{P(A_1 \cap E) + P(A_2 \cap E)} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{6}} = \frac{12}{17}$$

64. (4) A: Doctor finds a rash

B_1: Child has measles

S: Sick children

$P(S/F) = 0.9$

B_2: Child has flu $\Rightarrow P(B_2) = 9/10$

$P(S/M) = 0.10$

$P(A/B_1) = 0.95$

$P(R/M) = 0.95$

$P(A/B_2) = 0.08$

$P(R/F) = 0.08$

$$P(B_1/A) = \frac{0.1 \times 0.95}{0.1 \times 0.95 + 0.9 \times 0.08}$$

$$= \frac{0.095}{0.095 + 0.072}$$

$$= \frac{0.095}{0.167} = \frac{95}{167}$$

65. (3) A: car met with an accident

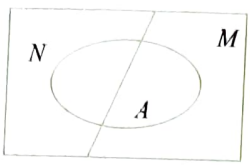
B_1: driver was alcoholic, $P(B_1) = 1/5$

B_2: driver was sober, $P(B_2) = 4/5$

$P(A/B_1) = 0.001;\ P(A/B_2) = 0.0001$

$$P(B_1/A) = \frac{(0.2)(0.001)}{(0.2)(0.001)+(0.8)(0.0001)} = 5/7$$

66. (1) Let N be the event of picking up a normal die; $P(N) = 1/4$. Let M be the event of picking up a magnetic die; $P(M) = 3/4$. Let A be the event that die shows up 3. Then



$$P(A) = P(A \cap N) + P(A \cap M)$$
$$= P(N)P(A/N) + P(M)\,P(A/M)$$
$$= \frac{1}{4}\times\frac{1}{6}+\frac{3}{4}\times\frac{2}{3} = \frac{7}{24}$$

$$P(N/A) = \frac{P(N \cap A)}{P(A)}$$
$$= \frac{(1/4)(1/6)}{7/24} = \frac{1}{7}$$

67. (1) Consider the following events.

A: ball drawn is black

E_1: bag I is chosen

E_2: bag II is chosen

E_3: bag III is chosen

Then,

$$P(E_1) = (E_2) = P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = \frac{3}{5},\ P(A/E_2) = \frac{1}{5},\ P(A/E_3) = \frac{7}{10}$$

Therefore, the required probability is

$$P(E_3/A) = \frac{P(E_3)P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)} = \frac{7}{15}$$

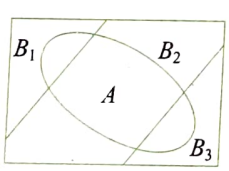
68. (2) Let us consider the following events.

A: card shows up black

B_1: card with both sides black

B_2: card with both sides white

B_3: card with one side white and one black



$$P(B_1) = \frac{2}{10},\ P(B_2) = \frac{3}{10},\ P(B_3) = \frac{5}{10}$$

$$P(A/B_1) = 1,\ P(A/B_2) = 0,\ P(A/B_3) = \frac{1}{2}$$

$$P(B_1/A) = \frac{\frac{2}{10}\times 1}{\frac{2}{10}\times 1+\frac{3}{10}\times(0)+\frac{5}{10}\times\frac{1}{2}} = \frac{4}{4+5} = \frac{4}{9}$$

Multiple Correct Answers Type

1. (1), (3), (4)

Since A and B are independent events, we have

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{2}\times\frac{1}{5} = \frac{1}{10}$$

$$P(A/B) = P(A) = \frac{1}{2}$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{1}{2}+\frac{1}{5}-\frac{1}{10} = \frac{3}{5}$$

Now,

$$P(A/A \cup B) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{1/2}{3/5} = \frac{5}{6}$$

$$P[(A \cap B)/(\bar{A} \cup \bar{B})] = P(A \cap B/\overline{(A \cap B)}) = 0$$

2. (1), (2), (3)

We are given that

$$P(A \cap B') = 0.20,\ P(A' \cap B) = 0.15,\ P(A \cap B) = 0.10$$

Now,

$$P(B) = P(A' \cap B) + P(A \cap B) = 0.15 + 0.20 = 0.35$$

$$\therefore\quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.10}{0.35} = \frac{2}{7}$$

$$P(A) = P(A \cap B') + P(A \cap B) = 0.3$$
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.55$$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1}{3}$$

3. (1), (2), (3)

Let 'H' be the event that married man watches the show and 'W' be the probability that married woman watches the show.

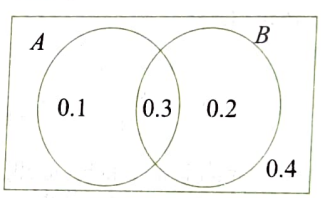
$$\therefore\quad P(H) = 0.4,\ P(W) = 0.5,\ P(H/W) = 0.7$$

a. $P(H \cap W) = P(W)\,P(H/W) = 0.5 \times 0.7 = 0.35$

b. $P(W/H) = \frac{P(H \cap W)}{P(H)} = \frac{0.35}{0.4} = \frac{7}{8}$

c. $P(H \cup W) = P(H) + P(W) - P(H \cap W)$
$= 0.4 + 0.5 - 0.35 = 0.55$

4. (2), (3)



$$P(A) = 0.4,\ P(B) = 0.5,\ P(A \cap B) = 0.3$$

$$\Rightarrow\quad P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$$

Now,

$$P(E_1) = \frac{P(A \cap B)}{P(A \cup B)} = \frac{0.3}{0.6} = \frac{1}{2}$$

$$P(E_2) = \frac{P(A \cap \bar{B}) + P(\bar{A} \cap B)}{P(A \cup B)} = \frac{0.1+0.2}{0.6} = \frac{1}{2}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = \frac{3}{5} = 0.60;$$

$$P(B/A) = \frac{0.3}{0.4} = \frac{3}{4} = 0.75$$

$$P(A/(A \cup B)) = \frac{P(A)}{P(A \cup B)} = \frac{0.4}{0.6} = \frac{2}{3}$$

$$\frac{P(B)}{P(A \cup B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

5. (3), (4)

$P(E_1) = 1 - P$ (unit's place in both is 1, 2, 3, 4, 6, 7, 8, 9)

$$P(E_1 = 0 \text{ or } 5) = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$$

$$P(E_2: 5) = P(1\ 3\ 5\ 7\ 9) - P(1\ 3\ 7\ 9)$$
$$= \frac{1}{4} - \frac{4}{25}$$
$$= \frac{25-16}{100} = \frac{9}{100}$$
$$\frac{P(E_2)}{P(E_1)} = \frac{9}{100} \times \frac{25}{9} = \frac{1}{4}$$
$$P(E_1) = 4P(E_2)$$
$$P(E_2/E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{P(E_2)}{P(E_1)} = \frac{1}{4}$$
$$P(E_1/E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_2)}{P(E_2)} = 1$$

6. (1), (2)

The probability that both will be alive for 10 years, hence, i.e., the probability that the man and his wife both will be alive 10 years hence is $0.83 \times 0.87 = 0.7221$. The probability that at least one of them will be alive is

$1 - P$ {That none of them remains alive 10 years hence}
$= 1 - (1 - 0.83)(1 - 0.87) = 1 - 0.17 \times 0.13$
$= 0.9779$

7. (2), (3), (4)

(1) False.

$$P(TTT \text{ or } HHH) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

(2) $$\frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B^c)}{1 - P(B)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$
$$P(A \cap B)[1 - P(B)] = P(B)\,P(A) - P(B)\,P(A \cap B)$$
$$P(A \cap B) = P(A)\,P(B)$$

Hence, the given statement is true.

(3) Let E_1 be the white ball drawn in first draw; E_2 be the event that black ball is drawn in second draw; E be the event that white ball is drawn in second draw. Then

$$P(E) = P(E/E_1)\,P(E_1) + P(E/E_2)\,P(E_2)$$
$$= \frac{d+w}{w+b+d}\left(\frac{w}{w+b}\right) + \frac{w}{w+b+d}\left(\frac{b}{w+b}\right)$$
$$= \left(\frac{w}{w+b}\right)\left(\frac{d+w}{w+b+d} + \frac{b}{w+b+d}\right)$$
$$= \left(\frac{w}{w+b}\right)$$

which is independent of d.

(4) To prove that A, B, C are pairwise independent only. Now,

$$P(A \cap B) = P((A \cap B \cap C) \cup (A \cap B \cap C))$$
$$= P(A \cap B \cap C) + P(A \cap B \cap C)$$
$$= P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(B)\,P(C) \quad \text{(given)}$$
$$= P(A) \times P(B)\,[P(C') + P(\bar{C})]$$
$$= P(A) \times P(B)$$

Similarly, for the other two. Hence, this statement is correct.

8. (1), (2), (3), (4)

(1) $$P(E_1) = 1 - P(RRR) = 1 - \left[\frac{1}{3} \times \frac{2}{4} \times \frac{3}{5}\right] = 0.9$$

(2) $$P(E_2) = 3P(BRR) = 3 \times \frac{2}{3} \times \frac{1}{4} \times \frac{2}{5} = 0.2$$

(3) $$P(E_3) = P(RRR/(RRR \cup BBB))$$
$$= \frac{P(RRR)}{P(RRR) + P(BBB)}$$
$$= \frac{0.1}{0.1 + \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5}}$$
$$= \frac{0.1}{0.1 + 0.4} = 0.2$$

(4) $$P(E_4) = 1 - P(BBB) = 1 - \frac{2}{5} = 0.6$$

9. (1), (2), (3), (4)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
or $$\frac{5}{8} = \frac{3}{8} + \frac{4}{8} - P(A \cap B)$$
or $$P(A \cap B) = \frac{2}{8} = \frac{1}{4}$$

Now,

$$P(A^c/B) = \frac{P(A^c \cap B)}{P(B)}$$
$$= \frac{P(B) - P(A \cap B)}{P(B)}$$
$$= 1 - 2\left(\frac{1}{4}\right)$$
$$= \frac{1}{2}$$

$$2P(A/B^c) = \frac{2P(A \cap B^C)}{P(B^C)}$$
$$= \frac{2(P(A) - P(A \cap B))}{1 - P(B)}$$
$$= 4\left(\frac{3}{8} - \frac{2}{8}\right) = \frac{1}{2}$$

Hence, option (1) is correct.

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4} \times \frac{2}{1} = \frac{1}{2} = P(B)$$

Hence, option (2) is correct. Again,

$$P(A^c/B^c) = \frac{P(A^C \cap B^C)}{P(B^C)}$$
$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$
$$= 2\left(1 - \frac{5}{8}\right) = \frac{3}{4}$$

$$P(B/A^c) = \frac{P(B \cap A^C)}{1 - P(A)}$$
$$= \frac{P(B) - P(A \cap B)}{5/8}$$
$$= \frac{1/2 - 1/4}{5/8}$$
$$= \frac{1}{4} \times \frac{8}{5}$$
$$= \frac{2}{5}$$

Hence,

$$8P(A^c/B^c) = 15P(B/A^c)$$

Hence, option (3) is not correct. Again,

$$2P(A/B^c) = \frac{1}{2}$$

or $$P(A/B^c) = \frac{1}{4} = P(A \cap B)$$

Hence, option (4) is correct.

10. (1), (2), (4)

We have, the probability that the bomb strikes the target is $p = 1/2$. Let n be the number of bombs which should be dropped to ensure 99% chance or better of completely destroying the target. Then, the probability that out of n bombs at least two bombs strike the target is greater than 0.99. Let X denote the number of bombs striking the target. Then

$$P(X = r) = {}^nC_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{n-r} = {}^nC_r\left(\frac{1}{2}\right)^n, r = 0, 1, 2, \ldots, n$$

We should have

$$P(X \geq 2) \geq 0.99$$

$$\Rightarrow \quad \{1 - P(X < 2)\} \geq 0.99$$

$$\Rightarrow \quad 1 - \{P(X = 0) + P(X = 1)\} \geq 0.99$$

$$\Rightarrow \quad 1 - \left\{(1+n)\frac{1}{2^n}\right\} \geq 0.99$$

$$\Rightarrow \quad 0.001 \geq \frac{1+n}{2^n}$$

$$\Rightarrow \quad 2^n > 100 + 100n$$

$$\Rightarrow \quad n \geq 11$$

Thus, the minimum number of bombs is 11.

11. (1), (2), (3)

Option (4) is true if and only if A and B are independent.

12. (1), (2), (3)

$$P(A/B) = P(A) = \frac{1}{2}$$

$$P\{A/(A \cup B)\} = \frac{P[A \cap (A \cup B)]}{P(A \cup B)}$$

$$[\because A \cap (A \cup B) = A \cap (A - B - A \cap B)]$$

$$= A - A \cap B - A \cap B = a\}$$

$$\Rightarrow \quad P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{5} - \frac{1}{10}} = \frac{\frac{1}{2}}{\frac{6}{10}} = \frac{5}{6}$$

and similarly,

$$P\left(\frac{A \cap B}{A' \cup B'}\right) = 0$$

13. (2), (3)

Let $P(A) = x$ and $P(B) = y$. Since A and B are independent events, therefore,

$$P(\bar{A} \cap B) = 2/15$$

$$\Rightarrow \quad P(\bar{A})P(B) = 2/15$$

$$\Rightarrow \quad (1 - P(A))\,P(B) = 2/15$$

$$\Rightarrow \quad (1 - x)y = 2/15 \qquad (1)$$

and $$P(A \cap \bar{B}) = \frac{1}{6} \Rightarrow P(A)\,P(\bar{B}) = \frac{1}{6}$$

$$\Rightarrow \quad x(1 - y) = \frac{1}{6}$$

$$\Rightarrow \quad x - xy = \frac{1}{6} \qquad (2)$$

Subtracting Eq. (1) from Eq. (2), we get

$$x - y = \frac{1}{30} \Rightarrow x = \frac{1}{30} + y$$

Putting this value of x in Eq. (1), we get

$$y - y\left(\frac{1}{30} + y\right) = \frac{2}{15}$$

or $$30y - y - 30y^2 = 2/5$$

or $$30y^2 - 29y + 4 = 0$$

or $$(6y - 1)(5y - 4) = 0$$

$$\Rightarrow \quad y = 1/6 \text{ or } y = 4/5$$

$$\Rightarrow \quad P(B) = 1/6 \text{ or } P(B) = 4/5$$

14. (1), (2), (3)

(1) $P(A) = \frac{1}{5}, P(B) = \frac{7}{25}, P(B/A) = \frac{9}{10}$

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - P(A)\,P(B/A)\right]$$

$$= 1 - \left[\frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{7}{25}\right] = \frac{7}{10}$$

(2) $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$$= \frac{P(A)\,P(B/A)}{P(B)}$$

$$= \frac{\frac{1}{5} \times \frac{9}{10}}{\frac{7}{25}}$$

$$= \frac{9}{50} \times \frac{25}{7} = \frac{9}{14} = \frac{18}{28}$$

(3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= P(A) + P(B) - P(A)P(B/A)$$

$$= \frac{1}{5} + \frac{7}{25} - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{10 + 14 - 9}{50}$$

$$= \frac{3}{10}$$

(4) $P(A' \cup B') = 1 - P(A \cap B)$

$$= 1 - P(A)P(B/A)$$

$$= 1 - \frac{1}{5} \times \frac{9}{10}$$

$$= \frac{41}{50}$$

15. (1), (4)

The probability that head appears r times is

$$^{99}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{99-r}$$

which is maximum when $r = 49$ or 50.

16. (1), (3)

Let one probability of choosing one integer k be $P(k) = \lambda/k^4$. (λ is one constant of proportionality). Then

$$\sum_{k=1}^{2m}\frac{\lambda}{k^4}=1$$

or $$\lambda\sum_{k=1}^{2m}\frac{1}{k^4}=1$$

Let x_1 be the probability of choosing the odd number. Then

$$x_1=\sum_{k=1}^{m}P(2k-1)=\lambda\sum_{k=1}^{m}\frac{1}{(2k-1)^4}$$

Also,

$$1-x_1=\sum_{k=1}^{m}P(2k)$$

$$=\lambda\sum_{k=1}^{m}\frac{1}{(2k)^4}$$

$$<\lambda\sum_{k=1}^{m}\frac{1}{(2k-1)^4}$$

$\Rightarrow \quad 1-x_1<x_1$

$\Rightarrow \quad x_1>1/2$

$\Rightarrow \quad x_2<1/2$

17. (1), (2), (3)

Let X = defective and Y = non-defective.

Then all possible outcomes are $\{XX, XY, YX, YY\}$

Also, $P(XX)=\frac{50}{100}\times\frac{50}{100}=\frac{1}{4}$

$$P(XY)=\frac{50}{100}\times\frac{50}{100}=\frac{1}{4}$$

$$P(YX)=\frac{50}{100}\times\frac{50}{100}=\frac{1}{4}$$

$$P(YY)=\frac{50}{100}\times\frac{50}{100}=\frac{1}{4}$$

Here, $A=(XX)\cup(XY)$; $B=(XY)\cup(YY)$; $C=(XX)\cup(YY)$

$\therefore \quad P(A)=P(XX)+P(XY)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

$\therefore \quad P(B)=P(XY)+P(YX)=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

Now, $P(A\times B)=P(XY)=\frac{1}{4}=P(A)\times P(B)$

Thus, A and B are independent events.

$$P(B\times C)=P(YX)=\frac{1}{4}=P(B)\times P(C)$$

Thus, B and C are independent events.

$$P(C\times A)=P(XY)=\frac{1}{4}=P(C)\times P(A)$$

Thus, C and A are independent events.

$$P(A\times B\times C)=0 \text{ (impossible events)}$$

$$\neq P(A)\times P(B)\times P(C)$$

Thus, A, B and C are dependent events.

Therefore, we can conclude that A, B, C are pairwise independent but A, B, C are dependent.

Linked Comprehension Type

1. (2), 2. (3), 3. (1)

Let $P(i)$ be the probability that exactly i students are passing an examination. Now given that

$$P(A_i)=\lambda i^2 \text{ (where } \lambda \text{ is constant)}$$

$$\Rightarrow \quad \sum_{i=1}^{10}P(A_i)=\sum_{i=1}^{10}\lambda i^2=\lambda\frac{10\times11\times21}{6}=\lambda\times385=1$$

$\Rightarrow \quad \lambda = 1/385$

Now, $P(5) = 25/385 = 5/77$.

Let A represent the event that selected students have passed the examination. Therefore,

$$P(A)=\sum_{i=1}^{10}P(A/A_i)P(A_i)$$

$$=\sum_{i=1}^{10}\frac{i}{10}\frac{i^2}{385}$$

$$=\frac{1}{3850}\sum_{i=1}^{10}i^3$$

$$=\frac{10^2\times11^2}{4\times3850}=\frac{11}{14}$$

Now,

$$P(A_1/A)=\frac{P(A/A_1)P(A_1)}{P(A)}$$

$$=\frac{\frac{1}{385}\times\frac{1}{10}}{\frac{11}{14}}$$

$$=\frac{1}{11\times55}\times\frac{1}{5}$$

$$=\frac{1}{3025}$$

4. (4) Let p_1 be the probability of being an answer correct from section 1. Then $p_1 = 1/5$. Let p_2 be the probability of being an answer correct from section 2. Then $p_2 = 1/15$. Hence, the required probability is $1/5 \times 1/15 = 1/75$.

5. (4) Scoring 10 marks from four questions can be done in 3 + 3 + 3 + 1 = 10 ways so as to answer 3 questions from section 2 and 1 question from section 1 correctly. Hence, the required probability is

$$\frac{^{10}C_3\,^{10}C_1}{^{20}C_4}=\frac{1}{5}\left(\frac{1}{15}\right)^3$$

6. (2) To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10}\,(1/15)^{10}$. Hence, probability of getting a score less than 40 is

$$1-\left(\frac{1}{5}\right)^{10}\left(\frac{1}{15}\right)^{10}=1-\left(\frac{1}{75}\right)^{10}$$

7. (3), 8. (2), 9. (4)

A: She gets a success

T: She studies 10 h: $P(T) = 0.1$

S: She studies 7 h: $P(S) = 0.2$

F: She studies 4 h: $P(F) = 0.7$

$P(A/T) = 0.8,\ P(A/S) = 0.6,\ P(A/F) = 0.4$

$$P(A) = P(A \cap T) + P(A \cap S) + P(A \cap F)$$
$$= P(T)\,P(A/T) + P(S)\,P(A/S) + P(F)\,P(A/F)$$
$$= (0.1)(0.8) + (0.2)(0.6) + (0.7)(0.4)$$
$$= 0.08 + 0.12 + 0.28 = 0.48$$

$$= P(F/A) = \frac{P(F \cap A)}{P(A)}$$

$$= \frac{(0.7)\,(0.4)}{0.48}$$

$$= \frac{0.28}{0.48} = \frac{7}{12}$$

$$P(F/\bar{A}) = \frac{P(F \cap \bar{A})}{P(\bar{A})}$$

$$= \frac{P(F) - P(F \cap A)}{0.52}$$

$$= \frac{(0.7) - 0.28}{0.52}$$

$$= \frac{0.42}{0.52} = \frac{21}{26}$$

10. (2) 11. (1), 12. (3)

$$P(S/T) = \frac{P(S \cap T)}{P(T)}$$

or $$0.5 = \frac{P(S \cap T)}{0.69}$$

or $$P(S \cap T) = 0.5 \times 0.69 = P(S)\,P(T)$$

Therefore, S and T are independent.

$$\therefore \quad P(S \text{ and } T) = P(S)P(T)$$
$$= 0.69 \times 0.5 = 0.345$$
$$P(S \text{ or } T) = P(S) + P(T) - P(S \cap T)$$
$$= 0.5 + 0.69 - 0.345$$
$$= 0.8450$$

13. (4) Let E be the event that all the amoeba population dies out;

E_1 be the event that after first second amoeba splits into two;

E_2 be the event that after first second amoeba remains the same.

Then

$$P(E) = P(E_1)P(E/E_1) + P(E_2)P(E/E_2)$$
$$= \frac{1}{2} \times \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4}$$
$$= \frac{3}{32}$$

14. (2) After 2 s, exactly 4 amoeba are alive, i.e., initially amoeba must split into two and in 2nd second again amoeba must split into two. Hence, the required probability is $(1/2) \times (1/2) \times (1/2) = 1/8$.

15. (1) In first second, mother amoeba splits into two with the probability 1/2.

In 2nd second, two amoeba split into two with probability

$(1/2) \times (1/2) = 1/4$

In 3rd second, four amoeba split into two with probability

$(1/2) \times (1/2) \times (1/2) \times (1/2) = 1/16$

Hence, the probability that the population is maximum after 3 s is $(1/2) \times (1/4) \times (1/16) = 1/2^7$.

16. (1) $P(A_2) = \frac{18}{36}$;

$$P(A_3) = \frac{12}{36} = \frac{1}{3};$$

$$P(A_4) = \frac{9}{36} = \frac{1}{4}$$

$$P(A_5) = \frac{7}{36}$$

$$P(A_6) = \frac{6}{36} = \frac{1}{6}$$

Hence, A_3 is most probable.

17. (3) $P(A_2) = \frac{1}{2}, P(A_3) = \frac{1}{3}, P(A_6) = \frac{1}{6}$

$$\therefore \quad P(A_2 \cap A_3) = P(A_2)\,P(A_3)$$
$$\Rightarrow \quad P(A_6) = P(A_2)\,P(A_3)$$
$$\frac{6}{36} = \frac{1}{2} \times \frac{1}{3}$$

Hence, A_2 and A_3 are independent.

18. (4) Note that A_1 is independent with all events $A_1, A_2, A_3, A_4, \ldots, A_{12}$. Now, total number of ordered pairs is 23.

$$\underbrace{(1,1), (1,2), (1,3), \ldots, (1,11)}_{22} + (1,12)$$

Also, A_2, A_3 and A_3, A_2 are independent. Hence, there are 25 ordered pairs.

19. (1) 20. (3), 21. (3)

The scores of n can be reached in the following two mutually exclusive events:

(i) by throwing a head when the score is $(n-1)$;

(ii) by throwing a tail when the score is $(n-2)$

Hence,

$$P_n = P_{n-1} \times \frac{1}{2} + P_{n-2} \times \frac{1}{2} \quad [\because P(\text{head}) = P(\text{tail}) = 1/2]$$
$$= \frac{1}{2}[P_{n-1} + P_{n-2}] \qquad (1)$$

$$\Rightarrow \quad P_0 + \frac{1}{2}P_{n-1} = P_{n-1} + \frac{1}{2}P_{n-2}$$

(adding $(1/2)\,P_{n-1}$ on both sides)

$$= P_{n-2} + \frac{1}{2}P_{n-3}$$
$$\vdots$$
$$= P_2 + \frac{1}{2}P_1 \qquad (2)$$

Now, a score of 1 can be obtained by throwing a head at a single toss. Therefore,

$$P_1 = \frac{1}{2}$$

And a score of 2 can be obtained by throwing either a tail at a single toss or a head at the first toss as well as second toss. Therefore,

$$P_2 = \frac{1}{2} + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{3}{4}$$

From Eq. (2), we have

$$P_n + \frac{1}{2}P_{n-1} = \frac{3}{4} + \frac{1}{2}\left(\frac{1}{2}\right) = 1$$

or $$P_n = 1 - \frac{1}{2}P_{n-1}$$

or $$P_n - \frac{2}{3} = 1 - \frac{1}{2}P_{n-1} - \frac{2}{3}$$

or $$P_n - \frac{2}{3} = -\frac{1}{2}\left(P_{n-1} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^2\left(P_{n-2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^3\left(P_{n-3} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1}\left(P_1 - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{2} - \frac{2}{3}\right)$$

$$= \left(-\frac{1}{2}\right)^{n-1}\left(-\frac{1}{6}\right)$$

$$= \left(-\frac{1}{2}\right)^n \frac{1}{3}$$

or $$P_n = \frac{2}{3} + \frac{(-1)^n}{2^n}\frac{1}{3} = \frac{1}{3}\left\{2 + \frac{(-1)^n}{2^n}\right\}$$

Now,

$$P_{100} = \frac{2}{3} + \frac{1}{3\times 2^{100}} > \frac{2}{3}$$

and $$P_{101} = \frac{2}{3} - \frac{1}{3\times 2^{101}} < \frac{2}{3}$$

$$\Rightarrow \quad P_{101} < \frac{2}{3} < P_{100}$$

Matrix Match Type

1. a → p; b → r; c → q; d → p.

$$P(A) = \frac{{}^4C_1\,{}^8C_2}{{}^{12}C_3} = \frac{4\,28}{220} = \frac{112}{220} = \frac{28}{55}$$

$$P(B) = \frac{{}^4C_3 + {}^8C_3}{{}^{12}C_3} = \frac{4+56}{220} = \frac{60}{220} = \frac{3}{11}$$

$P(C) = P(WBB$ or BWB or WWB or $BBB)$

$$= \frac{8}{12}\times\frac{4}{11}\times\frac{3}{10}\times\frac{4}{12}\times\frac{8}{11}\times\frac{3}{10}\times\frac{8}{12}\times\frac{7}{11}\times\frac{4}{10}\times\frac{4}{12}\times\frac{3}{11}\times\frac{2}{10}$$

$$= \frac{96+96+224+24}{12\times 110} = \frac{440}{12\times 110} = \frac{1}{3}$$

$A \cap B = \phi \Rightarrow A$ and B are mutually exclusive

$$P(B \cap C) = P(BBB) = \frac{4\times 3\times 2}{12\times 110} = \frac{1}{55}$$

Also,

$$P(B)\,P(C) = \frac{3}{11}\times\frac{1}{3} = \frac{1}{11}$$

Hence, B and C are neither independent nor mutually exclusive.

$$P(C \cap A) = P(WWB)\,\frac{8\times 7\times 4}{12\times 11\times 10} = \frac{28}{3\times 55}$$

$$P(C)\,P(A) = \frac{1}{3}\times\frac{112}{220} = \frac{28}{3\times 55}$$

Thus, C and A are independent.

Also, A, B, C are mutually exclusive as A and B are mutually exclusive.

2. a → q, r, s; b → r; c → p, q; d → p, q, r, s.

a. Suppose the coin is tossed n times. The probability of getting head or tail is 1/2. The probability of not getting any head in n tosses is $(1/2)^n$. The probability of getting at least one head is $1 - (1/2)^n$. Now given that

$$1 - (1/2)^n \geq 0.8$$

or $$\left(\frac{1}{2}\right)^n \leq 0.2$$

or $$2^n \geq 5$$

Therefore, the least value of n is 3.

b. The total number of mappings is n^n. The number of one-one mappings is $n!$ Hence the probability is

$$\frac{n!}{n^n} = \frac{3}{32} = \frac{6}{64} = \frac{3!}{4^3} = \frac{4!}{4^4}$$

Comparing, we get $n = 4$.

c. Given equation is

$$2x^2 + 2mx + m + 1 = 0$$

$$D = 4m^2 - 8(m+1) \geq 0$$

$$m^2 - 2m - 2 \geq 0$$

$$(m-1)^2 - 3 \geq 0$$

$$\Rightarrow \quad m = 3, 4, 5, 6, 7, 8, 9, 10$$

Also, the number of ways of choosing m is 10. Therefore, the required probability is 4/5.

$$\therefore \quad 5k = 4$$

d. $20P^2 - 13P + 2 \leq 0$

$$\Rightarrow \quad (4P-1)(5P-2) \leq 0$$

$$\Rightarrow \quad \frac{1}{4} \leq P \leq \frac{2}{5}$$

$$\Rightarrow \quad \frac{1}{4} \leq \frac{1}{5} + \frac{1}{5}\left(\frac{4}{5}\right) + \frac{1}{5}\left(\frac{4}{5}\right)^2 + \cdots + \frac{1}{5}\left(\frac{4}{5}\right)^{n-1} \leq \frac{2}{5}$$

$$\Rightarrow \quad n = 2$$

Hence, maximum as well as minimum value of n is 2.

3. a → r, s; b → p, q, r, s; c → p, q, r, s; d → p.

a. P(success) = 1/2; P(failure) = 1/2

Suppose 'n' bombs are to dropped. Let E be the event that the bridge is destroyed. Then,

$$P(E) = 1 - P(0 \text{ or } 1 \text{ success})$$

$$= 1 - \left(\left(\frac{1}{2}\right)^n + {}^nC_1\,\frac{1}{2}\left(\frac{1}{2}\right)^{n-1}\right)$$

$$= 1 - \left(\frac{1}{2^n} + \frac{n}{2^n}\right) \geq 0.9$$

$$\Rightarrow \quad \frac{1}{10} \geq \frac{n+1}{2^n} \text{ or } \frac{2^n}{10(n+1)} \geq 1$$

b. The bag contains 2 red, 3 white and 5 black balls. Hence $P(S) = 1/5$; $P(F) = 4/5$; Let E be the event of getting a red ball.

$$P(E) = P(S \text{ or } FS \text{ or } FFS \text{ or } \ldots] \geq \frac{1}{2}$$

$$\therefore \quad P(F)^n \leq \frac{1}{2}; \left(\frac{4}{5}\right)^n \leq \frac{1}{2}$$

The value of n consistent is 4.

c. Let there be x red socks and y blue socks and $x > y$. Then

$$\frac{{}^xC_2 + {}^yC_2}{{}^{x+y}C_2} = \frac{1}{2}$$

or $$\frac{x(x-1) + y(y-1)}{(x+y)(x+y-1)} = \frac{1}{2}$$

Multiplying both sides by $2(x+y)(x+y-1)$ and expanding, we get

$$2x^2 - 2x + 2y^2 - 2y = x^2 + 2xy + y^2 - x - y$$

Rearranging, we have

$$x^2 - 2xy + y^2 = x + y$$

or $(x-y)^2 = x + y$

or $|x-y| = x + y$

Now, $x + y \leq 17$

$$x - y \leq \sqrt{17}$$

As $x - y$ must be an integer, so

$$x - y = 4$$

$\therefore \quad x + y = 16$

Adding both together and dividing by 2 yields $x \leq 10$.

d. Let the number of green socks be $x > 0$. Let E: be the event that two socks drawn are of the same colour.

$$P(E) = P(RR \text{ or } BB \text{ or } WW \text{ or } GG)$$

$$= \frac{3}{{}^{6+x}C_2} + \frac{{}^xC_2}{{}^{6+x}C_2}$$

$$= \frac{6}{(x+6)(x+5)} + \frac{x(x-1)}{(x+6)(x+5)}$$

$$= \frac{1}{5}$$

$\Rightarrow \quad 5(x^2 - x + 6) = x^2 + 11x + 30$

or $4x^2 - 16x = 0$

or $x = 4$

4. a → q, r; b → r; c → p; d → a, b, c, d.

a. $$\frac{{}^rC_2}{{}^{r+b}C_2} = \frac{1}{2} \; 2r(r-1) = (r-b)(r+b-1)$$

$$= 2r(r-1) = (r+b)(r+b-1)$$

or $2r^2 - 2r = r^2 + (2b-1)r + b^2 - 1$

or $r^2 - (1+2b)r + 1 - b^2 = 0$

or $b^2 + 2br + r - r^2 - 1 = 0$

or $$b = \frac{-2r \pm \sqrt{4r^2 - 4(r - r^2 - 1)}}{2}$$

$$= -r \pm \sqrt{2r^2 - r + 1}$$

Since b is integer, possible values of r are 3 and 8.

b. $$^4C_2 \left(\frac{r}{r+10}\right)^2 \left(\frac{10}{r+10}\right)^2 = \frac{3}{8}$$

c. $$\left(\frac{r}{r+10}\right)^2 \left(\frac{10}{r+10}\right)^2 = \frac{1}{16}$$

$\Rightarrow \quad r = 10$

d. Probability of getting exactly n red balls in $2n$ draws is always equal to probability of getting exactly n black balls in $2n$ draws for any value of r and b, hence the ratio r/b can be 10, 3, 8, 2.

5. (2) We have,

$$P(A \cap B) = P(A)\,P(B) = \frac{1}{12}$$

a. $$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2}$$

b. $$P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)} = \frac{2}{3}$$

c. $$(C)\; P\left(\frac{B}{A' \cap B'}\right) = \frac{P(B \cap (A' \cap B'))}{P(A' \cap B')} = \frac{P(\phi)}{P(A' \cap B')} = 0$$

d. $$P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B)}{P(B)} = P(A') = \frac{2}{3}$$

6. (4) Let E_i denote the event that the bag contains i black and $(12 - i)$ white balls ($i = 0, 1, 2, \ldots, 12$) and A denote the event that the four balls drawn are all black. Then

$$P(E_i) = \frac{1}{13} \; (i = 0, 1, 2, \ldots, 12)$$

$$P\left(\frac{A}{E_i}\right) = 0 \text{ for } i = 0, 1, 2, 3$$

$$P\left(\frac{A}{E_i}\right) = \frac{{}^iC_4}{{}^{12}C_4} \text{ for } i \geq 4$$

a. $$P(A) = \sum_{i=0}^{12} P(E_i)\,P\left(\frac{A}{E_i}\right)$$

$$= \frac{1}{13} \times \frac{1}{{}^{12}C_4}\left[{}^4C_4 + {}^5C_4 + \cdots + {}^{12}C_4\right]$$

$$= \frac{{}^{13}C_5}{13 \times {}^{12}C_4} = \frac{1}{5}$$

b. Clearly,

$$P\left(\frac{A}{E_{10}}\right) = \frac{{}^{10}C_4}{{}^{12}C_4} = \frac{14}{33}$$

c. By Bayer's theorem,

$$P\left(\frac{E_{10}}{A}\right) = \frac{P(E_{10})\left(\frac{A}{E_{10}}\right)}{P(A)}$$

$$= \frac{\frac{1}{13} \times \frac{14}{33}}{\frac{1}{5}} = \frac{70}{429}$$

d. Let B denote the probability of drawing 2 white and 2 black balls. Then

$$P\left(\frac{B}{E_i}\right) = 0 \text{ if } i = 0, 1 \text{ or } 11, 12$$

$$P\left(\frac{B}{E_i}\right)=\frac{{}^{i}C_2\times{}^{12-i}C_2}{{}^{12}C_4} \text{ for } i=2,3,\ldots,10$$

$$\therefore \quad P(B)=\sum_{i=0}^{12}P(E_i)\,P\left(\frac{B}{E_i}\right)$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}[2\{{}^{2}C_2\times{}^{10}C_2+{}^{3}C_2+\cdots+{}^{10}C_2\times{}^{2}C_2]$$

$$=\frac{1}{13}\times\frac{1}{{}^{12}C_4}[2\{{}^{2}C_2\,{}^{10}C_2+{}^{3}C_2\,{}^{9}C_2+\cdots+{}^{5}C_2\times{}^{7}C_2\}+{}^{6}C_2\times{}^{6}C_2]$$

$$=\frac{1}{13}\times\frac{1}{495}(1287)$$

$$=\frac{1}{5}$$

Numerical Value Type

1. (7) Let the probability of the faces 1, 3, 5 or 6 be p for each face.

Hence, probability of each of the faces 2 or 4 is $3p$

Now according to the question $4p+6p=1$

$$\Rightarrow \quad p=\frac{1}{10}$$

$$\therefore \quad P(1)=P(3)=P(5)=P(6)=\frac{1}{10}$$

and $\quad P(2)=P(4)=\dfrac{3}{10}$

$\Rightarrow$ Required probability

$p=P$(total of 7 with a draw of dice)

$=P(16, 61, 25, 52, 43, 34)$

$$=2\left(\frac{1}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)+2\left(\frac{3}{10}\cdot\frac{1}{10}\right)$$

$$=\frac{2+6+6}{100}=\frac{14}{100}=\frac{7}{50}$$

2. (1) There are n white balls in the turn.

$\Rightarrow$ Probability of Mr. A to draw two balls of same color is

$$\frac{{}^{3}C_2+{}^{n}C_2}{{}^{n+3}C_2}=\frac{1}{2}\text{ (given)}$$

or $\quad \dfrac{6+n(n-1)}{(n+3)(n+2)}=\dfrac{1}{2}$

or $\quad n^2-7n+6=0$

$\Rightarrow \quad n=1$ or 6 (1)

Also required probability for Mr. B according to the question is

$$\frac{3}{n+3}\,\frac{3}{n+3}+\frac{n}{n+3}\,\frac{n}{n+3}=\frac{5}{8} \quad \text{(given)}$$

Solving, we get $n^2-10n+9=0$; $n=1$ or 9 (2)

From (1) and (2), $n=1$

3. (2) When A and B are mutually exclusive, $P(A\cap B)=0$

$\therefore \quad P(A\cup B)=P(A)+P(B)$ (1)

$\Rightarrow \quad 0.8=0.5+p$

or $\quad p=0.3$ (2)

$P(A\cup B)=P(A)+P(B)$

$=P(A)+P(B)-P(A\cup B)$

$=P(A)+P(B)-P(A)P(B)$

or $\quad 0.8=0.5+q-(0.5)\,q$

or $\quad 0.3=q/2$

or $\quad q=0.6$

$\Rightarrow \quad q/p=2$ (3)

4. (3) For ranked 1 and 2 players to be winners and runners up, respectively, they should not be paired with each other in any round:

$$\Rightarrow \quad p=\frac{30}{31}\times\frac{14}{15}\times\frac{6}{7}\times\frac{2}{3}=\frac{16}{31}$$

5. (6) $P(A/B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.6}{0.8}=\dfrac{3}{4}$

(Maximum value of $P(A\cap B)=P(A)=0.6$)

6. (5) Highest number in three throws is 4, which means at least one of the throws must be equal to 4.

Number of ways when three blocks are filled from $\{1,2,3,4\}=4^3$

$\therefore$ Number of ways when filled from $\{1,2,3\}=3^3$

$\therefore$ Required number of ways $=4^3-3^3$

$$\therefore \quad \text{Probability, } p=\frac{4^3-3^3}{6^3}=\frac{37}{216}$$

7. (4) Let event A: Card is of heart but not king (12 cards)

Event B : King but not heart (3 cards)

Event C: Heart and king (1 card)

$\therefore$ Required probability

$$p=P(E)=\frac{{}^{12}C_1\cdot{}^{3}C_1+{}^{3}C_1\cdot{}^{1}C_1+{}^{1}C_1\cdot{}^{12}C_1}{{}^{52}C_2}=\frac{2}{52}$$

$\therefore \quad 104\,p=4$

8. (2) Here $P(E)=\dfrac{1}{2}$ and $P(F_k)={}^{n}C_k\cdot\dfrac{1}{2^n}$

Also $P(E\cap F_k)=P$(exactly k heads are obtained and head obtained in first flip)

$$=\frac{1}{2}\cdot{}^{n-1}C_{k-1}\left(\frac{1}{2}\right)^{n-1}$$

Events E and F_k are independent. Therefore,

$P(E\cap F_k)=P(E)\cdot P(F_k)$

or $\quad {}^{n-1}C_{k-1}\times\dfrac{1}{2^n}=\dfrac{1}{2}\times{}^{n}C_k\dfrac{1}{2^n}$

or $\quad 2\times{}^{n-1}C_{k-1}={}^{n}C_k$

or $\quad n=2k$

9. (3) $P(E_1)=1-[P\text{ (all heads)}+P\text{ (all tails)}]$

$$=1-\left[\frac{1}{2^n}+\frac{1}{2^n}\right]$$

$$=1-\frac{1}{2^{n-1}}$$

$P(E_2)=P$ (no head) + P(exactly one head)

$$=\frac{1}{2^n}+{}^{n}C_1\times\frac{1}{2^n}=\frac{n+1}{2^n}$$

$P(E_1\cap E_2)=$ (exactly one head and $(n-1)$ tail)

$$={}^{n}C_1\times\frac{1}{2}\times\frac{1}{2^{n-1}}=\frac{n}{2^n}$$

If E_1 and E_2 are independent, then

$$\frac{n}{2^n}=\left(1-\frac{1}{2^{n-1}}\right)\left(\frac{n+1}{2^n}\right)$$

or $n=\left(1-\frac{1}{2^{n-1}}\right)(n+1)$

or $n=n+1-\frac{n+1}{2^{n-1}}$

or $n+1=2^{n-1}$

or $n=3$

10. (6) Given S_2 reaches the semifinals.

Since all other players (2^n-1) are equally likely to win the finals with probability p. We have

$$(2^n-1)p+\frac{1}{4}=1$$

or $(2^n-1)p=\frac{3}{4}$

or $p=\frac{3}{4(2^n-1)}$

If $p=\frac{1}{84}$, then

$$\frac{1}{84}=\frac{3}{4(2^n-1)}$$

or $2^n-1=63$

or $2^n=64$

or $n=6$

Archives

JEE Main

Singe Correct Answer Type

1. (1) $S=\{00, 01, 02, \ldots, 49\}$

Let A be the event that the sum of the digits on the selected ticket is 8. Then

$A=\{08, 17, 26, 35, 44\}$

Let B be the event that the product of the digits is zero. Than

$B=\{00, 01, 02, 03, \ldots, 09, 10, 20, 30, 40\}$

$\therefore$ $A\cap B=\{8\}$

The required probability is

$$P(A/B)=\frac{P(A\cap B)}{P(B)}$$

$$=\frac{\frac{1}{50}}{\frac{14}{50}}=\frac{1}{14}$$

2. (1) According to the question

$$1-q^n\geq\frac{9}{10}$$

$$\Rightarrow \left(\frac{3}{4}\right)^n\leq\frac{1}{10}$$

$$\Rightarrow n\geq -\log_{\frac{3}{4}}10$$

$$\Rightarrow n\geq\frac{1}{\log_{10}4-\log_{10}3}$$

3. (4) Probability of at least one failure

$$=1-\text{no failure}\geq\frac{31}{32}$$

$$\Rightarrow 1-p^5\geq\frac{31}{32}$$

$$\Rightarrow p^5\leq\frac{1}{32}$$

$$\Rightarrow p\geq\frac{1}{2}$$

Also $p\leq 0$

Hence $p\in\left[0,\frac{1}{2}\right]$

4. (3) We have

$C\subset D$

$\Rightarrow$ $C\cap D=C$

$$\Rightarrow P\left(\frac{C}{D}\right)=\frac{P(C\cap D)}{P(D)}=\frac{P(C)}{P(D)}\geq P(C)$$

As $0<P(D)\leq 1$

5. (2) Given set of numbers is $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

We have to select three numbers

Let us assume that

event $A\rightarrow$ in selected three numbers minimum is 3

event $B\rightarrow$ in selected three numbers maximum is 6

We have to find $P(A/B)$.

Now $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{n(A\cap B)}{n(B)}$

$A\cap B$ = in selected three numbers, minimum is 3 and maximum is 6

$\therefore$ The remaining number is either 4 or 5

$\therefore$ $n(A\cap B)=2$

$n(B)$ = selecting two numbers from 1, 2, 3, 4, 5 (as maximum of numbers is 6)

$={}^5C_2$

$\therefore$ $P(A/B)=\frac{n(A\cap B)}{n(B)}=\frac{2}{{}^5C_2}=\frac{1}{5}$

6. (3) Answer for any question can be guessed in 3 ways.

$\therefore$ Probability that any question is correct $=\frac{1}{3}$

From five questions if at least four questions are correctly answered then required probability

= Probability that four questions are correctly answered + Probability that five questions are correctly answered

$$={}^5C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)+{}^5C_5\left(\frac{1}{3}\right)^5=\frac{11}{3^5}$$

7. (3) $P(\overline{A\cup B})=\frac{1}{6}$

$$P(A\cup B)=\frac{5}{6},\ P(A)=\frac{3}{4}$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{5}{6}$$

$$\Rightarrow P(B)=\frac{5}{6}-\frac{3}{4}+\frac{1}{4}=\frac{1}{3}$$

$$\Rightarrow P(A\cap B)=P(A)\cdot P(B)=\frac{1}{4}$$

8. (1) Since the 3 boxes are identical, probability of selecting a box (in which we are to put 3 balls) is 1/3.

$$\Rightarrow p=\frac{1}{3},\ q=1-\frac{1}{3}=\frac{2}{3}$$

Probability that the selected box should contain exactly 3 balls is the same as probability of exactly 3 success in 12 trials (putting the 12 balls in 3 boxes).

$\therefore$ Required probability $= {}^{12}C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^9 = \frac{55}{3}\left(\frac{2}{3}\right)^{11}$

9. (3) E_1 : {(4, 1), ..., (4, 6)} $\rightarrow$ 6 cases

E_2 : {(1, 2), ..., (6, 2)} $\rightarrow$ 6 cases

E_3 : 18 cases (sum of both is odd)

$\therefore \quad P(E_1) = \frac{3}{6} = \frac{1}{6} = P(E_2)$

$$P(E_3) = \frac{18}{36} = \frac{1}{2}$$

$$P(E_1 \cap E_2) = \frac{1}{36}$$

$$P(E_2 \cap E_3) = \frac{3}{36} = \frac{1}{12}$$

Similarly $P(E_3 \cap E_1) = \frac{1}{12}$

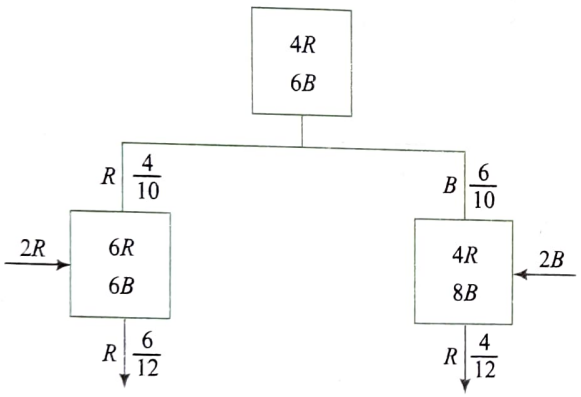
$$P(E_1 \cap E_2 \cap E_3) = 0$$

$\therefore \quad E_1, E_2, E_3$ are not independent.

10. (2) $p = \frac{15}{25}, q = \frac{10}{25}, n = 10$

$$\sigma^2 = npq = 10 \cdot \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{5}$$

11. (3)



From the figure, required probability,

$$P(E) = \frac{4}{10} \times \frac{6}{12} + \frac{6}{10} \times \frac{4}{12} = \frac{48}{120} = \frac{2}{5}$$

JEE ADVANCED

Singe Correct Answer Type

1. (3) Even G = original signal is green

$E_1 = A$ receives the signal correct

$E_2 = B$ receives the signal correct

E = Signal received *by B* is green

P (Signal received *by B* is green)

$$= P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2) + P(\bar{G}E_1\bar{E}_2) + P(\bar{G}\bar{E}_1E_2)$$

$$P(E) = \frac{46}{5 \times 16}$$

$$P(G/E) = \frac{40/5 \times 16}{46/5 \times 16} = \frac{20}{23}$$

2. (1) P (at least one of them solves correctly) $= 1 - P$ (none of them solves correctly)

$$= 1 - \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} \times \frac{7}{8}\right) = \frac{235}{256}$$

3. (3) $P(T_1) = \frac{1}{5}$

$$P(T_2) = \frac{4}{5}$$

$$P(D) = \frac{7}{100},$$

where D is the event that computers produced in the factory turn out to be defective.

Given $P\left(\frac{D}{T_1}\right) = 10 \, . \, P\left(\frac{D}{T_2}\right)$

Let $P\left(\frac{D}{T_2}\right) = x$

$\therefore \quad P(T_1) \times P\left(\frac{D}{T_1}\right) + P(T_2) \, . \, P\left(\frac{D}{T_2}\right) = P(D) = \frac{7}{100}$

$\Rightarrow \quad \frac{1}{5} \times 10x + \frac{4}{5} \times x = \frac{7}{100}$

$\Rightarrow \quad x = \frac{1}{40}$

$$\therefore \quad P\left(\frac{T_2}{D'}\right) = \frac{P\left(\frac{D'}{T_2}\right)P(T_2)}{P\left(\frac{D'}{T_1}\right)P(T_1) + P\left(\frac{D'}{T_2}\right)P(T_2)}$$

$$= \frac{\frac{39}{40} \times \frac{4}{5}}{\frac{30}{40} \times \frac{1}{5} + \frac{39}{40} \times \frac{4}{5}}$$

$$= \frac{\frac{4}{5} \times \frac{39}{40}}{\frac{93}{100}} = \frac{78}{93}$$

Multiple Correct Answers Type

1. (1), (4)

Let $P(E) = e$ and $P(F) = f$

$$P(E \cup F) - P(E \cap F) = \frac{11}{25}$$

$\Rightarrow \quad e + f - 2ef = \frac{11}{25}$ (1)

$$P(\bar{E} \cap \bar{F}) = \frac{2}{25}$$

$\Rightarrow \quad (1 - e)(1 - f) = \frac{2}{25}$

or $\quad 1 - e - f + ef = \frac{2}{25}$ (2)

From (1) and (2),

$$ef = \frac{12}{25} \text{ and } e + f = \frac{7}{5}$$

Solving, we get

$$e = \frac{4}{5}, f = \frac{3}{5} \text{ or } e = \frac{3}{5}, f = \frac{4}{5}$$

2. (2), (4)

$$P(X_1)=\frac{1}{2},\ P(X_2)=\frac{1}{4},\ P(X_3)=\frac{1}{4}$$

$$P(X)=P(X_1\cap X_2\cap X_3)+P(X_1\cap X_2^C\cap X_3)$$
$$+P(X_1^C\cap X_2\cap X_3)+P(X_1\cap X_2\cap X_3^C)=\frac{1}{4}$$

(1) $P(X_1^C|X)=\dfrac{P(X\cap X_1^C)}{P(X)}=\dfrac{1/32}{1/4}=\dfrac{1}{8}$

(2) P [exactly two engines of the ship are functioning

$$|X]=\frac{7/32}{1/4}=\frac{7}{8}$$

(3) $P\left(\dfrac{X}{X_2}\right)=\dfrac{P(X\cap X_2)}{P(X_2)}=\dfrac{5/32}{1/4}=\dfrac{5}{8}$

(4) $P\left(\dfrac{X}{X_1}\right)=\dfrac{P(X\cap X_1)}{P(X_1)}=\dfrac{7/32}{1/2}=\dfrac{7}{16}$

3. (1), (2)

We have $P(X)=\dfrac{1}{3},\ \dfrac{P(X\cap Y)}{P(Y)}=\dfrac{1}{2}$ and $\dfrac{P(Y\cap X)}{P(X)}=\dfrac{2}{5}$

$$\therefore\quad P(X\cap Y)=\frac{P(Y)}{2}=\frac{2}{5}P(X)=\frac{2}{5}\cdot\frac{1}{3}=\frac{2}{15}$$

$$\Rightarrow\quad P(Y)=\frac{4}{15}$$

$$P(X'|Y)=\frac{P(X'\cap Y)}{P(Y)}$$

$$=\frac{P(Y)-P(X\cap Y)}{P(Y)}=\frac{\frac{4}{15}-\frac{2}{15}}{\frac{4}{15}}=\frac{2}{4}=\frac{1}{2}$$

$$P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$$
$$=\frac{1}{3}+\frac{4}{15}-\frac{2}{15}=\frac{7}{15}$$

Linked Comprehension Type

1. (1) $P(X=3)=\left(\dfrac{5}{6}\right)\left(\dfrac{5}{6}\right)\dfrac{1}{6}=\dfrac{25}{216}$

2. (2) Given that

$$P(X\le 2)=\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}=\frac{11}{36}$$

Hence, the required probability is $1-(11/36)=25/36$.

3. (4) For $X\ge 6$, the probability is

$$\frac{5^5}{6^6}+\frac{5^5}{6^7}+\cdots\infty=\frac{5^5}{6^6}\left(\frac{1}{1-5/6}\right)=\left(\frac{5}{6}\right)^5$$

For $X>3$,

$$\frac{5^3}{6^4}+\frac{5^4}{6^5}+\frac{5^5}{6^6}+\cdots\infty=\left(\frac{5}{6}\right)^3$$

Hence, the conditional probability is

$$\frac{(5/6)^6}{(5/6)^3}=\frac{25}{36}$$

4. (2) $H\to$ 1 ball form U_1 to U_2

$T\to$ 2 ball form U_1 to U_2

E : 1 ball drawn from U_2

$P(W$ from $U_2)$

$$=\frac{1}{2}\times\left(\frac{3}{5}\times 1\right)+\frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right)+\frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times\frac{1}{3}\right)$$
$$+\frac{1}{2}\times\left(\frac{{}^3C_1\cdot{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right)=\frac{23}{30}$$

5. (4) $P\left(\dfrac{H}{W}\right)=\dfrac{P(W/H)\times P(H)}{P(W/T).P(T)+(W/H).P(H)}$

$$=\frac{\frac{1}{2}\left(\frac{3}{2}\times 1+\frac{2}{5}\times\frac{1}{2}\right)}{23/30}=\frac{12}{23}$$

6. (1) P(required)

$=P$ (all are white) + P (all are red) + P (all are black)

$$=\frac{1}{6}\times\frac{2}{9}\times\frac{3}{12}+\frac{3}{6}\times\frac{3}{9}\times\frac{4}{12}+\frac{2}{6}\times\frac{4}{9}\times\frac{5}{12}$$
$$=\frac{6}{648}+\frac{36}{648}+\frac{40}{648}=\frac{82}{648}$$

7. (4) Let A : one ball is white and other is red

E_1 : both balls are from box B_1,

E_2 : both balls are form box B_2,

E_3 : both balls are form box B_3

Here, P(required) $=P\left(\dfrac{E_2}{A}\right)$

$$=\frac{P\left(\frac{A}{E_2}\right).P(E_2)}{P\left(\frac{A}{E_1}\right).P(E_1)+P\left(\frac{A}{E_2}\right).P(E_2)+P\left(\frac{A}{E_3}\right).P(E_3)}$$

$$=\frac{\frac{{}^2C_1\times{}^3C_1}{{}^9C_2}\times\frac{1}{3}}{\frac{{}^1C_1\times{}^3C_1}{{}^6C_2}\times\frac{1}{3}+\frac{{}^2C_1\times{}^3C_1}{{}^9C_2}\times\frac{1}{3}+\frac{{}^3C_1\times{}^4C_1}{{}^{12}C_2}\times\frac{1}{3}}.$$

$$=\frac{\frac{1}{6}}{\frac{1}{5}+\frac{1}{6}+\frac{2}{11}}=\frac{55}{181}$$

8. (1), (2)

Box I $<$ Red $\to n_1$, Black $\to n_2$

Box II $<$ Red $\to n_3$, Black $\to n_4$

$$P(R)=\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}$$

$$P(\text{II}/R)=\frac{\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}{\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}$$

$$= \frac{\frac{n_3}{n_3+n_4}}{\frac{n_1}{n_1+n_2}+\frac{n_3}{n_3+n_4}}$$

by option $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

$$P(II/R) = \frac{\frac{5}{20}}{\frac{3}{6}+\frac{5}{20}} = \frac{\frac{1}{4}}{\frac{1}{2}+\frac{1}{4}} = \frac{1}{4} \times \frac{4}{2+1} = \frac{1}{3}$$

Also, when $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$; $P(II/R) = 1/3$

9. (3), (4)

Given $\frac{n_1}{n_1+n_2} \cdot \frac{n_1-1}{n_1+n_2-1} + \frac{n_2}{n_1+n_2} \cdot \frac{n_1}{n_1+n_2-1} = \frac{1}{3}$

$\Rightarrow \quad 3(n_1^2 - n_1 + n_1n_2) = (n_1 + n_2)(n_1 + n_2 - 1)$

$\Rightarrow \quad 3n_1(n_1 + n_2 - 1) = (n_1 + n_2)(n_1 + n_2 - 1)$

$\Rightarrow \quad 2n_1 = n_2$

10. (2) $P(X > Y) = T_1T_1 + DT_1 + T_1D$ (where T_1 represents wins and D represents draw)

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{6} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{6}\right) = \frac{5}{12}$$

11. (3) $P(X = Y) = DD + T_1T_2 + T_2T_1$

$$= \left(\frac{1}{6} \times \frac{1}{6}\right) + \left(\frac{1}{2} \times \frac{1}{3}\right) + \left(\frac{1}{3} \times \frac{1}{2}\right) = \frac{13}{36}$$

Numerical Value Type

1. (6) Let $P(E_1) = x$, $P(E_2) = y$ and $P(E_3) = z$, then

$(1 - x)(1 - y)(1 - z) = p$

$x(1 - y)(1 - z) = \alpha$,

$(1 - x)y(1 - z) = \beta$,

$(1 - x)(1 - y)(z) = \gamma$

So $\frac{1-x}{x} = \frac{p}{\alpha} = x = \frac{\alpha}{\alpha + p}$,

Similarly, $z = \frac{\gamma}{\gamma + p}$.

So $\frac{P(E_1)}{P(E_3)} = \frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}} = \frac{\frac{\gamma+p}{\gamma}}{\frac{\alpha+p}{\alpha}} = \frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}$

Also given $\frac{\alpha\beta}{\alpha - 2\beta} = p = \frac{2\beta\gamma}{\beta - 3\gamma} \Rightarrow \beta = \frac{5\alpha\gamma}{\alpha + 4\gamma}$

Substituting in given relation

$$\left(\alpha - 2\left(\frac{5\alpha\gamma}{\alpha + 4\gamma}\right)\right)p = \frac{\alpha \times 5\alpha\gamma}{\alpha + 4\gamma}$$

or $\left(\frac{p}{\gamma} + 1\right) = 6\left(\frac{p}{\alpha} + 1\right)$

or $\frac{\frac{p}{\gamma}+1}{\frac{p}{\alpha}+1} = 6.$

2. (8) Let coin be tossed n times.

P(at least two heads) = 1 – P(no heads) – P(exactly one head)

P(at least two heads) $= 1 - \left(\frac{1}{2}\right)^n - {}^nC_1 \cdot \left(\frac{1}{2}\right)^n \geq 0.96$

$\Rightarrow \quad \frac{4}{100} \geq \frac{n+1}{2^n}$

$\Rightarrow \quad \frac{2^n}{n+1} \geq 25$

Therefore, least value of n is 8.

1 APPENDIX

Chapterwise Solved January 2019 JEE Main Questions (All Sets)

Chapter 1 (Set Theory and Real Number System)

1. In a class of 140 students numbered 1 to 140, all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is

(1) 102 (2) 42 (3) 1 (4) 38

Chapter 2 (Theory of Equations)

2. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$. Then $\alpha^{15} + \beta^{15}$ is equal to

(1) 512 (2) -512 (3) -256 (4) 256

3. If both the roots of the quadratic equation $x^2 - mx + 4 = 0$ are real and distinct and they lie in the interval $[1, 5]$, then m lies in the interval

(1) $(4, 5]$ (2) $(3, 4)$ (3) $(5, 6)$ (4) $(-5, -4)$

4. The number of all possible positive integral values of α for which the roots of the quadratic equation $6x^2 - 11x + \alpha = 0$ are rational numbers is

(1) 2 (2) 5 (3) 3 (4) 4

5. Consider the quadratic equation $(c - 5)x^2 - 2cx + (c - 4) = 0$, $c \neq 5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval $(0, 2)$ and its other root lies in the interval $(2, 3)$. Then the number of elements in S is

(1) 11 (2) 18 (3) 10 (4) 12

6. The values of λ such that sum of the squares of the roots of the quadratic equation $x^2 + (3 - \lambda)x + 2 = \lambda$ has the least value is

(1) 2 (2) $\frac{4}{9}$ (3) $\frac{15}{8}$ (4) 1

7. If one real root of the quadratic equation $81x^2 + kx + 256 = 0$ is cube of the other root, then a value of k is

(1) -81 (2) 100 (3) -300 (4) 144

8. Let α and β be the roots of the quadratic equation $x^2 \sin\theta - x(\sin\theta\cos\theta + 1) + \cos\theta = 0$ $(0 < \theta < 45^\circ)$, and $\alpha < \beta$. Then $\sum_{n=0}^{\infty}\left(\alpha^n + \frac{(-1)^n}{\beta^n}\right)$ is equal to

(1) $\frac{1}{1-\cos\theta} + \frac{1}{1+\sin\theta}$ (2) $\frac{1}{1+\cos\theta} + \frac{1}{1-\sin\theta}$

(3) $\frac{1}{1-\cos\theta} - \frac{1}{1+\sin\theta}$ (4) $\frac{1}{1+\cos\theta} - \frac{1}{1-\sin\theta}$

9. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2 + m(m - 4)x + 2 = 0$, then the least value of m for which $\lambda + \frac{1}{\lambda} = 1$ is

(1) $2 - \sqrt{3}$ (2) $4 - 3\sqrt{2}$

(3) $-2 + \sqrt{2}$ (4) $4 - 2\sqrt{3}$

10. The number of integral values of m for which the quadratic expression $(1 + 2m)x^2 - 2(1 + 3m)x + 4(1 + m)$, $x \in R$, is always positive is

(1) 8 (2) 7 (3) 6 (4) 3

Chapter 3 (Complex Numbers)

11. Let

$$A = \left\{0 \in \left(-\frac{\pi}{2}, \pi\right) : \frac{3 + 2i\sin\theta}{1 - 2i\sin\theta} \text{ is purely imaginary}\right\}.$$

Then the sum of the elements in A is

(1) $\frac{5\pi}{6}$ (2) $\frac{2\pi}{3}$ (3) $\frac{3\pi}{4}$ (4) π

12. Let z_0 be a root of the quadratic equation $x^2 + x + 1 = 0$. If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then arg z is equal to

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{3}$ (3) 0 (4) $\frac{\pi}{6}$

13. Let z_1 and z_2 be any two non-zero complex numbers such that $3|z_1| = 2|z_2|$. If $z = \frac{3z_1}{2z_2} + \frac{2z_2}{3z_1}$, then

(1) $|z| = \frac{1}{2}\sqrt{\frac{17}{2}}$ (2) $\text{Re}(z) = 0$

(3) $|z| = \sqrt{\frac{5}{2}}$ (4) $\text{Im}(z) = 0$

14. Let $z=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5+\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$. If $R(z)$ and $I(z)$, respectively, denote the real and imaginary parts of z, then

(1) $R(z)>0$ and $I(z)>0$ (2) $R(z)<0$ and $I(z)>0$

(3) $R(z)=-3$ (4) $I(z)=0$

15. Let $\left(-2-\frac{1}{3}i\right)^3=\frac{x+iy}{27}$ $(i=\sqrt{-1})$, where x and y are real numbers. Then $y-x$ equals

(1) 85 (2) 85 (3) -91 (4) 91

16. If $\frac{z-\alpha}{z+\alpha}$ $(\alpha\in R)$ is a purely imaginary number and $|z|=2$, then a value of α is

(1) 1 (2) 2 (3) $\sqrt{2}$ (4) $\frac{1}{2}$

17. Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1|=9$ and $|Z_2-3-4i|=4$. Then the minimum value of $|Z_1-Z_2|$ is

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

Chapter 4
(Principle of Mathematical Induction)

18. Consider the statement: "$P(n)$: n^2-n+41 is prime". Then which one of the following is true?

(1) $P(5)$ is false but $P(3)$ is true

(2) Both $P(3)$ and $P(5)$ are false

(3) $P(3)$ is false but $P(5)$ is true

(4) Both $P(3)$ and $P(5)$ are true

Chapter 5
(Progression and Series)

19. If a, b and c are three distinct real numbers in G. P. and $a+b+c=xb$, then x cannot be

(1) 4 (2) -3 (3) -2 (4) 2

20. Let $a_1, a_2, ..., a_{30}$ be an A. P., $S=\sum_{i=1}^{30}a_i$ and $T=\sum_{i=1}^{15}a_{(2i-1)}$.

If $a_5=27$ and $S-2T=75$, then a_{10} is equal to

(1) 57 (2) 47 (3) 42 (4) 52

21. The sum of series

$$1+6+\frac{9(1^2+2^2+3^2)}{7}+\frac{12(1^2+2^2+3^2+4^2)}{9}+\frac{15(1^2+2^2+...+5^2)}{11}+...$$

up to 15 terms is

(1) 7820 (2) 7830 (3) 7520 (4) 7510

22. Let a, b and c be the 7th, 11th and 13th terms, respectively, of a non-constant A.P.. If these are also the three consecutive terms of a G.P., then $\frac{a}{c}$ is equal to

(1) 1/2 (2) 4 (3) 2 (4) 7/13

23. The sum of all two-digit positive numbers which when divided by 7 yield 2 or 5 as remainder is

(1) 1365 (2) 1256 (3) 1465 (4) 1356

24. If 5, $5r$ and $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to

(1) $\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $\frac{5}{4}$ (4) $\frac{7}{4}$

25. The sum of an infinite geometric series with positive terms is 3 and the sums of the cubes of its terms is $\frac{27}{19}$. Then the common ratio of this series is

(1) $\frac{4}{9}$ (2) $\frac{2}{9}$ (3) $\frac{2}{3}$ (4) $\frac{1}{3}$

26. Let $a_1, a_2, ..., a_{10}$ be a G.P.. If $\frac{a_3}{a_1}=25$, then $\frac{a_9}{a_5}$ equals

(1) $2(5^2)$ (2) $4(5^2)$ (3) 5^4 (4) 5^3

27. If 19^{th} term of a non-zero A.P. is zero, then (49^{th} term) : (29^{th} term) is

(1) 3 : 1 (2) 4 : 1 (3) 2 : 1 (4) 1 : 3

28. The product of three consecutive terms of a G.P. is 512. If 4 is added to each of the first and the second of these terms, the three terms now form an A.P.. Then the sum of the original three terms of the given G.P. is

(1) 36 (2) 24 (3) 32 (4) 28

29. Let $S_k=\frac{1+2+3+...+k}{k}$. If $S_1^2+S_2^2+...+S_{10}^2=\frac{5}{12}A$, then A is equal to

(1) 303 (2) 283 (3) 156 (4) 301

30. If the sum of the first 15 terms of the series $\left(\frac{3}{4}\right)^3+\left(1\frac{1}{2}\right)^3+\left(2\frac{1}{4}\right)^3+3^3+\left(3\frac{3}{4}\right)^3+...$ is equal to $225k$, then k is equal to

(1) 9 (2) 27 (3) 108 (4) 54

Chapter 6
(Inequalities Involving Means)

31. Let x, y be positive real numbers and m, n be positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{m+n}{6mn}$ (4) 1

Chapter 7
(Permutation and Combination)

32. Consider a class of 5 girls and 7 boys. The number of different teams consisting of 2 girls and 3 boys that can be formed from this class, if there are two specific boys A and B, who refuse to be the members of the same team, is

(1) 200 (2) 300 (3) 500 (4) 350

33. Let S be the set of all triangles in the xy–plane, each having one vertex at the origin and the other two vertices lie on coordinate axes with integral coordinates. If each triangle in S has area 50 sq. units, then the number of elements in the set S is

(1) 9 (2) 18 (3) 32 (4) 36

34. The number of natural numbers less than 7,000 which can be formed by using the digits 0, 1, 3, 7, 9 (repetition of digits allowed) is equal to

(1) 250 (2) 374 (3) 372 (4) 375

35. The number of functions f from $\{1, 2, 3, \ldots, 20\}$ onto $\{1, 2, 3, \ldots, 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is

(1) $(15)! \times 6!$ (2) $5^6 \times 15$

(3) $5! \times 6!$ (4) $6^5 \times (15)!$

36. Let $S = \{1, 2, 3, \ldots, 100\}$. Then number of non-empty subsets A of S such that the product of elements in A is even is

(1) $2^{50}(2^{50} - 1)$ (2) $2^{100} - 1$

(3) $2^{50} - 1$ (4) $2^{50} + 1$

37. Consider three boxes, each containing 10 balls labelled 1, 2, ..., 10. Suppose one ball is randomly drawn from each of the boxes denoted by n_i, the label of the ball drawn from the i^{th} box, $(i = 1, 2, 3)$. Then, the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is

(1) 82 (2) 240 (3) 164 (4) 120

38. Let Z be the set of integers. If $A = \{x \in Z : 2(x + 2)(x^2 - 5x + 6)\} = 1$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$ is

(1) 2^{18} (2) 2^{10} (3) 2^{15} (4) 2^{12}

39. There are m men and two women participating in a chess tournament. Each participant plays two games with every other participant. If the number of games played by the men between themselves exceeds the number of games played between the men and the women by 84, then the value of m is

(1) 9 (2) 11 (3) 12 (4) 7

Chapter 8 (Binomial Theorem)

40. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$ then k is equal to

(1) 14 (2) 6 (3) 4 (4) 8

41. The coefficient of t^4 in the expansion of $\left(\frac{1 - t^6}{1 - t}\right)^3$ is

(1) 12 (2) 15 (3) 10 (4) 14

42. If $\sum_{i=1}^{20}\left(\frac{{}^{20}C_{i-1}}{{}^{20}C_i + {}^{20}C_{i-1}}\right)^3 = \frac{k}{21}$, then k equals

(1) 200 (2) 50 (3) 100 (4) 400

43. If the third term in the binomial expansion of $\left(1 + x^{\log_2 x}\right)^5$ equals 2560, then a possible value of x is

(1) $2\sqrt{2}$ (2) $\frac{1}{8}$ (3) $4\sqrt{2}$ (4) $\frac{1}{4}$

44. The positive value of λ for which the coefficient of x^2 in the expression $x^2\left(\sqrt{x} + \frac{\lambda}{x^2}\right)^{10}$ is 720 is

(1) $\sqrt{5}$ (2) 4 (3) $2\sqrt{2}$ (4) 3

45. If $\sum_{r=0}^{25}({}^{50}C_r \cdot {}^{50-r}C_{25-r}) = K({}^{50}C_{25})$, then K is equal to

(1) $2^{25} - 1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}

46. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3} + \frac{3}{x}\right)^8$ equals 5670 is

(1) 6 (2) 8 (3) 0 (4) 4

47. The value of r for which ${}^{20}C_r \times {}^{20}C_0 + {}^{20}C_{r-1} \times {}^{20}C_1 + {}^{20}C_{r-2} \times {}^{20}C_2 + \ldots + {}^{20}C_0 \times {}^{20}C_r$ is maximum is

(1) 20 (2) 15 (3) 11 (4) 10

48. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + \ldots + a_{50}x^{50}$, for all $x \in R$, then $\frac{a_2}{a_0}$ is equal to

(1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25

49. Let $S_n = 1 + q + q^2 + \ldots + q^n$ and

$$T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots + \left(\frac{q+1}{2}\right)^n,$$

where q is a real number and $q \neq 1$.

If ${}^{101}C_1 + {}^{101}C_2 \times S_1 + \ldots + {}^{101}C_{101} \times S_{100} = \alpha T_{100}$, then α is equal to

(1) 2^{100} (2) 200 (3) 2^{99} (4) 202

50. Ratio of the 5^{th} term from the beginning to the 5^{th} term from the end in the binomial expansion of $\left(2^{1/3} + \frac{1}{2(3)^{1/3}}\right)^{10}$ is

(1) $1 : 4(16)^{\frac{1}{3}}$ (2) $1 : 2(6)^{\frac{1}{3}}$

(3) $2(36)^{\frac{1}{3}} : 1$ (4) $4(36)^{\frac{1}{3}} : 1$

51. If ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in A.P., then n can be

(1) 14 (2) 11 (3) 9 (4) 12

52. The total number of irrational terms in the binomial expansion of $(7^{1/5} - 3^{1/10})^{60}$ is

(1) 55 (2) 49 (3) 48 (4) 54

Chapter 9 (Probability I)

53. Two integers are selected at random from the set $\{1, 2, \ldots, 11\}$. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is

(1) $\frac{2}{5}$ (2) $\frac{1}{2}$ (3) $\frac{3}{5}$ (4) $\frac{7}{10}$

54. Let $S = \{1, 2, \ldots, 20\}$. A subset B of S is said to be "nice", if the sum of the elements of B is 203. Then the probability that a randomly chosen subset of S is "nice" is

(1) $\frac{6}{2^{20}}$ (2) $\frac{5}{2^{20}}$ (3) $\frac{4}{2^{20}}$ (4) $\frac{7}{2^{20}}$

55. In a class of 60 students, 40 opted for NCC, 30 opted for NSS and 20 opted for both NCC and NSS. If one of these students is selected at random, then the probability that the student selected has opted neither for NCC nor for NSS is

(1) $\frac{2}{3}$ (2) $\frac{1}{6}$ (3) $\frac{1}{3}$ (4) $\frac{5}{6}$

56. In a game, a man wins Rs. 100 if he gets 5 of 6 on a throw of a fair die and loses Rs. 50 for getting any other number on the die. If he decides to throw the die either till he gets a five or a six or to a maximum of three throws, then his expected gain/loss (in rupees) is

(1) $\frac{400}{3}$ gain (2) $\frac{400}{3}$ loss

(3) 0 (4) $\frac{400}{9}$ loss

Chapter 10 (Mathematical Reasoning)

57. If the Boolean expression $(p \oplus q) \wedge (\sim p \odot q)$ is equivalent to $p \wedge q$, where $\oplus, \odot \in \{\vee, \wedge\}$, then the ordered pair $(\oplus, \odot)$ is

(1) $(\wedge, \vee)$ (2) $(\vee, \vee)$ (3) $(\wedge, \wedge)$ (4) $(\vee, \wedge)$

58. The logical statement $[\sim(\sim p \vee q) \vee (p \wedge r) \wedge (\sim q \wedge r)]$ is equivalent to

(1) $(p \wedge r) \wedge \sim q$ (2) $(\sim p \wedge \sim q) \wedge r$

(3) $\sim p \vee r$ (4) $(p \wedge \sim q) \vee r$

59. Consider the following three statements:

p : 5 is a prime number.

q : 7 is a factor of 192.

r : L.C.M. of 5 and 7 is 35.

Then the truth value of which one of the following statements is true?

(1) $(p \wedge q) \vee (\sim r)$ (2) $(\sim p) \wedge (\sim q \wedge r)$

(3) $(\sim p) \vee (q \wedge r)$ (4) $p \vee (\sim q \wedge r)$

60. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology?

(1) $(p \vee r) \rightarrow (p \wedge r)$ (2) $p \vee r$

(3) $p \wedge r$ (4) $(p \wedge r) \rightarrow (p \vee r)$

61. The Boolean expression $((p \wedge q) \vee (p \vee \sim q)) \wedge (\sim p \wedge \sim q)$ is equivalent to

(1) $p \wedge (\sim q)$ (2) $p \vee (\sim q)$

(3) $(\sim p) \wedge (\sim q)$ (4) $p \wedge q$

62. The expression $\sim(\sim p \rightarrow q)$ is logically equivalent to

(1) $\sim p \wedge \sim q$ (2) $p \wedge q$

(3) $\sim p \wedge q$ (4) $p \wedge \sim q$

Chapter 11 (Statistics)

63. Five students of a class have an average height 150 cm and variance 18 cm. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is

(1) 22 (2) 20 (3) 16 (4) 18

64. A data consists of n observations $x_1, x_2, \ldots, x_n$. If $\sum_{i=1}^{n} (x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n} (x_i - 1)^2 = 5n$, then the standard deviation of this data is

(1) 5 (2) $\sqrt{5}$ (3) $\sqrt{7}$ (4) 2

65. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is

(1) 4 : 9 (2) 6 : 7 (3) 5 : 8 (4) 10 : 3

66. If mean and standard deviation of 5 observations x_1, x_2, x_3, x_4, x_5 are 10 and 3, respectively, then the variance of 6 observations $x_1, x_2, \ldots, x_5$ and -50 is equal to

(1) 582.5 (2) 507.5 (3) 586.5 (4) 509.5

67. The outcome of each of 30 items was observed; 10 items gave an outcome $\frac{1}{2} - d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2} + d$ each. If the variance of this outcome data is $\frac{4}{3}$, then $|d|$ equals

(1) 2 (2) $\frac{\sqrt{5}}{2}$ (3) $\frac{2}{3}$ (4) $\sqrt{2}$

68. Contrapositive of the statement "If two numbers are not equal, then their squares are not equal." is

(1) If the squares of two numbers are equal, then the numbers are equal.

(2) If the squares of two numbers are equal, then the numbers are not equal.

(3) If the squares of two numbers are not equal, then the numbers are equal.

(4) If the squares of two numbers are not equal, then the numbers are not equal.

69. A bag contains 30 white balls and 10 red balls. 16 balls are drawn one by one randomly from the bag with replacement. If X is the number of white balls drawn, then $\left(\frac{\text{mean of } X}{\text{standard deviation of } X}\right)$ is equal to

(1) 4 (2) $\frac{4\sqrt{3}}{3}$ (3) $4\sqrt{3}$ (4) $3\sqrt{2}$

70. If the sum of the deviations of 50 observations from 30 is 50, then the mean of these observations is

(1) 50 (2) 51 (3) 30 (4) 31

71. The mean and the variance of five observations are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4, then absolute value of the difference of the other two observations is

(1) 1 (2) 3 (3) 7 (4) 5

Chapter 12 (Determinants)

72. The system of linear equations

$x + y + z = 2$

$2x + 3y + 2z = 5$

$2x + 3y + (a^2 - 1)z = a + 1$

(1) has infinitely many solutions for $a = 4$

(2) is inconsistent when $|a| = \sqrt{3}$

(3) is inconsistent when $a = 4$

(4) has a unique solution for $|a| = \sqrt{3}$

73. If the system of linear equations

$x - 4y + 7z = g$

$3y - 5z = h$

$-2x + 5y - 9z = k$

is consistent, then

(1) $g + h + k = 0$ (2) $2g + h + k = 0$

(3) $g + h + 2k = 0$ (4) $g + 2h + k = 0$

74. If the system of equations

$x + y + z = 5$

$x + 2y + 3z = 9$

$x + 3y + \alpha z = \beta$

has infinitely many solutions, then $\beta - \alpha$ equals

(1) 5 (2) 18 (3) 21 (4) 8

75. Let $a_1, a_2, a_3, ..., a_{10}$ be in G.P. with $a_i > 0$ for $i = 1, 2, ..., 10$ and S be the set of pairs (r, k); $r, k \in N$ (the set of natural numbers)

for which $\begin{vmatrix} \log_e a_1^r a_2^k & \log_e a_2^r a_3^k & \log_e a_3^r a_4^k \\ \log_e a_4^r a_5^k & \log_e a_5^r a_6^k & \log_e a_6^r a_7^k \\ \log_e a_7^r a_8^k & \log_e a_8^r a_9^k & \log_e a_9^r a_{10}^k \end{vmatrix} = 0$. Then

the number of elements in S is

(1) Infinitely many (2) 4

(3) 10 (4) 2

76. If the system of linear equations

$2x + 2y + 3z = a$

$3x - y + 5z = b$

$x - 3y + 2z = c$

where a, b and c are non-zero real numbers, has more than one solution, then

(1) $b - c - a = 0$ (2) $a + b + c = 0$

(3) $b + c - a = 0$ (4) $b - c + a = 0$

77. If $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a + b + c)$

$\times (x + a + b + c)^2$, $x \neq 0$ and $a + b + c \neq 0$, then x is equal to

(1) $-(a + b + c)$ (2) $2(a + b + c)$

(3) abc (4) $-2(a + b + c)$

78. An ordered pair (α, β) for which the system of linear equations

$(1 + \alpha)x + \beta y + z = 2$

$\alpha x + (1 + \beta)y + z = 3$

$\alpha x + \beta y + 2z = 2$

has a unique solution is

(1) $(1, -3)$ (2) $(-3, 1)$ (3) $(2, 4)$ (4) $(-4, 2)$

79. The set of all values of λ for which the system of linear equations

$x - 2y - 2z = \lambda x$

$x + 2y + z = \lambda y$

$-x - y = \lambda z$

has a non-trivial solution

(1) contains more than two elements

(2) is a singleton

(3) is an empty set

(4) contains exactly two elements

Chapter 13 (Matrices)

80. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, then the matrix A^{-50} when $\theta = \frac{\pi}{12}$, is equal to

(1) $\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (2) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$

(3) $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ (4) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

81. If $A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$,

then A is

(1) invertible only if $t = \frac{\pi}{2}$

(2) not invertible for any $t \in R$

(3) invertible for all $t \in R$

(4) invertible only if $t = \pi$

82. Let $d \in R$, and

$A = \begin{bmatrix} -2 & 4 + d & (\sin\theta) - 2 \\ 1 & (\sin\theta) + 2 & d \\ 5 & (2\sin\theta) - d & (-\sin\theta) + 2 + 2d \end{bmatrix}$, $\theta \in [0, 2\pi]$.

If the minimum value of det. (A) is 8, then a value of d is

(1) -7 (2) $2(\sqrt{2} + 2)$

(3) -5 (4) $2(\sqrt{2} + 1)$

83. Let $A=\begin{bmatrix}2 & b & 1\\ b & b^2+1 & b\\ 1 & b & 2\end{bmatrix}$ where $b>0$. Then the minimum value of $\frac{\det.(A)}{b}$ is

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $-2\sqrt{3}$ (4) $2\sqrt{3}$

84. Let $A=\begin{pmatrix}0 & 2q & r\\ p & q & -r\\ p & -q & r\end{pmatrix}$. If $AA^T=I_3$, then $|p|$ is

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{1}{\sqrt{6}}$ (4) $\frac{1}{\sqrt{3}}$

85. Let A and B be two invertible matrices of order 3×3. If $\det.(ABA^T)=8$ and $\det.(AB^{-1})=8$, then $\det.(BA^{-1}B^T)$ is equal to

(1) 16 (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

86. Let $P=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ 9 & 3 & 1\end{bmatrix}$ and $Q=[q_{ij}]$ be two 3×3 matrices such that $Q-P^5=I_3$. Then $\frac{q_{21}+q_{31}}{q_{32}}$ is equal to

(1) 15 (2) 9 (3) 135 (4) 10

87. If $A=\begin{bmatrix}1 & \sin\theta & 1\\ -\sin\theta & 1 & \sin\theta\\ -1 & -\sin\theta & 1\end{bmatrix}$, then for all $\theta\in\left(\frac{3\pi}{4},\frac{5\pi}{4}\right)$, $\det.(A)$ lies in the interval

(1) $\left[\frac{5}{2},4\right)$ (2) $\left(\frac{3}{2},3\right]$ (3) $\left(0,\frac{3}{2}\right]$ (4) $\left(1,\frac{5}{2}\right]$

Chapter 14 (Probability II)

88. Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Let X denote the random variable of number of aces obtained in the two drawn cards. Then $P(X=1)+P(X=2)$ equals

(1) 52/169 (2) 25/169 (3) 49/169 (4) 24/169

89. An urn contains 5 red and 2 green balls. A ball is drawn at random from the urn. If the drawn ball is green, then a red ball is added to the urn and if the drawn ball is red, then a green ball is added to the urn; the original ball is not returned to the urn. Now, a second ball is drawn at random from it. The probability that the second ball is red is

(1) $\frac{26}{49}$ (2) $\frac{32}{49}$ (3) $\frac{27}{49}$ (4) $\frac{21}{49}$

90. An unbiased coin is tossed. If the outcome is a head, then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail, then a card from a well-shuffled pack of nine cards numbered 1, 2, 3, ..., 9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is

(1) $\frac{13}{36}$ (2) $\frac{19}{36}$ (3) $\frac{19}{72}$ (4) $\frac{15}{72}$

91. If the probability of hitting a target by a shooter, in any shot is 1/3, then the minimum number of independent shots at the target required by him so that the probability of hitting the target at least once is greater than $\frac{5}{6}$ is

(1) 6 (2) 5 (3) 4 (4) 3

92. In a random experiment, a fair die is rolled until two fours are obtained in succession. The probability that the experiment will end in the fifth throw of the die is equal to

(1) $\frac{150}{6^5}$ (2) $\frac{175}{6^5}$ (3) $\frac{200}{6^5}$ (4) $\frac{225}{6^5}$

Answers Key

1. (4)	**2.** (3)	**3.** (1)	**4.** (3)	**5.** (1)
6. (1)	**7.** (3)	**8.** (1)	**9.** (2)	**10.** (2)
11. (2)	**12.** (1)	**13.** (4)	**14.** (4)	**15.** (4)
16. (2)	**17.** (1)	**18.** (4)	**19.** (4)	**20.** (4)
21. (1)	**22.** (2)	**23.** (4)	**24.** (4)	**25.** (3)
26. (3)	**27.** (1)	**28.** (4)	**29.** (1)	**30.** (2)
31. (2)	**32.** (2)	**33.** (4)	**34.** (2)	**35.** (1)
36. (1)	**37.** (4)	**38.** (3)	**39.** (3)	**40.** (4)
41. (2)	**42.** (3)	**43.** (4)	**44.** (2)	**45.** (3)
46. (3)	**47.** (1)	**48.** (4)	**49.** (1)	**50.** (4)
51. (1)	**52.** (4)	**53.** (1)	**54.** (2)	**55.** (2)
56. (3)	**57.** (1)	**58.** (1)	**59.** (4)	**60.** (4)
61. (3)	**62.** (1)	**63.** (2)	**64.** (2)	**65.** (1)
66. (2)	**67.** (4)	**68.** (1)	**69.** (3)	**70.** (4)
71. (3)	**72.** (2)	**73.** (2)	**74.** (4)	**75.** (1)
76. (1)	**77.** (4)	**78.** (3)	**79.** (2)	**80.** (1)
81. (3)	**82.** (3)	**83.** (4)	**84.** (1)	**85.** (2)
86. (4)	**87.** (2)	**88.** (2)	**89.** (2)	**90.** (3)
91. (2)	**92.** (2)			